Diffusion of particle velocity in the dense wave spectrum limit

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Ann. Appl. Prob. 20 (2010) 2022-2039
J. Stat. Phys. 148 (2012) 591-605
Plasma Phys. Control. Fusion 53 (2011) 025012 (36 pp.)
& to appear

with Nicolas Besse, Pierre Bertrand (Nancy)
Dominique Escande, Etienne Pardoux (Marseilles)

Statistical mechanics of self-gravitating particles

Two problems

- 1.5 deg. freedom, « controlled » dynamics :

  stochastic processes
  convergence of N particles to ensemble

- Self-consistent many-body dynamics :

  history, physical motivation, heuristics
  numerical simulations
  analytical estimates in some regimes

  space-time phase ?

Test particles in many waves

Hamiltonian dynamics (toy model, paradigm)

$$\ddot{q} = \mathfrak{m}^{-1} \sum_{m=-\mu}^{\mu'-1} A_m \sin(q - mt + \varphi_m)$$
$$= \mathfrak{m}^{-1} \sum_m A_m \cos(mt - \varphi_m) \sin q - \mathfrak{m}^{-1} \sum_m A_m \sin(mt - \varphi_m) \cos q$$

with $A_m \cos \varphi_m, A_m \sin \varphi_m$ random i.i.d., std normal

$$\mathbb{E}(A_m \cos \varphi_m)^2 = \mathbb{E}(A_m \sin \varphi_m)^2 = 1/2 \text{ for all } m \in \mathbb{Z}$$

(Y.E., E. Pardoux, *Ann. Appl. Prob.* ***20*** (2010) 2022-2039)

Test particles in many waves

Scaling : $P = mv/A$, $Q = q$

$M \to \infty$: Stratonovich integration (Doss, Sussman, 1977-78) :

$$\begin{aligned} \mathrm{d}Q_t &= AP_t \,\mathrm{d}t \,, \\ \mathrm{d}P_t &= \sin(Q_t) \circ \mathrm{d}C_t + \cos(Q_t) \circ \mathrm{d}S_t \end{aligned}$$

C and S are martingales, with quadratic variation

$$\langle C \rangle_t = \langle S \rangle_t = \pi t \,,\ \langle C, S \rangle_t = 0 \,.$$

i.e. $C^2 - \langle C \rangle$ is a martingale, etc... but periodized :

$$\mathrm{d}C_{t+2k\pi} = \mathrm{d}C_t \quad , \quad \mathrm{d}S_{t+2k\pi} = \mathrm{d}S_t$$

Test particles in many waves

THEOREM 3.1. *For any $\mathcal{N} > 0$, the momentum processes $P^{(\nu)}$ defined by*

$$\mathrm{d}Q_t^{(\nu)} = AP_t^{(\nu)}\,\mathrm{d}t\,, \quad Q_0^{(\nu)} = q_0^{(\nu)}\,, \tag{3.5}$$

$$\mathrm{d}P_t^{(\nu)} = (\sin Q_t^{(\nu)})\mathrm{d}C_t + (\cos Q_t^{(\nu)})\mathrm{d}S_t\,, \quad P_0^{(\nu)} = p_0^{(\nu)}\,, \tag{3.6}$$

with $\mathcal{N}$ different initial data $(q_0^{(\nu)}, p_0^{(\nu)}) \in \mathbb{T}\times\mathbb{R}$, $1 \leq \nu \leq \mathcal{N}$, converge as $A \to \infty$ to $\mathcal{N}$ independent Wiener processes with variance πt, and convergence is in law in $C(\mathbb{R}_+, \mathbb{R}^{\mathcal{N}})$.

… supports using ensemble picture for large *N*

Test particles in many waves

$N = 1,\ 0 < t < 2\pi$: straightforward... for experts;-))

P is a martingale (w.r.t. filtration ass. C,S), and

$$dP^2 = 2\,P\,dP + \sin^2(Q)\,d\langle C\rangle + \cos^2(Q)\,d\langle S\rangle$$
$$= (\text{martingale}) + (\sin^2(Q) + \cos^2(Q))\,\pi\,dt$$

i.e. $d\langle P\rangle = \pi\,dt$

$\Rightarrow$ P is brownian for any $A>0$

$N = 2,\ 0 < t < 2\pi$: $P^{(1)}, P^{(2)}$ martingale in $\mathbb{R}^2$, so prove that

$$\langle P^{(1)}, P^{(2)}\rangle \to 0 \text{ as } A \to \infty$$

Test particles in many waves

$N = 2$, $0 < t < 2\pi$: to prove that $\langle P^{(1)}, P^{(2)} \rangle \to 0$ as $A \to \infty$

Let $V_t = (P_t^{(1)} - P_t^{(2)}) / (2\,\pi^{1/2})$, $U_t = (Q_t^{(1)} - Q_t^{(2)}) / 2$:

$$\langle P^{(\nu)}, P^{(\nu')} \rangle_t \;=\; \int_0^t \cos(2U_s^n)\pi \mathrm{d}s = \int_0^t (1 - 2\sin^2 U_s^n)\pi \mathrm{d}s$$

$$\frac{\mathrm{d}U_t^n}{\mathrm{d}t} = nV_t^n\,,\; U_0 = u\,, \qquad n \;=\; \pi^{1/2} A$$

$$\mathrm{d}V_t^n = \sin(U_t^n)\mathrm{d}W_t\,,\; V_0 = v$$

Test particles in many waves

THEOREM 3.2. *As* $n \to \infty$,

$$V^n \Rightarrow v + \frac{1}{\sqrt{2}} B \,,$$

where $\{B_t,\ t \geq 0\}$ *is a standard one–dimensional Brownian motion, and the convergence is in law in* $C(\mathbb{R}_+, \mathbb{R})$.

Test particles in many waves

$N > 0$, $K > 0$, $0 < t < 2K\pi$: let

$$P_t^{(n+(k-1)N)} = P_{t-2(k-1)\pi}^{(n)}$$

$$Q_t^{(n+(k-1)N)} = Q_{t-2(k-1)\pi}^{(n)}$$

reduces to previous problem.

Test particles in many waves

Non-«Wiener» wave fields, e.g. finite bandwidth :

work in progress...

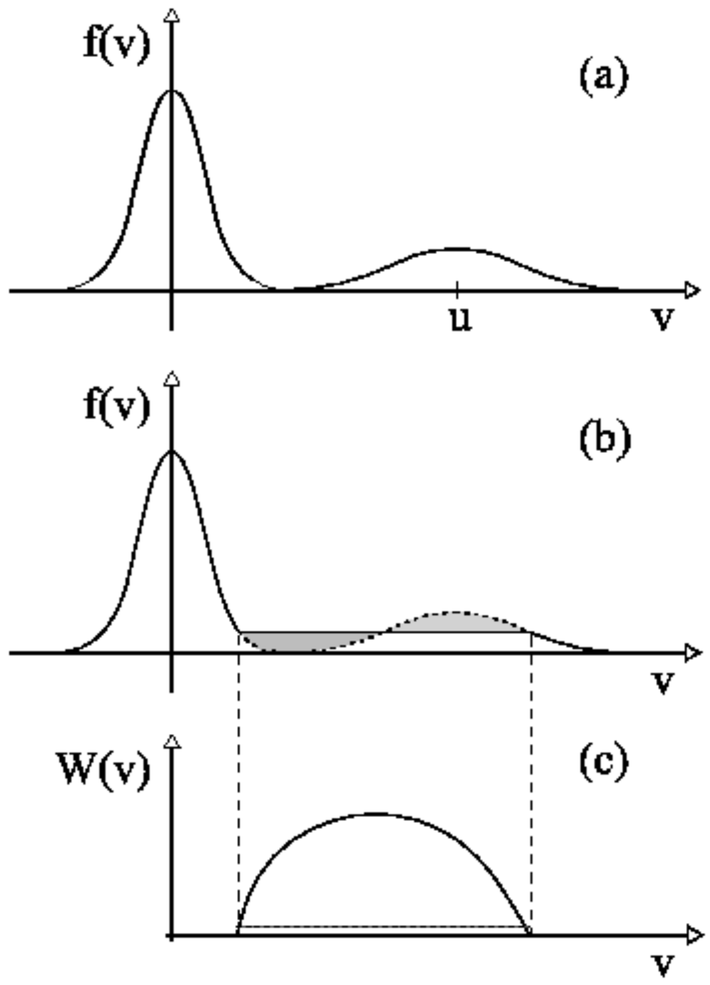
Two problems

- 1.5 deg. freedom, « controlled » dynamics :
  - stochastic processes
  - convergence of N particles to ensemble
- Self-consistent many-body dynamics :
  - history, physical motivation, heuristics
  - numerical simulations
  - analytical estimates in some regimes

