GRAVITATION IN THE MICROCANONICAL ENSEMBLE: APPROPRIATE SCALING LEADING TO EXTENSIVITY AND THERMALIZATION

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Hard spheres with gravitational interactions

- We consider a **classical gravitational model** made with
  - N identical hard spheres (m,σ)
  - enclosed in a spherical box ($\Lambda = 4\pi R^3/3$)

- The corresponding **Hamiltonian** reads

$$H_N = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} + \frac{1}{2} \sum_{i \neq j} v(|\mathbf{r}_i - \mathbf{r}_j|)$$

$$v(r) = \infty \text{ for } r < \sigma \quad , \quad v(r) = -Gm^2/r \text{ for } r > \sigma$$

◇ No **dispersion** in shapes, sizes and masses
◇ No **sticking** leading to agregation
◇ Status of the **box** versus **Self-confinement** ?

Microcanonical description

Isolated system, with **fixed energy** E and no other **conserved quantity**.

♠ Microcanonical ensemble :

- Distribution in phase space

$$f_{\mathrm{micro}}(\mathbf{r}_1, ..., \mathbf{r}_N, \mathbf{p}_1, ..., \mathbf{p}_N) = \delta(E - H_N)$$

- Number of microstates

$$\Omega(E, N, \Lambda) = C_N \int_{\Lambda^N \times R^{3N}} \prod_i \mathrm{d}^3\mathbf{r}_i \mathrm{d}^3\mathbf{p}_i \delta(E - H_N)$$

◇ f_{micro} is a **stationary** solution of evolution equations
◇ $\Omega(E, N, \Lambda)$ is **finite** for $\sigma > 0$; it **diverges** for $\sigma = 0$ and $N \geq 3$ [POMEAU, 2007]

♠ Dynamical limitations :

- Existence of **quasi-stationary states**
  [ANTONI-RUFFO-TORCINI,2004],[CHAVANIS,2005]
- Possible **ergodicity breaking**

The scaling continuous limit

♠ Consider the **scaling limit** N $\rightarrow \infty$ with :

- $R = N^{1/5}\ell_0$ with ℓ_0 **fixed** $\longrightarrow$ particle density $\rho_{\rm p} = (3/(4\pi\ell_0^3))N^{2/5}$
- $m = N^{-2/5}m_0$ with m_0 **fixed** $\longrightarrow$ mass density $\rho = m\rho_{\rm p} = 3m_0/(4\pi\ell_0^3)$
- $\sigma = N^{-2/15}d_0$ with d_0 **fixed** $\longrightarrow$ packing fraction $\eta = \rho_{\rm p}\sigma^3 = 3d_0^3/(4\pi\ell_0^3)$
- $E = N(Gm_0^2/\ell_0)\varepsilon$ with ε **fixed** $\longrightarrow$ extensivity of energy

That scaling limit (SL) defines an infinite continuous medium in a stationary state controlled by two independent dimensionless parameters, namely the energy per particle ε in units of Gm_0^2/ℓ_0, and the packing fraction $\eta = d_0^3/(8\ell_0^3)$.

◇ **Different** from the usual thermodynamical limit N $\rightarrow \infty$ with E/N , $\rho_{\rm p}$ fixed
◇ **Different** from other scaling limits [de Vega-Sanchez, 2002], [Joyce, 2008]

Bounds for the potential energy

♠ For any allowed configuration, the potential energy

$$V_N = -\frac{1}{2} \sum_{i \neq j} \frac{Gm^2}{|\mathbf{r}_i - \mathbf{r}_j|}$$

is **larger** than that of the collapsed configuration where the N hard spheres make a single cluster with size $L_{\rm coll} \sim N^{1/3}\sigma$, which is of order $-Gm^2N^2/L_{\rm coll}$. In the scaling limit, this provides the **classical version of H-stability**

$$V_N > -V_0 N \text{ with } V_0 = C_{\rm HS}\frac{Gm_0^2}{d_0} > 0$$

♠ For any allowed configuration, the potential energy should be **smaller** than that of a homogeneous surface mass distribution $Nm/(4\pi R^2)$,

$$V_N < -N\frac{Gm_0^2}{2\ell_0}$$

Extensivity of potential energy

- Thanks to the extensivity of its upper and lower bounds, the average potential energy

$$\langle V_N \rangle = -\frac{1}{2}\int_{\Lambda^2} \mathrm{d}^3\mathbf{r}\mathrm{d}^3\mathbf{r}'\rho^{(2)}(\mathbf{r},\mathbf{r}')\frac{G}{|\mathbf{r}-\mathbf{r}'|}$$

should also be **extensive** in the scaling limit (like the potential energy of an homogeneous sphere with mass density ρ).

