Abrupt Bifurcation to chaotic scattering: view from the anti-integrable limit

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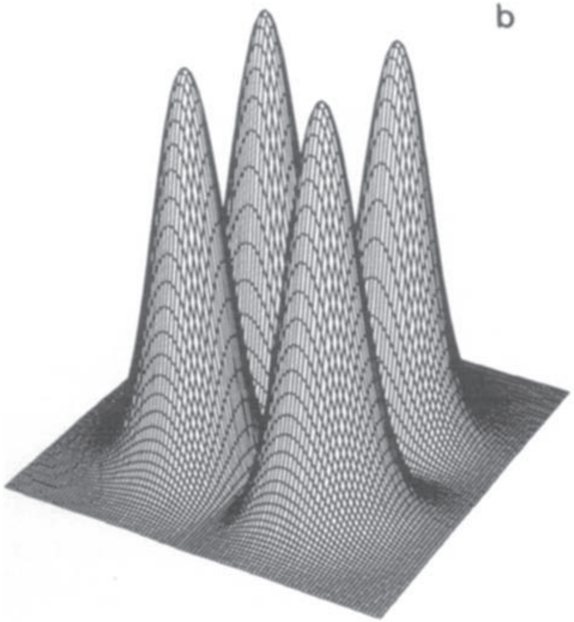
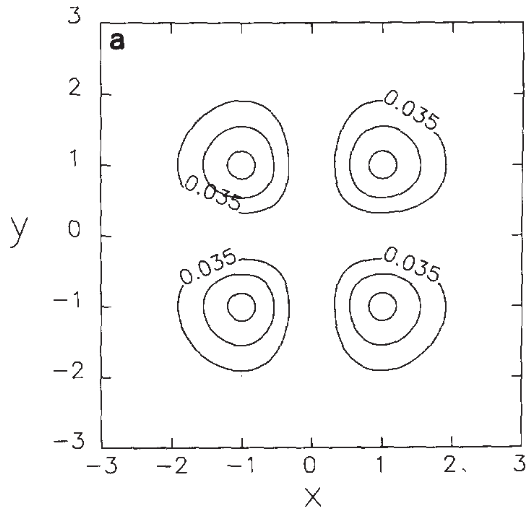
Bleher, Ott and Grebogi (1989) studied scattering of a particle from a 4 hill potential $V(x,y) = x^2 y^2 e^{-(x^2+y^2)}$





and observed numerically an "abrupt bifurcation" from regular scattering for $E > E_c$ to "fully developed" chaotic scattering for $E < E_c$, where $E_c = e^{-2}$ is the energy of the peaks.

S Bleher, E. Ott and C. Grebogi, Route to chaotic scattering, Phys Rev lett **63**, 919–922 (1989)