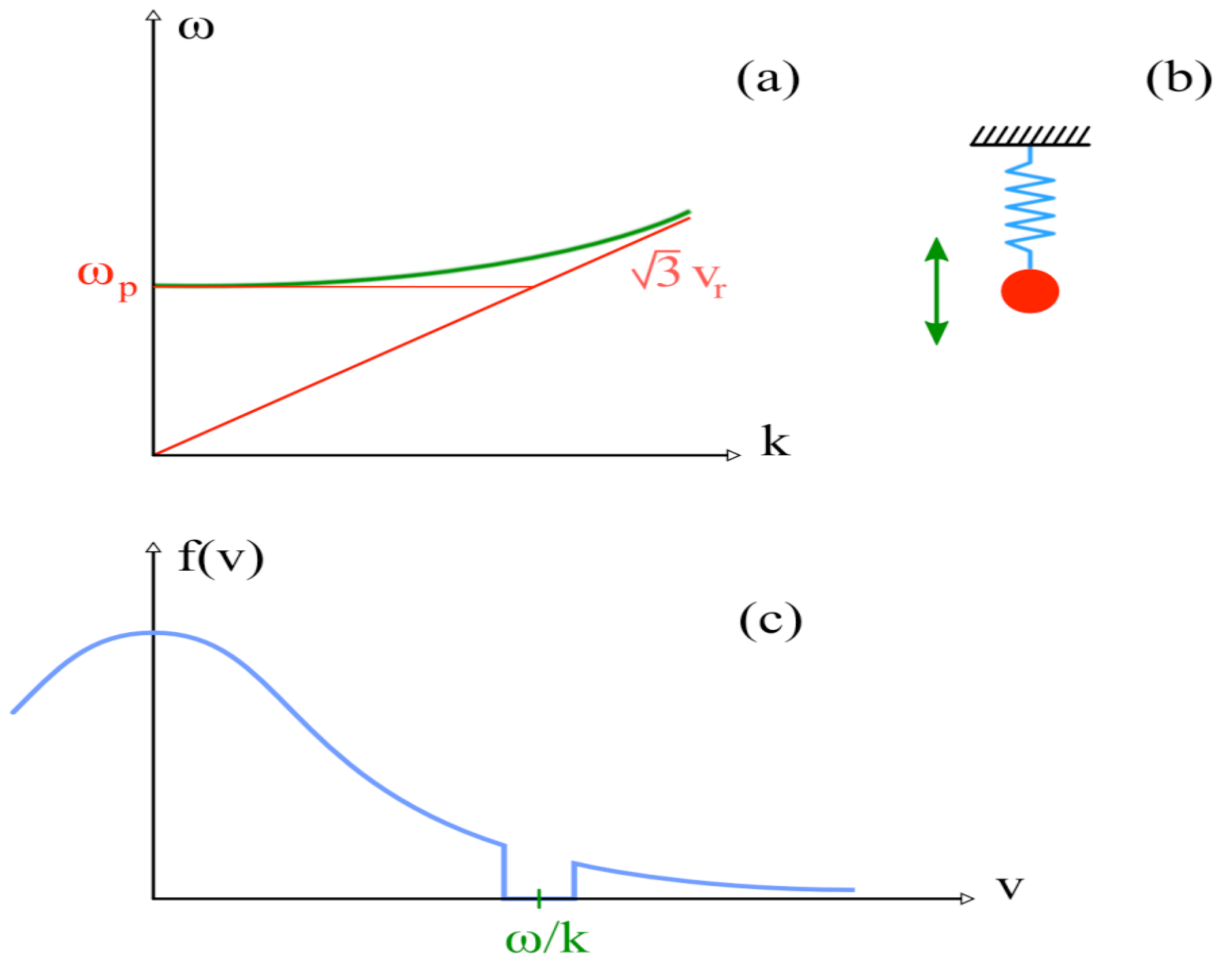
- Self-consistent wave-particle dynamics :
  Vlasov-wave and quasilinear approximation
- Bump-on-tail (weak warm beam) instability
- Conservation laws (global, local)
- Final wave & particle distribution prediction
- Statistical variability
- Microscopic dynamics in plateau regime
- Full time evolution : always quasilinear ?

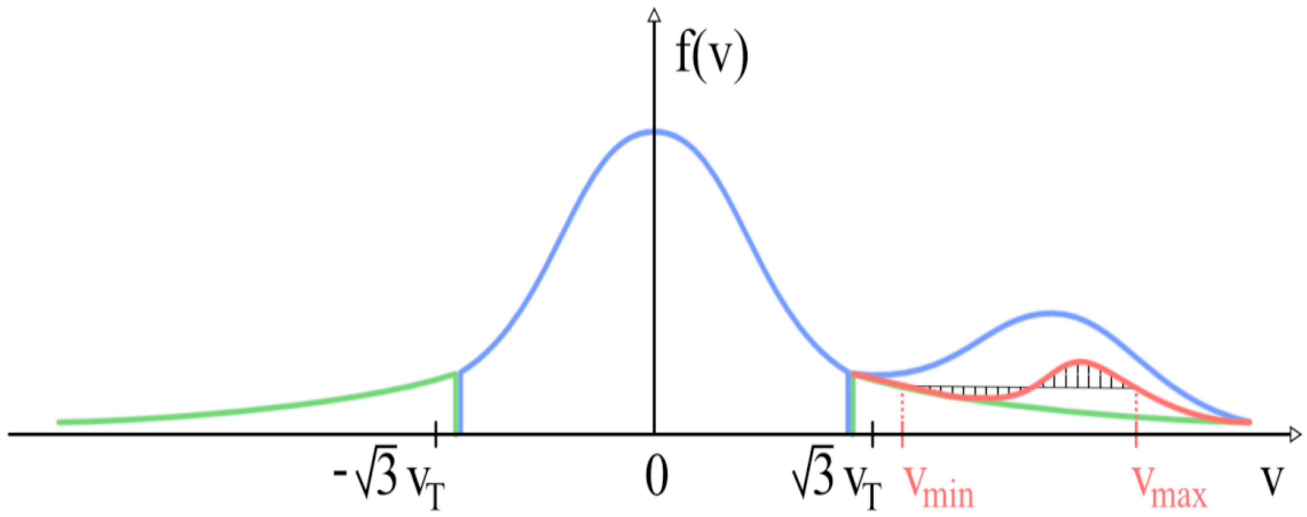
(BEEB, PPCF 53 (2011) 025012)