- Extensivity consistent with the expected scaling behaviours for $\mathbf{q}$,$\mathbf{q}'$,... fixed

$$\lim_{\mathrm{SL}} \rho^{(1)}(R\mathbf{q}) = \rho g^{(1)}(\mathbf{q};\varepsilon,\eta)$$

$$\lim_{\mathrm{SL}} \rho^{(2)}(R\mathbf{q},R\mathbf{q}') = \rho^2 g^{(2)}(\mathbf{q},\mathbf{q}';\varepsilon,\eta)$$

Fluctuations of the potential energy

♠ The fluctuations $\langle V_N^2 \rangle - [\langle V_N \rangle]^2$ can be expressed as spatial integrals of $1/|\mathbf{r}-\mathbf{r}'|^2$, $1/|\mathbf{r}-\mathbf{r}'||\mathbf{r}-\mathbf{r}''|$, and $1/|\mathbf{r}-\mathbf{r}'||\mathbf{r}''-\mathbf{r}'''|$ weighted respectively by **two-, three- and four-body** distribution functions. A simple estimation within the considered scaling limit provides

$$\langle V_N^2 \rangle - [\langle V_N \rangle]^2 = o(N^2)$$

♠ Accordingly, we will use in further estimations of averages involving V_N the ansatz :

$$V_N \rightarrow \langle V_N \rangle + W_N$$

for most contributing configurations with $W_N = o(N)$ when $N \rightarrow \infty$

◇ **Non-rigorous** although quite plausible (possible subtle correlations with other variables)

The inhomogeneous mass density

- The mass density is

$$\rho^{(1)}(\mathbf{r}) = \rho(\mathbf{r}) = m\langle\sum_{i=1}^{N}\delta(\mathbf{r}_i - \mathbf{r})\rangle$$

Using f_{micro}, the standard integration over the momenta $\mathbf{p}_i$ leads to

$$\rho(\mathbf{r}) = \mathrm{cst}\int_{\Lambda^{N-1},|\mathbf{r}_i-\mathbf{r}_j|>\sigma}\prod_{i=2}^{N}\mathrm{d}^3\mathbf{r}_i[E - V_N(\mathbf{r},\mathbf{r}_2,...,\mathbf{r}_N)]^{3N/2-1}$$

- Introduce the gravitational potential $\Phi(\mathbf{r}|\mathbf{r}_2,...,\mathbf{r}_N)$ at $\mathbf{r}$ created by the $(N-1)$ particles located at $\mathbf{r}_2,...,\mathbf{r}_N$. Since

$$V_N = V_{N-1} + m\Phi$$

we can exactly rewrite

$$\rho(\mathbf{r}) = \mathrm{cst}\left\langle\prod_{i=2}^{N}\theta(|\mathbf{r}_i - \mathbf{r}|/\sigma - 1)[E - V_{N-1}]^{3/2}[1 - \frac{m\Phi}{E - V_{N-1}}]^{3N/2-1}\right\rangle_{N-1}$$

Emergence of thermalization

- Rewrite

$$[1-\frac{m\Phi}{E-V_{N-1}}]^{3N/2-1}=\exp\left\{(3N/2-1)\ln[1-\frac{m\Phi}{E-V_{N-1}}]\right\}$$

Since $m\Phi = O(1)$ and $E - V_{N-1} = O(N)$, the expansion of the logarithm leads to

$$[1-\frac{m\Phi}{E-V_{N-1}}]^{3N/2-1}\sim\exp\left\{-\frac{(3N/2-1)m\Phi}{E-V_{N-1}}\right\}$$