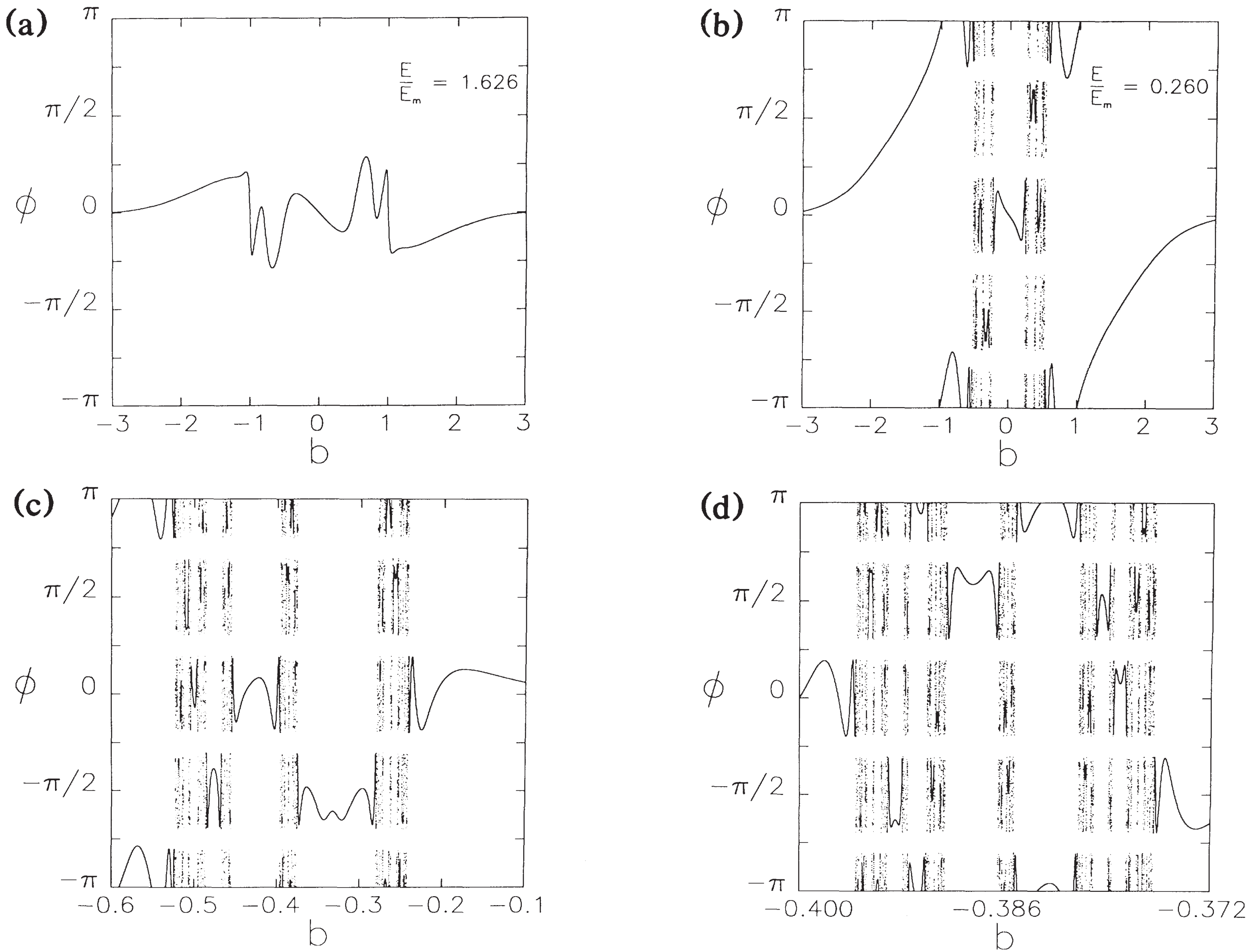


FIG. 1. Plot of deflection angle ϕ vs impact parameter b: (a) $E=1.626E_m$; (b) $E=0.260E_m$; (c) and (d) are blowups of (b).

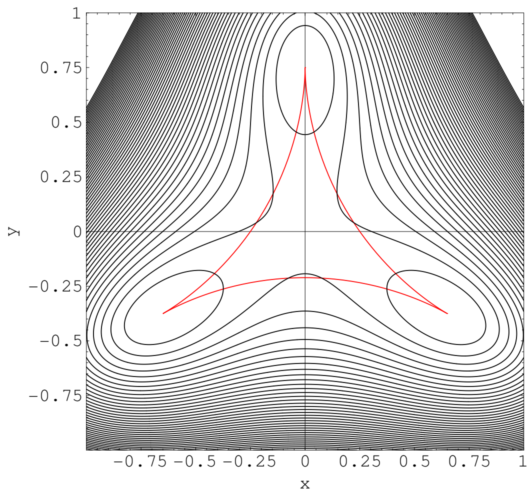
Our goal is to provide a rigorous mathematical explanation of their scenario.