Weak warm beam instability

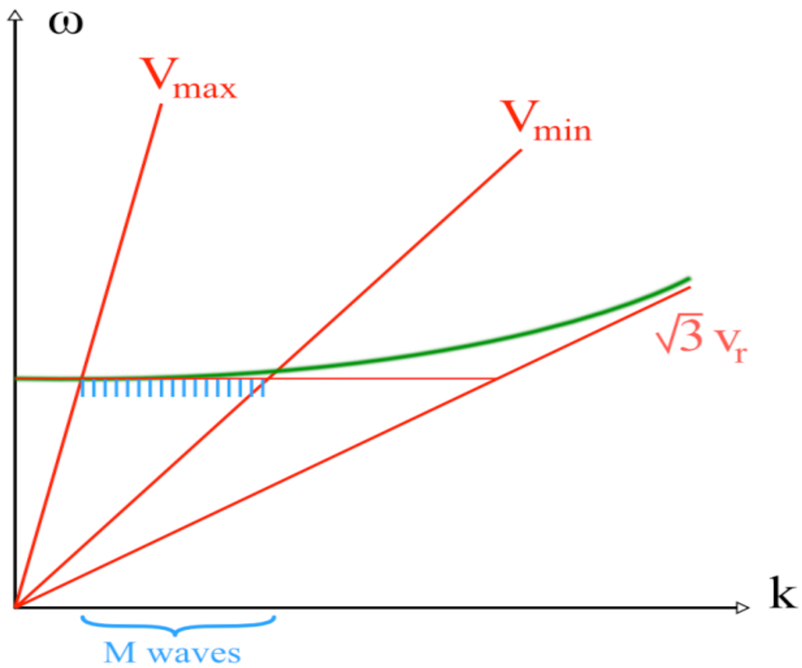


Description with a finite dimensional Hamiltonian system





Red bump OK
Green tail OK
Blue bump too big



Hamiltonian for
N particles + *M* harmonic oscillators + coupling

Keeps the genuine granular character of plasmas

Also describes
- traveling wave tube
- free-electron laser: *M*=1 Colson-Bonifacio

...

$$H_{sc} = \sum_{l=1}^{N} \frac{p_l^2}{2} + \sum_{j=1}^{M} \omega_{j0} \frac{X_j^2 + Y_j^2}{2} + \sum_{l=1}^{N} \sum_{j=1}^{M} k_j^{-1} \varepsilon_j (Y_j \sin k_j x_l - X_j \cos k_j x_l)$$

Conservation laws (global)

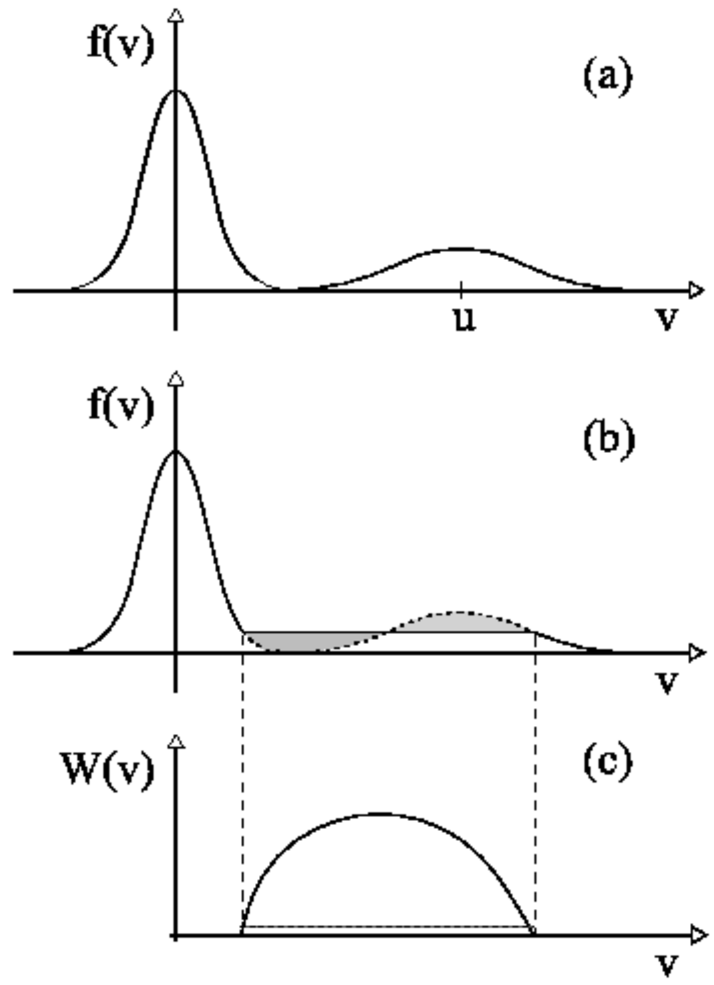
- Momentum $P_{\rm sc} = \sum_{r=1}^{N} p_r + \sum_{j=1}^{M} k_j I_j$

with $I_j = \dfrac{X_j^2 + Y_j^2}{2}$, $\varepsilon_j = \varepsilon \, \beta_j$

- Energy $$H_{\rm sc}^{N,M} = \sum_{r=1}^{N} \frac{p_r^2}{2} + \sum_{j=1}^{M} \omega_{j0} I_j - \varepsilon \sum_{r=1}^{N} \sum_{j=1}^{M} k_j^{-1} \beta_j \sqrt{2 I_j} \cos(k_j x_r - \theta_j)$$

Weak warm beam instability

- 1961-62 QL theory
- Perturbative theory + random phase approximation



Particle dynamics

$$\dot{x}_r = p_r$$

$$\dot{p}_r = -\varepsilon \sum_{j=1}^{M} \beta_j \sqrt{2I_j} \sin(k_j x_r - \theta_j)$$

with M>>1, incoherent phases θ_j

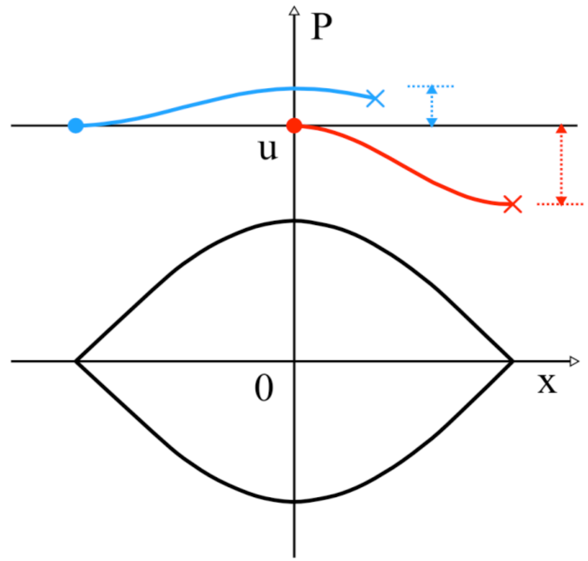
Approx. white noise, $D_{\mathrm{QL}}(p_m) = \dfrac{\pi \varepsilon^2 \beta_m^2 \langle I_m(t') \rangle}{k_m \Delta v_m}$

(1.5 deg. freedom)

Wave dynamics

Interaction local in *v* : for $Z_j = X_j + \mathrm{i}Y_j$

$$\dot{Z}_j = -\mathrm{i}\omega_{j0}Z_j + \mathrm{i}\varepsilon\beta_j k_j^{-1}\sum_{r=1}^{N} \mathrm{e}^{-\mathrm{i}k_j x_r}$$



Conservation law

$$P_{\mathrm{sc}} = \sum_{r=1}^{N} p_r + \sum_{j=1}^{M} k_j I_j$$

Local balance... $\gamma_{j\mathrm{L}} \equiv \frac{\pi}{2}\frac{\varepsilon^2\beta_j^2}{k_j^2} N f_{\mathrm{t}}'(\omega_j/k_j)$

Quasilinear approximation

M>>1 : force ~ white noise,
wave power spectrum

$$\psi(t, v_m) = k_m |\zeta_m|^2 / (2\Delta v_m)$$

coupled to only *f*(*v*)

$$\partial_t \bar{f} = \partial_v (D_{\mathrm{QL}}(t,v) \partial_v \bar{f}) \quad \text{and} \quad \partial_t \psi = 2\gamma_{\mathrm{L}}(t,v)\psi$$