- If we apply the fluctuation ansatz to V_{N-1} in the average defining $\rho(\mathbf{r})$, we can replace $E - V_{N-1}$ by $E - \langle V_{N-1}\rangle_{N-1}$ at leading order, and we find

$$\rho(\mathbf{r})\sim\text{cst}\left\langle\prod_{i=2}^{N}\theta(|\mathbf{r}_i-\mathbf{r}|/\sigma-1)\exp\left\{-\frac{(3N/2-1)m\Phi}{E-\langle V_{N-1}\rangle_{N-1}}\right\}\right\rangle_{N-1}$$

$\Rightarrow$ **THERMALIZATION** with T $\sim 2(E-\langle V_N\rangle)/(3N)$

Hydrostatic picture

- A hydrostatic approach is justified thanks to
  - **Local thermodynamical equilibrium** is ensured by hard-core repulsion entirely.
  - At the local scale, particles feel the **average gravitational potential**

$$\phi(\mathbf{r}) = \langle m\Phi \rangle = -\int_\Lambda \mathrm{d}^3\mathbf{r}' \rho(\mathbf{r}') \frac{Gm}{|\mathbf{r}' - \mathbf{r}|}$$

  - The local correlation length λ_{HS} is much **smaller** than the characteristic variation length R of $\rho(\mathbf{r})$.

- Accordingly, the hydrostatic equilibrium reads

$$\nabla P(\mathbf{r}) = -\rho(\mathbf{r}) \nabla \phi(\mathbf{r})$$

where $P(\mathbf{r})$ is the pressure for a homogeneous gas of hard spheres (**no gravitation**) at temperature T and number density $\rho(\mathbf{r})/m$

$$P(\mathbf{r}) = T \frac{\rho(\mathbf{r})}{m} p_{\mathrm{HS}}(\eta \rho(\mathbf{r})/\rho)$$

Limit of point-particles

♠ Once the SL has been taken, we can take the limit $\eta \to 0$ where $p_{\rm HS}(\eta\rho(\mathbf{r})/\rho) \to 1$. The integration of the hydrostatic equation provides

$$\rho(\mathbf{r}) = C \exp\left\{-\frac{\phi(\mathbf{r})}{T}\right\}$$

with

$$\phi(\mathbf{r}) = -\int_{\Lambda} \mathrm{d}^3\mathbf{r}' \rho(\mathbf{r}') \frac{Gm}{|\mathbf{r}' - \mathbf{r}|}$$

⇒ **ISOTHERMAL SPHERE**
[Emden,1907],[Antonov,1964],[Lynden-Bell,1968]

◇ **Continuity** of $\rho(\mathbf{r})$ when $\eta \to 0$ at fixed ε ?

Consistency checks of the a priori assumptions

♠ Within the hydrostatic approach :

- Mass distribution $\rho(\mathbf{r})$ varies on the scale R
- Correlations, like $[\rho^{(2)}(\mathbf{r},\mathbf{r}') - \rho(\mathbf{r})\rho(\mathbf{r}')]$, decay over the hard-sphere local correlation length λ_{HS} of order σ

♠ This implies :

- The average potential energy is indeed **extensive**, $\langle V_N \rangle = V_{\mathrm{self}} + V_{\mathrm{corr}}$ with $V_{\mathrm{self}} = O(N)$ and $V_{\mathrm{corr}} = O(N^{1/3})$.
- Fluctuations behave as $\langle V_N^2 \rangle - [\langle V_N \rangle]^2 = O(N)$

◇ Fluctuations similar to that of an **ordinary system** with short-range interactions at **thermodynamical equilibrium**.

Expected limitations at low negative energies

The hydrostatic approach should provide the **exact** mass distribution for ε positive large enough and η small enough. That approach should be no longer valid for ε sufficiently negative and/or η sufficiently large, because :

- Local **cristalisation** $\longrightarrow \rho(\mathbf{r})$ varies on the scale σ
- **Absence** of solutions for the hydrostatic equations
- **Multiplicity** of solutions for the hydrostatic equations $\longrightarrow$ **Phase transitions ?**

Concluding comments

- The scaling ensures the emergence of **local thermodynamical equilibrium**
- Gravitational interactions can be treated at the **mean-field** level
- Short-range repulsion **controls** the shape of the mass distribution

Open questions within the present scaling :

- States with ϵ sufficiently negative ?
- Evaporation ?
- Fragmentation ?
- Phase transitions [CHAVANIS, 2006]