Scattering is *regular* when the outgoing state is a smooth function of the ingoing state. It is (fully developed) *chaotic* when the outgoing state has a Cantor set of singularities as a function on the ingoing state. It is due to a hyperbolic suspension of a topological Markov chain (shift on doubly infinite paths in a graph).

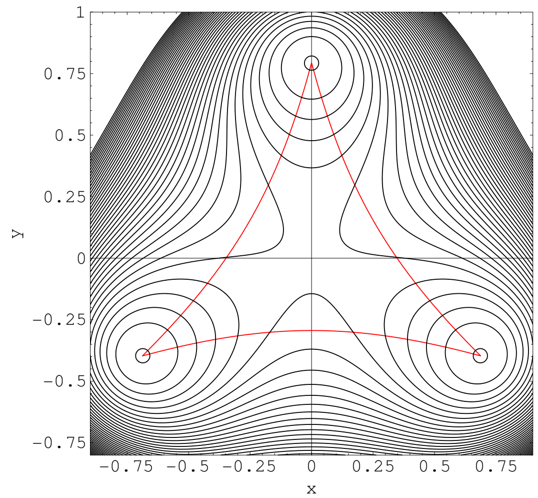
Second motivation: Dynamics of a typical particle in a potential in dimension > 1 is believed to contain chaos.
Most proofs construct a transverse homoclinic orbit to a hyperbolic periodic orbit by perturbation from an integrable limit and then obtain *exponentially weak* chaos.

Here, we'll show how to prove *strong* chaos in suitable circumstances, using Aubry's concept of **anti-integrable limit**.

Examples:



Example 1



Example 2

$$V(x,y) = y^3 - 3x^2y - (x^2+y^2)^2$$

$$E_c = \frac{27}{256}$$

$$V(x,y) = y^3 - 3x^2y - (x^2+y^2)^3$$

$$E_c = \frac{1}{4}$$

For each of them we prove:

- 3 maxima with height E_c
- $\exists$ precisely 6 heteroclinics between them (3×2 directions)

  - For $E > E_c$ there are no bounded orbits.
  - For $E = E_c$ the only bounded orbits are the 3 equilibria and 6 heteroclinics between them.
  - For $E \in (E_c - \varepsilon, E_c)$ (some $\varepsilon > 0$) there is a **hyperbolic set** shadowing all concatenations of the heteroclinics.

Therefore an **abrupt bifurcation**.

General setting:

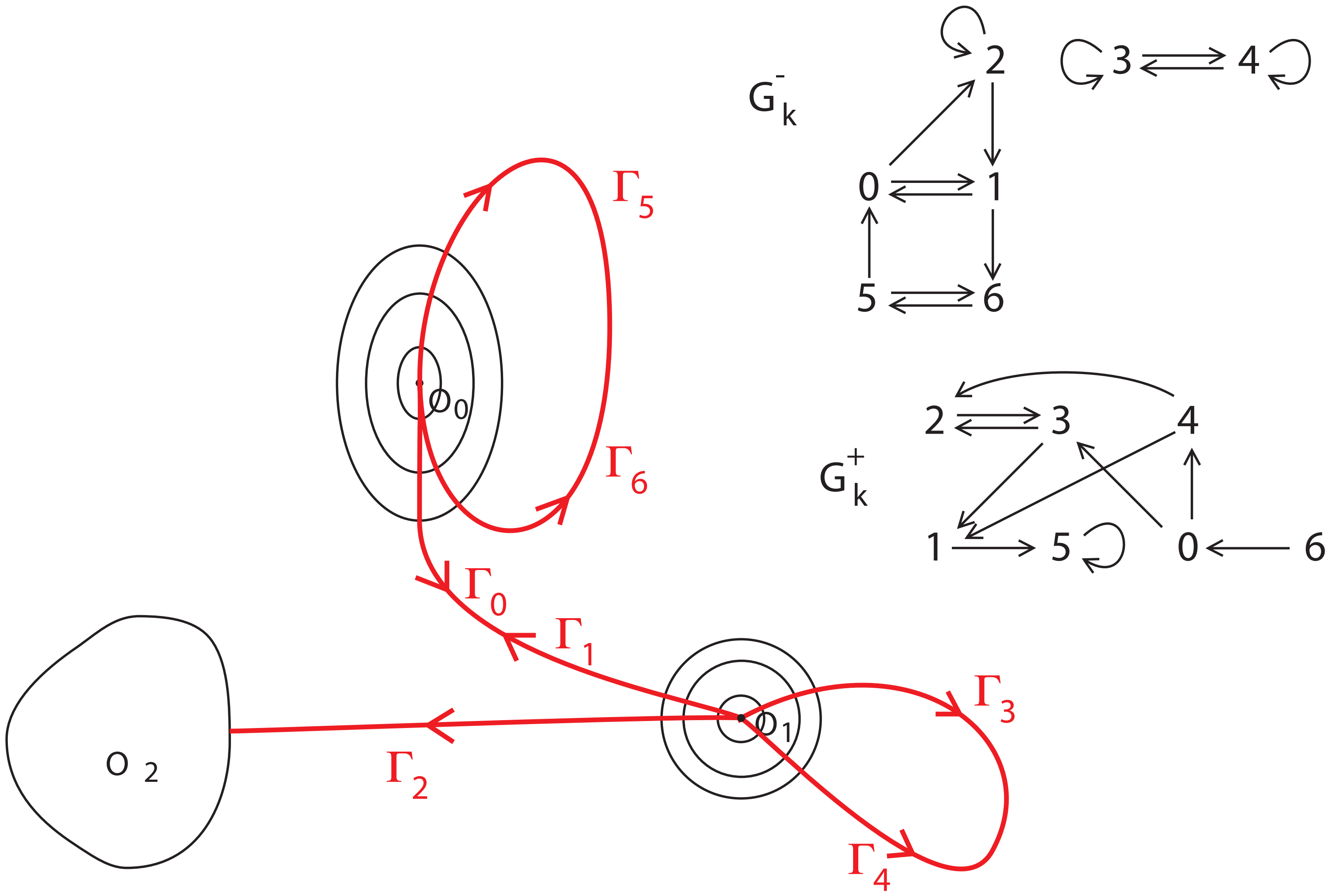
Two degrees of freedom Lagrangian system $L = T - V$, $T = \frac{1}{2}|\dot{q}|^2$, with one or more local maxima of V at same height and some non-degenerate (hyperbolic) connecting orbits between them.

Say a maximum is **circular** if the eigenvalues of the linearised dynamics are $\pm\mu$ twice, **elliptic** if $\pm\mu_1$, $\pm\mu_2$, with $\mu_1 < \mu_2$.

For connecting orbits to or from an elliptic maximum we suppose they come tangent to the long axis (slow direction) (=generic case).

Say a concatenation of two connecting orbits is **admissible** for $E < E_c$ (resp. $E > E_c$) if at a circular maximum it turns by angle $> \frac{\pi}{2}$ ($< \frac{\pi}{2}$) and at an elliptic maximum it turns by π (0).

Let $G^{\pm}$ be the graphs whose vertices are connecting orbits (distinguishing the two directions) and edges are admissible transitions for $E > E_c$ ($E < E_c$).
Let $\sigma^{\pm}$ be the shift on the space of doubly infinite paths in $G^{\pm}$.