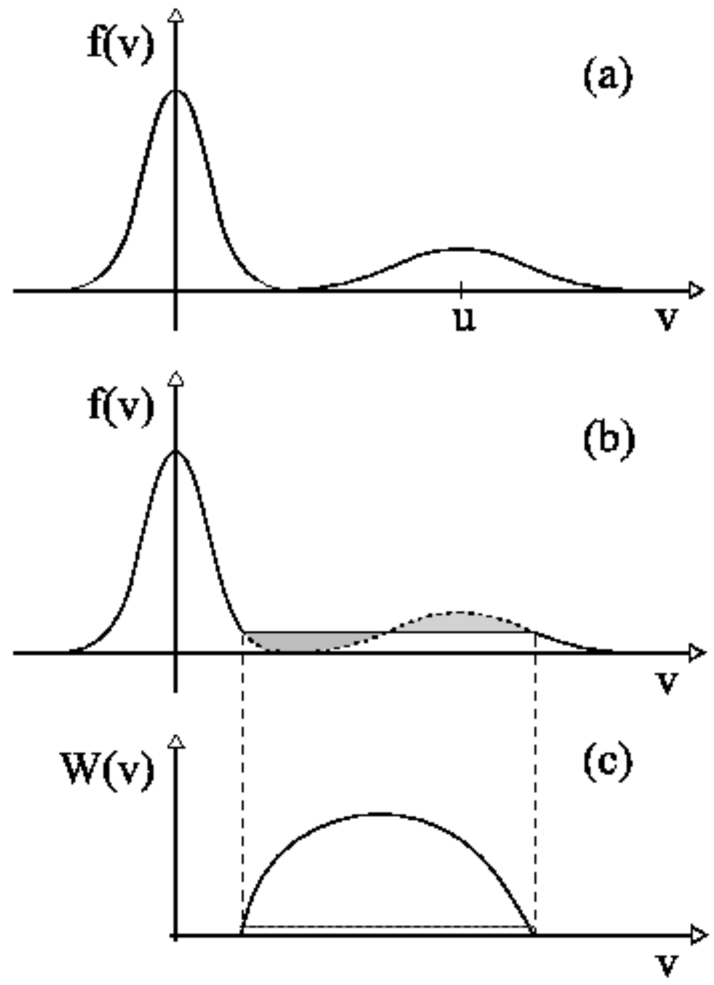
$$D_{\mathrm{QL}}(t,v) = \frac{\pi\varepsilon^2\beta^2}{k^2}\psi(t,v) \text{ and } \gamma_{\mathrm{L}}(t,v) = \frac{\pi\varepsilon^2\beta^2}{2k^2}\,\partial_v \bar{f}(t,v)$$

Weak warm beam instability

- 1961-62 QL theory
- Perturbative theory + random phase approximation

- Diffusion of particles
  Landau growth of waves

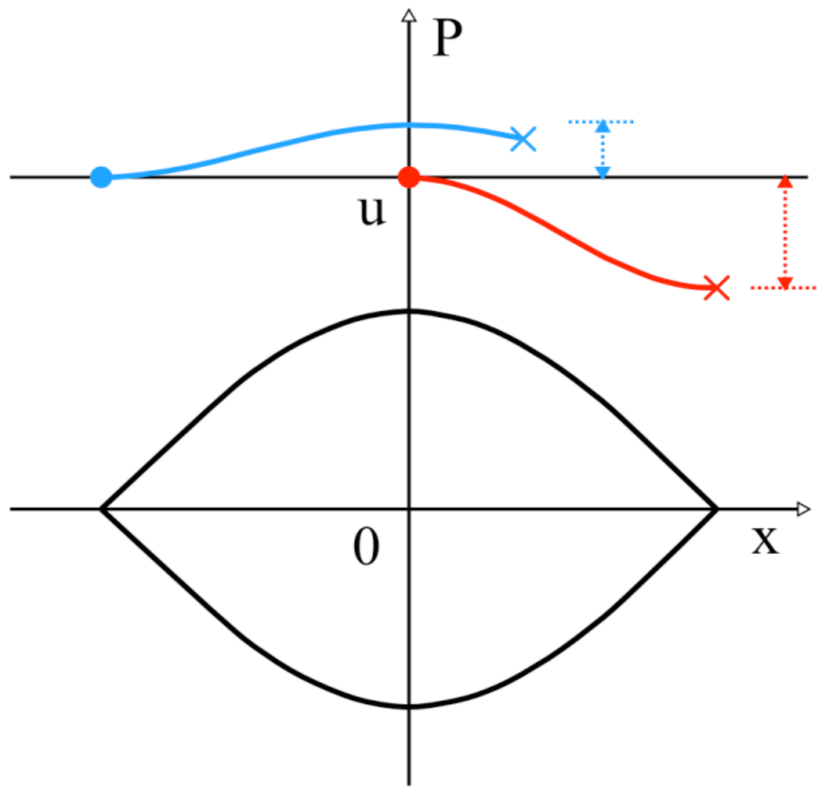
  …for $f(v,t)$ and $\psi(v,\mathrm{t})$ only, eliminating x and θ



Correct for the actual chaotic relaxation?

Current understanding (reformulation)
Resonant coupling

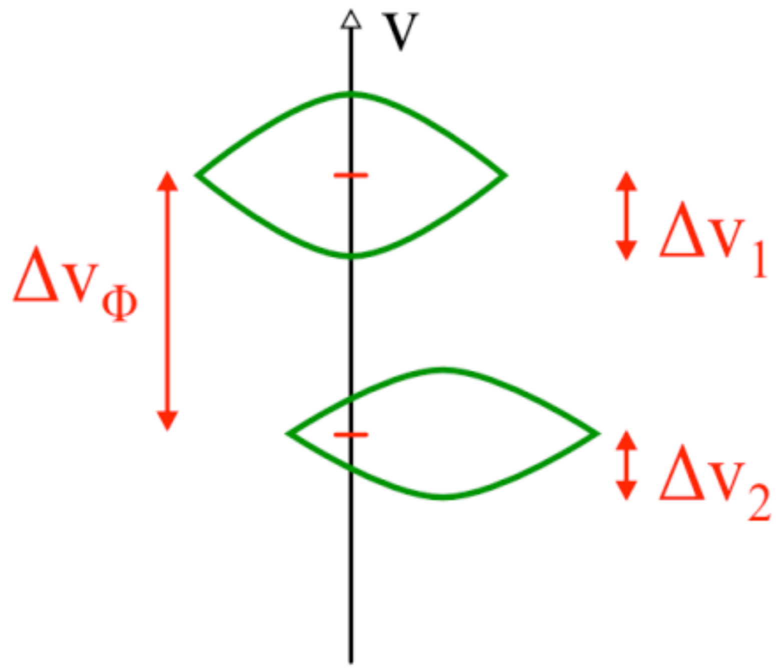
- synchronization



Resonant coupling :

- synchronization

- few waves :
  overlap, chaotic transport

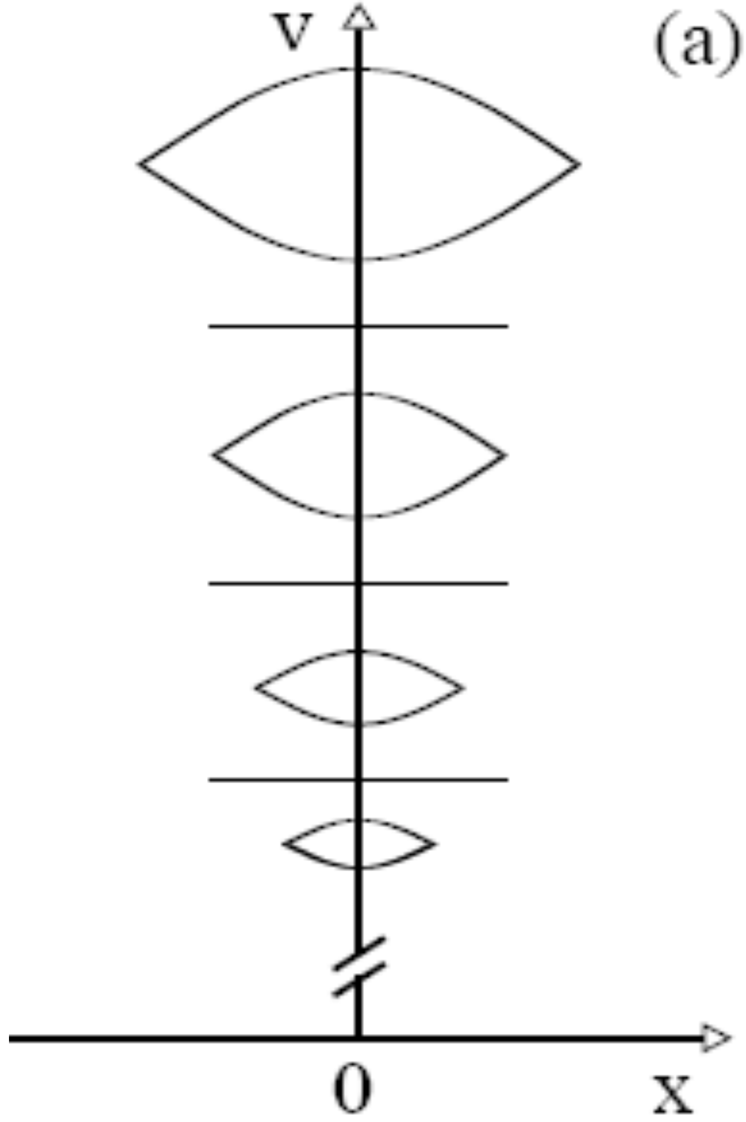


Resonant coupling :

- synchronization

- few waves :
  overlap, large-scale chaos

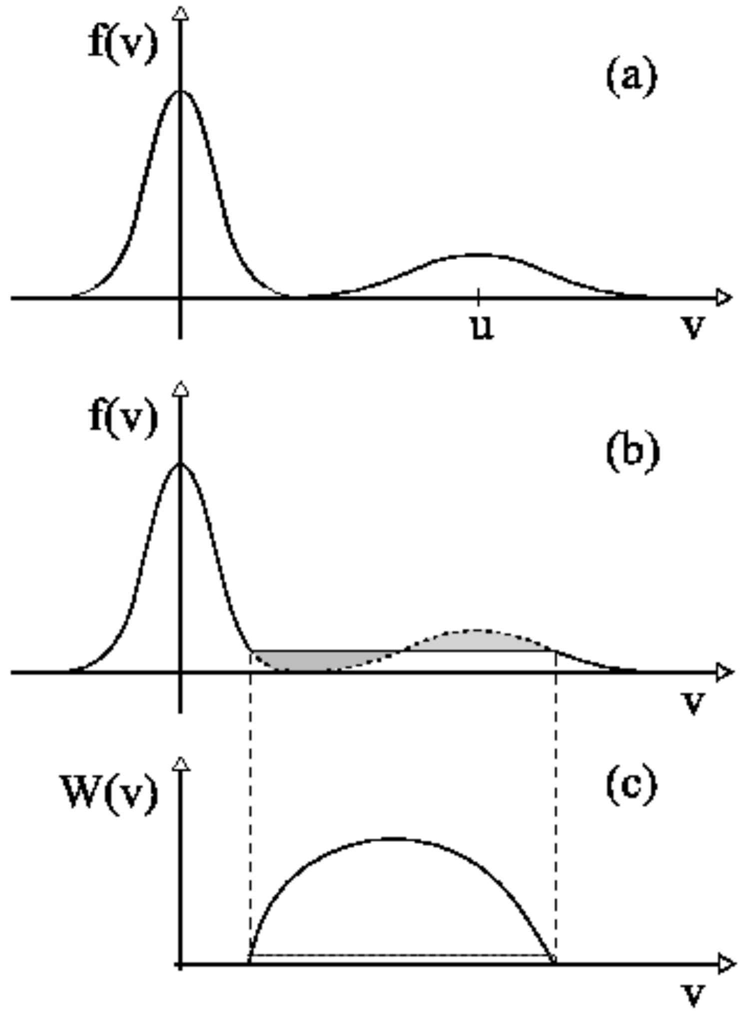
- dense spectrum :
  box $|v - \omega/k| \leq \Delta v_D \sim (D_{QL}/k)^{1/3} \sim |\zeta|^{4/3}$



Resonant coupling :

plateau formation

Landau growth



QL : physical assumptions

No gaps in power spectrum

Independent phases (repeatedly « random phase approx. »)

Vedenov, Velikhov, Sagdeev ; Drummond, Pines (1962)

Dupree, Adam, Laval, Pesme, Liang, Diamond, Galeev, Shapiro, Shevchenko, Doxas, Cary, ...

Experiment by Tsunoda, Doveil & Malmberg 1991:
QL predictions look right...
but QL assumptions are completely wrong

Vlasov-wave dynamics :

$$\partial_t f + v\partial_x f + \varepsilon \mathrm{Re}\Big(\mathrm{i}\sum_{m=1}^{M} \beta_m \zeta_m \mathrm{e}^{\mathrm{i}(k_m x - \omega_m t)}\Big)\partial_v f = 0,$$

$$\dot{\zeta}_m = \mathrm{i}\varepsilon \frac{\beta_m}{k_m}\frac{1}{L}\int_0^L \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}(k_m x - \omega_m t)} f(t,x,v)\, \mathrm{d}v \mathrm{d}x,$$

Particles : *f(x,v)*
Waves ($1 \le m \le M$) : $\zeta_m = (X_m + \mathrm{i}Y_m)\, \mathrm{e}^{\mathrm{i}\omega_m t}$

Hamiltonian dynamics, mean-field
Interaction is local in velocity

Semi-lagrangian simulation (GENCI)