G_k^-
G_k^+
O_0
O_1
O_2
Γ_0
Γ_1
Γ_2
Γ_3
Γ_4
Γ_5
Γ_6

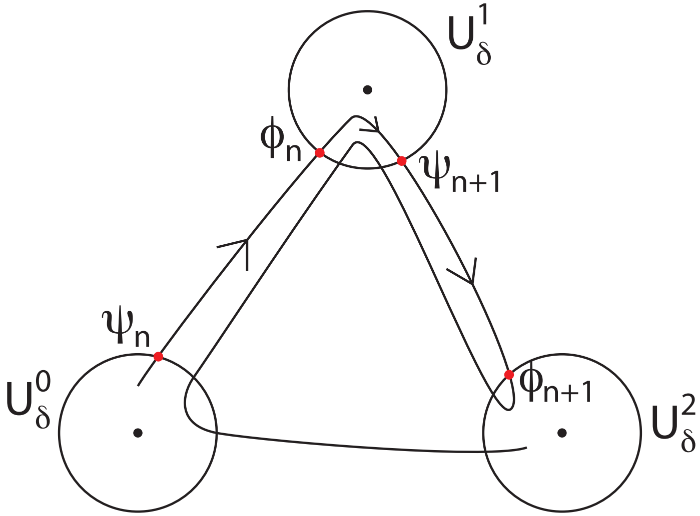
Theorem:

There exist $\epsilon^{\pm} > 0$ *such that for all* $E \in (E_c, E_c + \epsilon^{+})$*, respectively* $E \in (E_c - \epsilon^{-}, E_c)$*, there is a hyperbolic suspension of* $\sigma^{\pm}$ *whose trajectories follow uniformly closely the corresponding concatenation of connecting orbits.*

Idea of proof of existence of the hyperbolic set:

Trajectory of energy E corresponds to path γ of stationary Maupertuis action

$$J(\gamma) := \int_0^1 \sqrt{2(E - V(\gamma(s))} \; |\dot{\gamma}(s)| \; ds \, .$$



- Choose local neighbourhoods U_δ^i of the maxima i, e.g. of the form $E_c - V(x) \leq \frac{1}{2}\delta^2$, δ small.

- Divide paths close to a concatenation of connecting orbits into segments outside the U_δ and inside the U_δ.

- Label them by the points ψ_n of exit and ϕ_n of subsequent entry.

- The Maupertuis principle gives a locally unique solution for each outer segment $[\psi_n, \phi_n]$ if the corresponding connecting orbit is **non-degenerate**.

Denote the resulting action by $h_E(\psi_n, \phi_n)$.

- To analyse the inner segments: (wlog $E_c = 0$)

- for each $\phi_n = a$, $\psi_{n+1} = b$ and τ large enough there exists a unique orbit segment in U from a to b in time τ (by hyperbolicity of equilibrium)

- its energy $E(a, b, \tau) \sim -2\mu^2 \langle a, b\rangle e^{-\mu\tau}$ in the circular case

  $\sim -2\mu_1^2 a_1 b_1 e^{-\mu_1 \tau}$ in the elliptic case,

- $\frac{\partial E}{\partial \tau} \sim -\mu E$

So for $E > 0$ $\exists$ a connection if $\langle a, b\rangle \leq -\eta$ in the circular case or $a_1 b_1 \leq -\eta$ in the elliptic case, for some small η,
and for $E < 0$ . . . . $\geq \eta$,
and it takes time $\tau \sim \frac{1}{\mu} \log \left\{ \frac{-2\mu^2 \langle a,b\rangle}{E} \right\}$ for the circular case or similar for the elliptic case.
Denote the resulting Maupertuis action by $k_E(\phi_n, \psi_{n+1})$.

Then orbits near concatenation correspond to critical points of formal sum

$$\cdots + h_E(\psi_n, \phi_n) + k_E(\phi_n, \psi_{n+1}) + \ldots$$

k_E can be written $k_E = E\tau + \mathcal{L}$, with $\mathcal{L} = \int_0^\tau L dt$.
The dominant part of $\mathcal{L}$ is $S^+(\phi_n) + S^-(\psi_{n+1})$,

where $S^{\pm}$ are the generating functions for the stable / unstable manifolds $W^{\pm}$ of the equilibrium, i.e. $p = \mp\frac{\partial S^{\pm}}{\partial q}$, $S^{\pm}(\text{equil.}) = 0$. So let

$$\tilde{h}_E(\psi_n, \phi_n) = h_E(\psi_n, \phi_n) + S^-(\psi_n) + S^+(\phi_n)$$
$$\tilde{k}_E(\phi_n, \psi_{n+1}) = k_E(\phi_n, \psi_{n+1}) - S^-(\psi_{n+1}) - S^+(\phi_n) \; ,$$