$0.14 < \omega/k < 1$ $0 < v < 1.14$ $0 \leq x \leq 152\,\pi$

$M = 450$ $N_v = 768$ $N_x = 2112$

coupling $\eta = 2.55\ 10^{-3} = n_b/n_p$

$$\varepsilon = \sqrt{2\eta/(1+\eta)}$$

initial $f(x,v) = f(v)$ s.t. $\gamma = 10^{-3}$

initial ζ : random phases, small amplitude

$$|\zeta_m| = \epsilon_{\zeta^0}\sqrt{2\psi_\infty(v_m)\frac{\Delta v_m}{k_m}}$$

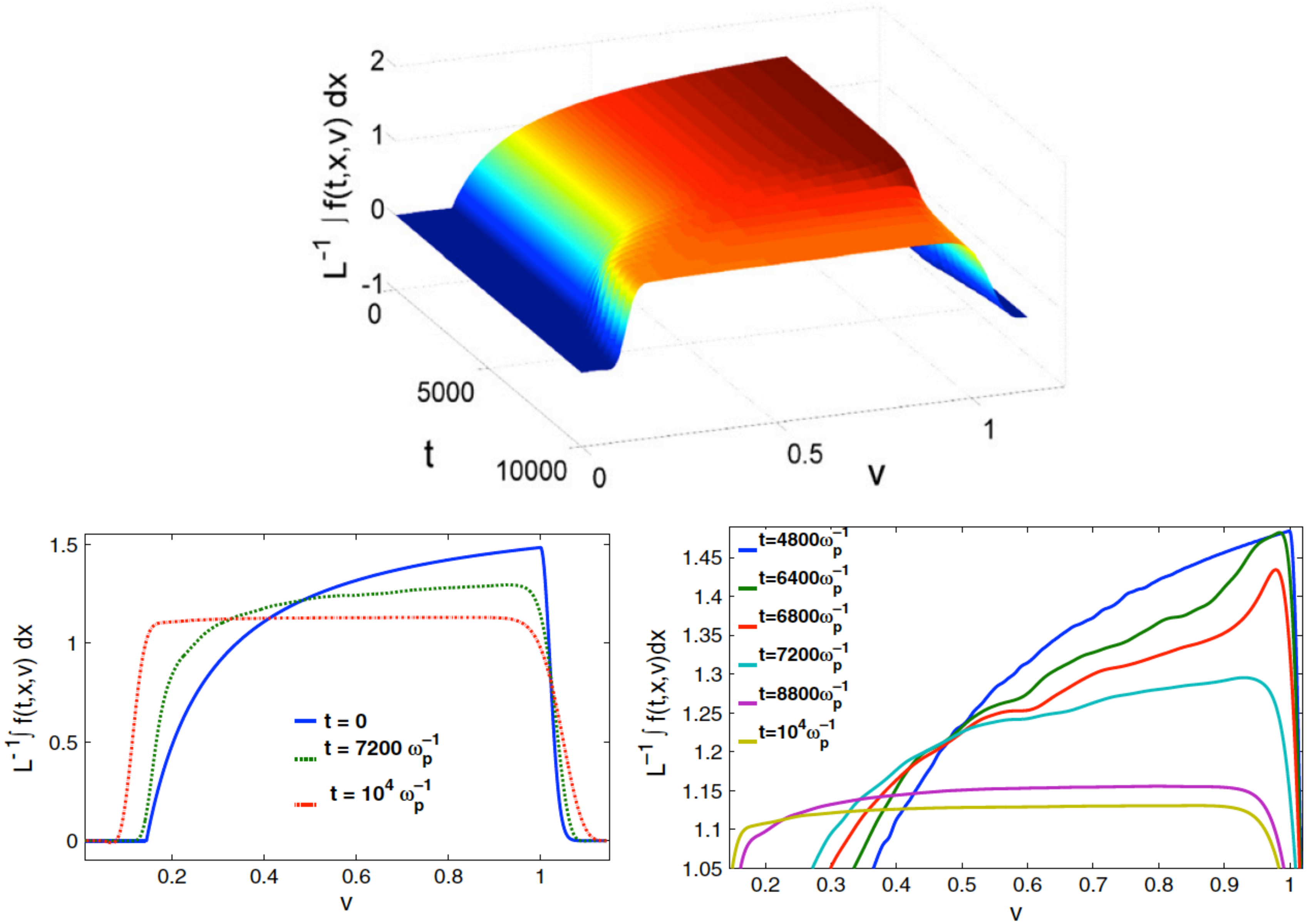


Figure 5. Plateau formation in the x-averaged velocity distribution function. (Colour online.)

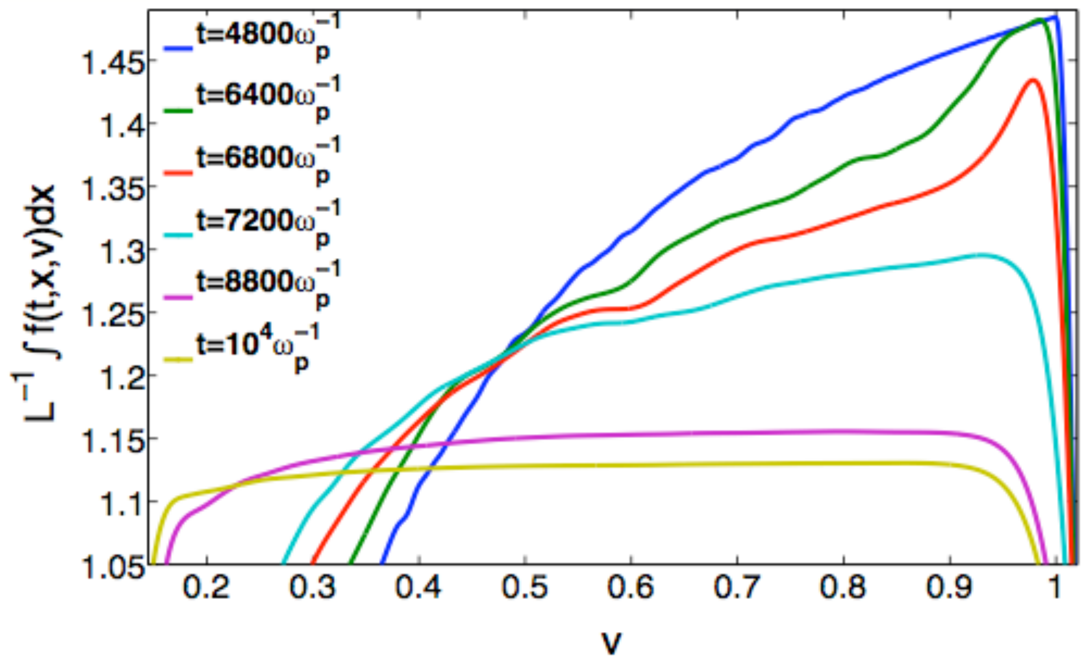
Conservation laws (local)

Momentum exchange : local in velocity

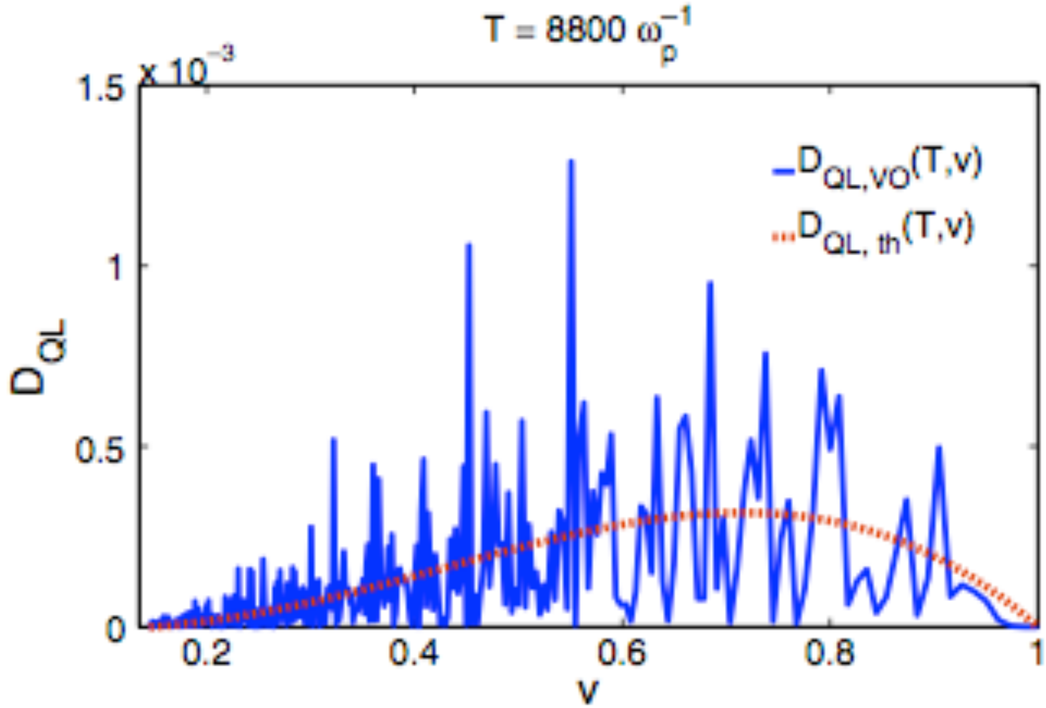
$$\partial_t \nu(p,t) = 0$$

$$\nu(p) \equiv f(p) - \partial_p \psi(p)$$

This predicts final power spectrum...
but agreement on final (ψ,f) does not warrant QL equations to hold.



t=4800ω_p^−1
t=6400ω_p^−1
t=6800ω_p^−1
t=7200ω_p^−1
t=8800ω_p^−1
t=10^4ω_p^−1
L^−1 ∫ f(t,x,v)dx
v



T = 8800 ω_p^−1
x 10^−3
D_QL
D_QL,VO(T,v)
D_QL, th(T,v)
v

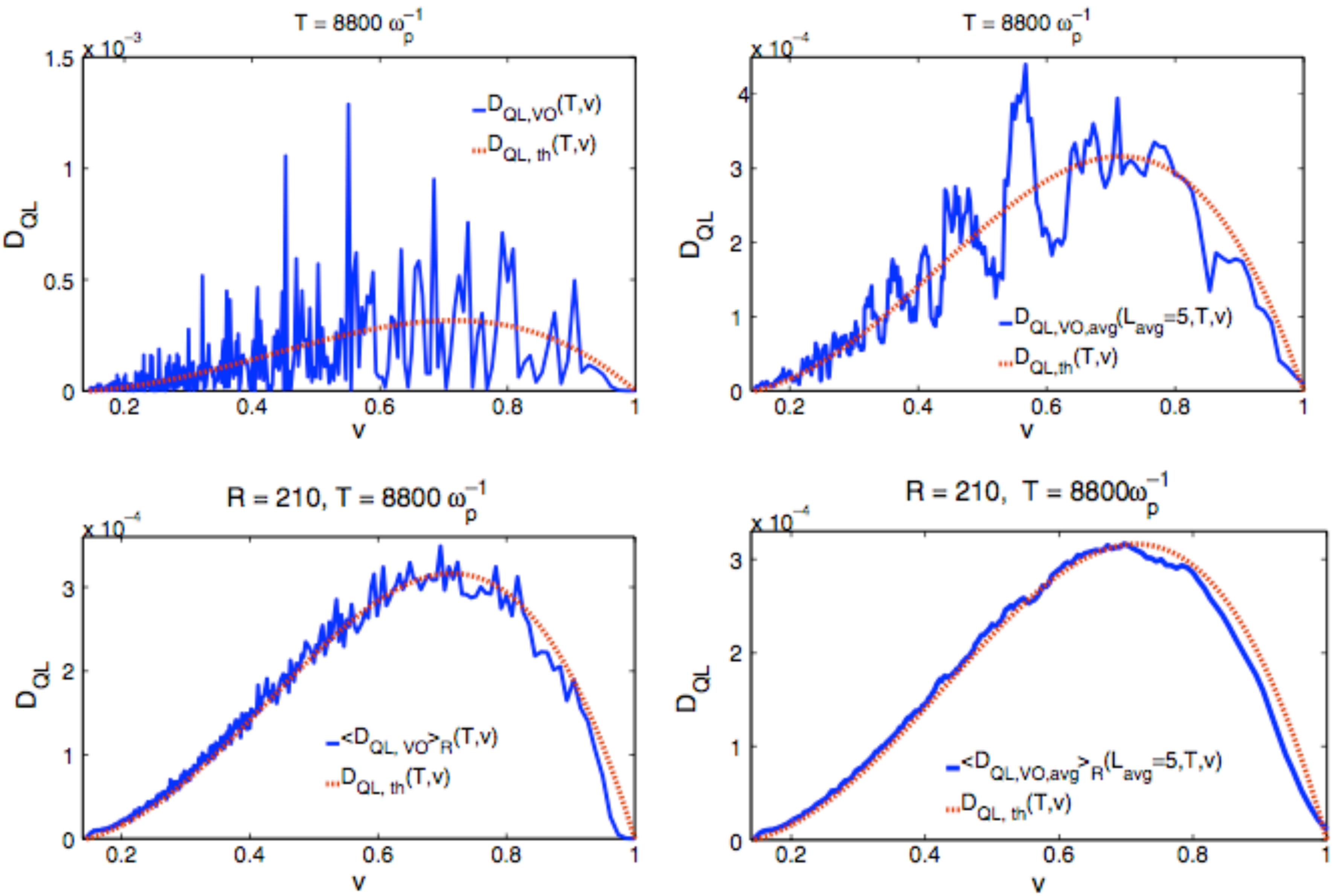
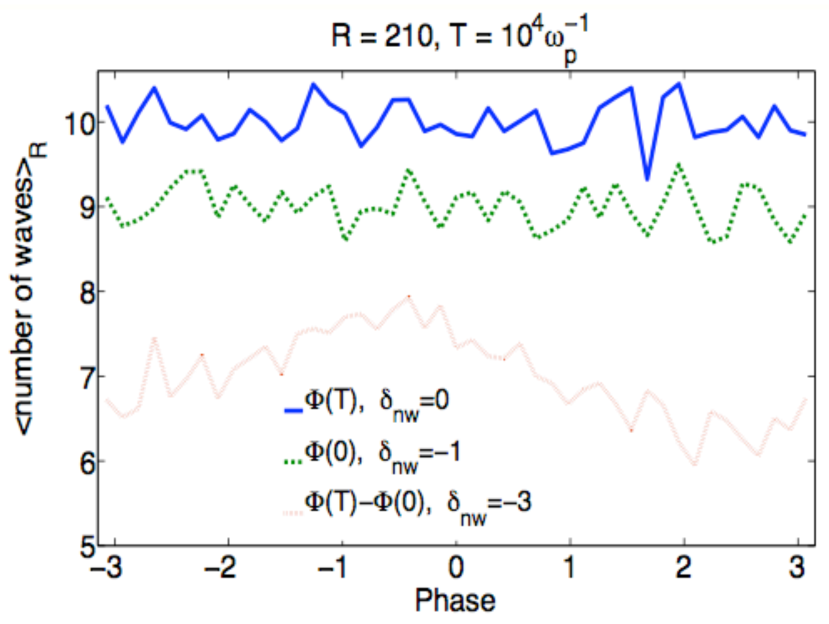
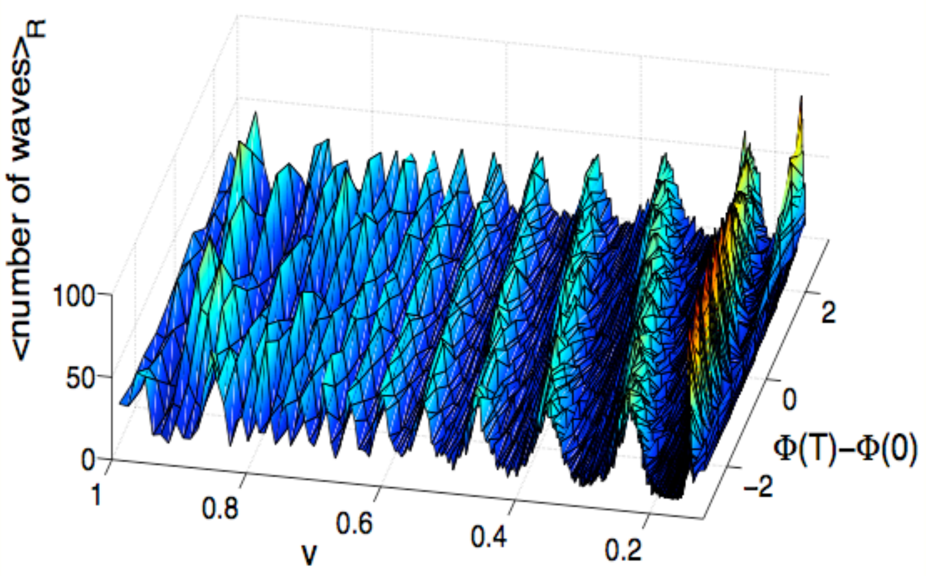


Figure 7. Quasilinear diffusion coefficient: (top) for a single realization; (bottom) ensemble averaged over $R = 210$ simulations; (left) for each velocity from the wave intensity; (right) averaged over $2L + 1 = 11$ neighbouring waves. Dashed red lines: theoretical QL prediction (4).

Waves phases :

initially random, finally random too, correlated ?





Initial randomness propagates...

Microscopic mechanism

Particle motion is chaotic...

but correlated !

and feeds back on waves !

Why should simple quasilinear approximation hold ?!