Then $\tilde{k}_{E_c} = 0$ and $\tilde{h}_{E_c}$ has a non-degenerate critical point for ψ_n, ϕ_n corresponding to the connecting orbit.

Can rewrite the variational problem as stationary points of

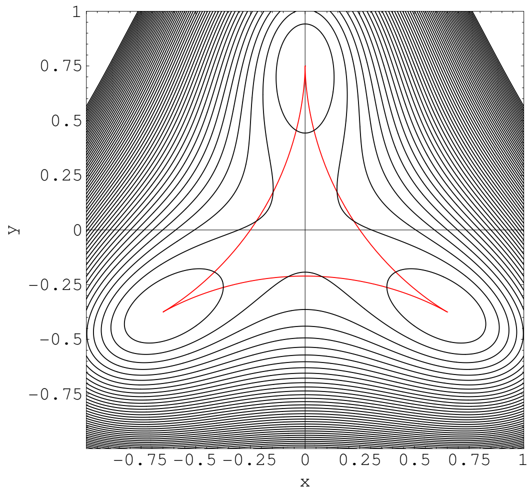
$$\cdots + \tilde{h}_E(\psi_n, \phi_n) + \tilde{k}_E(\phi_n, \psi_{n+1}) + \ldots .$$

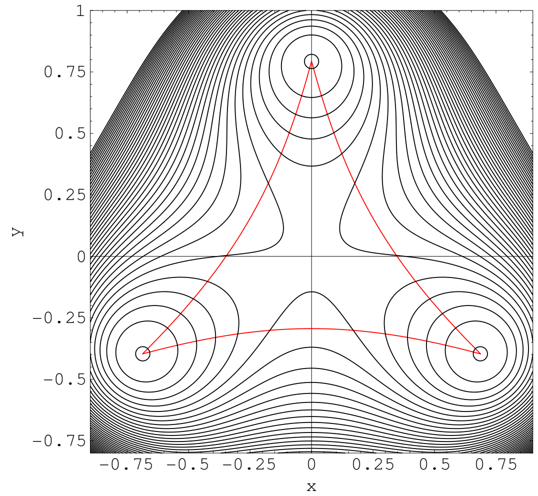
So the variational problem decouples at E_c into independent problems for each n (an **anti-integrable limit**) .

Making $E \neq E_c$ couples weakly as long as the angle condition holds, e.g. in the circular case $\tilde{k}_E(\phi, \psi) \sim \frac{E}{\mu}\left(\log(2\delta^2 \frac{\cos\theta}{E}) + 1\right)$ for $\frac{\cos\theta}{E} \geq \eta$, where $\theta = \pi - \psi + \phi$.

So there exists a locally unique critical point $((\psi_n, \phi_n))_{n\in\mathbb{Z}}$. $\square$

Application to Examples 1 and 2:





Example 1

$$V(x,y) = y^3 - 3x^2y - (x^2+y^2)^2$$

Example 2

$$V(x,y) = y^3 - 3x^2y - (x^2+y^2)^3$$

We proved existence of non-degenerate heteroclinic orbits as shown. Example 1 has elliptic hills and the heteroclinics come in same side of short axis so deduce existence of the chaotic set for $E \in (E_c - \epsilon, E_c)$.

Example 2 has circular hills and the heteroclinics meet at angle $\frac{\pi}{6}$ so deduce the same.

We also prove no bounded orbits for $E > E_c$ for both cases.

- What about Bleher, Ott, Grebogi's examples ?

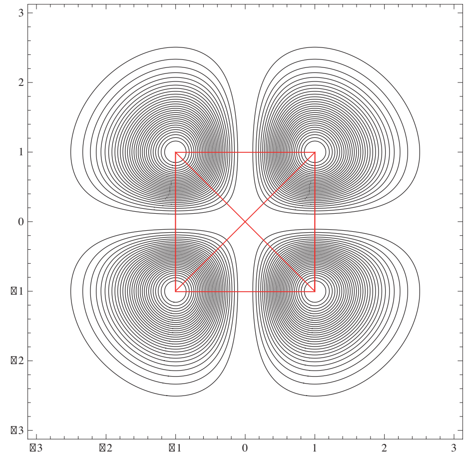
Example 3: (BOG 1989)

$V(x, y) = x^2 y^2 e^{-(x^2+y^2)}$,

- 4 *circular* maxima , $E_c = e^{-2}$

- $\geq$ 12 heteroclinics between them.

(easy: straight lines)

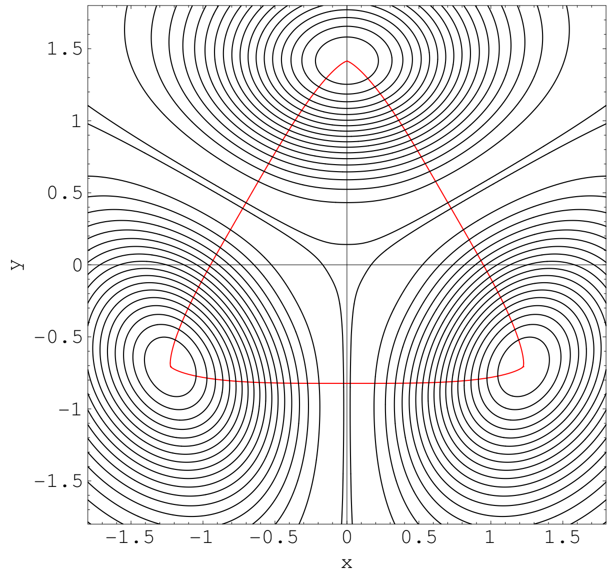


- For $E \in (E_c - \varepsilon, E_c)$ there is a **hyperbolic set** shadowing the subset of concatenations of the heteroclinics for which the angle turned at each hill top is $\frac{3}{4}\pi$ or π (i.e. not $\frac{\pi}{2}$). So our proof constructs only a subset of what BOG claimed (we are missing the transitions where angle change is $\frac{\pi}{2}$) but it is much better than what they actually proved.

Example 4: (BOG, Bifurcation to chaotic scattering, Physica D **46**, 87–121 (1990)

$V(x,y) = \frac{r}{2}(r^3 + y^3 - 3x^2y)e^{-r^2}$,

with $r = x^2 + y^2$



- 3 *elliptic* maxima

- $\geq$ 6 heteroclinics between them

but the orientation is such that the heteros come in on opposite sides of short axis.

So all our theorem gives is:

For $E \in (E_c, E_c + \varepsilon)$ there are 2 periodic orbits shadowing the concatenations of the heteroclinics with angle change 0 at each hill top and we don't believe their claim of abrupt bifurcation to chaos.

Conclusion:

We have made mathematical sense of BOG's results on the abrupt bifurcation to chaotic scattering, using an "anti-integrable" approach. We have constructed the hyperbolic chaotic set in some explicit examples.

These results are inspired by two papers of Bolotin and MacKay, proving existence of Poincaré second species orbits in the restricted circular three body problem and **could have other gravitational applications.**

S. Bolotin and R.S. MacKay, Periodic and chaotic trajectories of the second species for the n-centre problem. *Celestial Mechanics and Dynamical Astronomy* **77**, 49–75 (2000); Nonplanar second species periodic and chaotic trajectories for the circular restricted three-body problem, *CMDA* **94**, 433-449 (2006)