Microscopic mechanism

In plateau regime :

- strong overlap, strong nonlinearity
- yet wave-wave coupling negligible (EE08)
- test particles

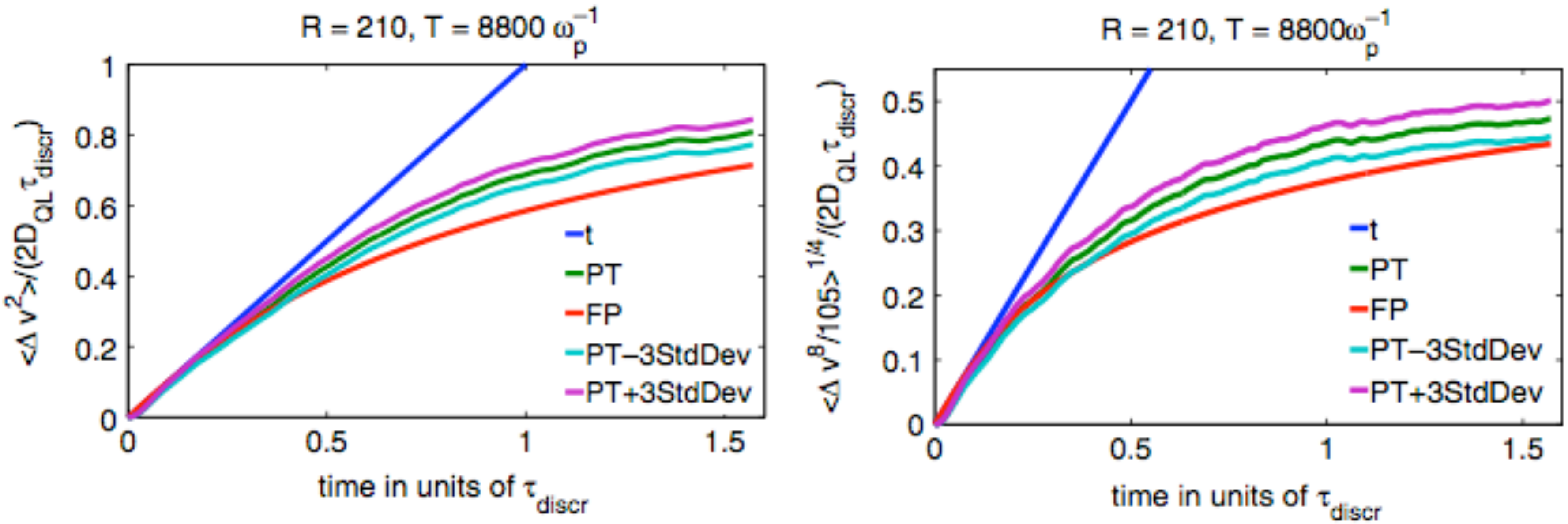


Figure 10. Scaled moments of orders 2 and 8 of the particle velocity deviation. FP: Fokker–Planck equation. PT: motion of 20 test particles in $R = 210$ realizations of the wave complex amplitudes $\zeta_m(T)$. (Colour online.)

Plateau regime (space-time phase ?)

f plateau (waterbag)

$\Rightarrow$ ζ_m's almost stationary :

wave-wave coupling negligible

$\Rightarrow$ particles individually in chaotic $H_{1.5\ \mathrm{deg.freedom}}$

chaotic particle transport obeys Liouville

$\Rightarrow$ plateau preserved

Plateau boundaries : KAM tori

Adiabatic corrections...

(EE : arXiv:0807.1839 ; BEEB : PPCF 2011)

Simulation validation

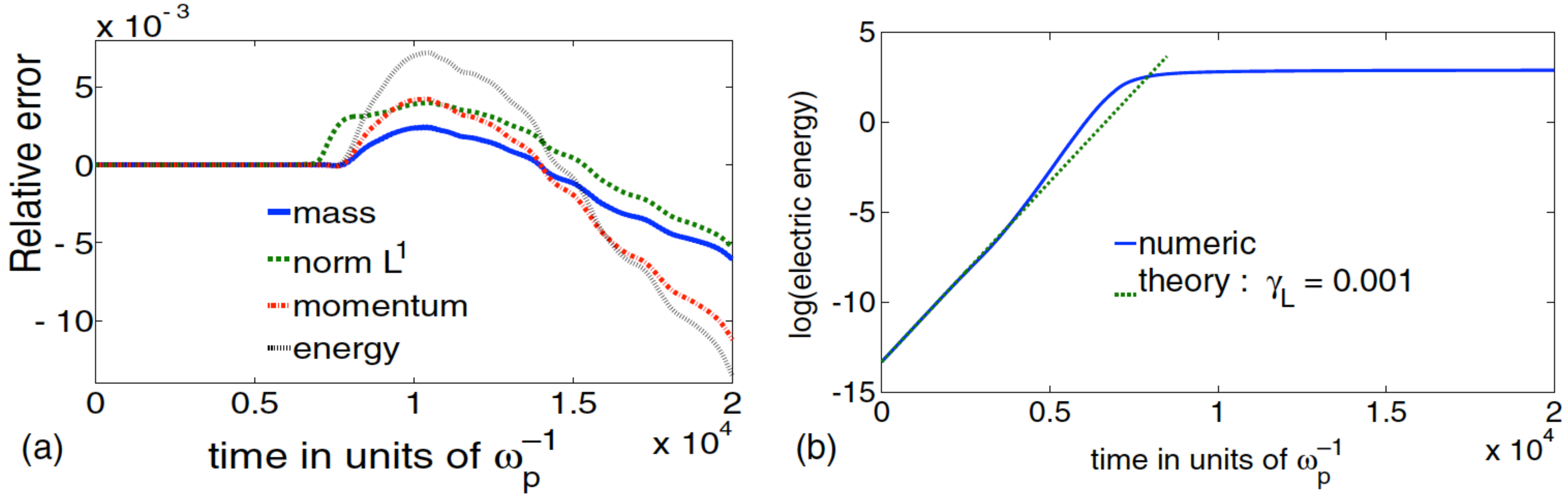
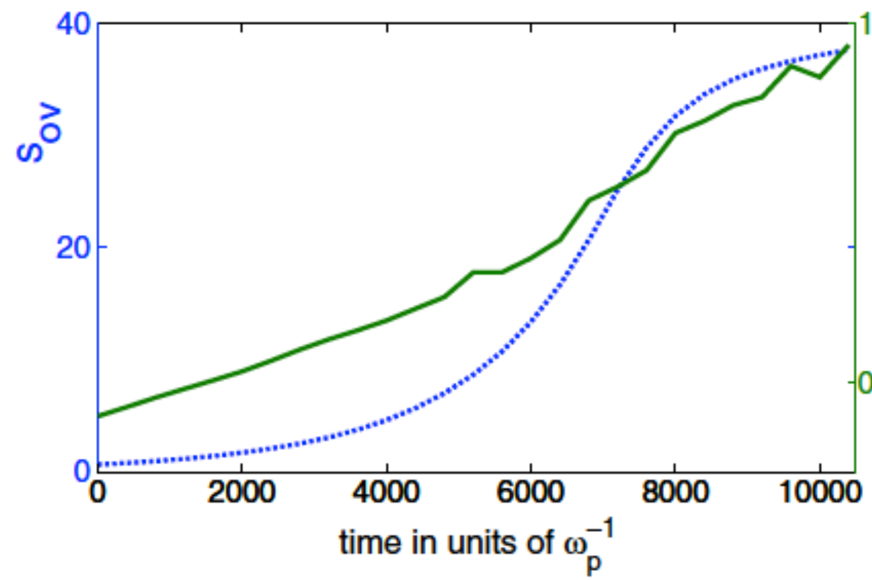


Figure 1. (*a*) Relative error in conservation laws. (*b*) Continuous blue line: logarithm of waves' total energy $\mathfrak{H}_w$; dotted green line: extrapolation at initial linear rate. (Colour online.)

Waves evolution : QL ?



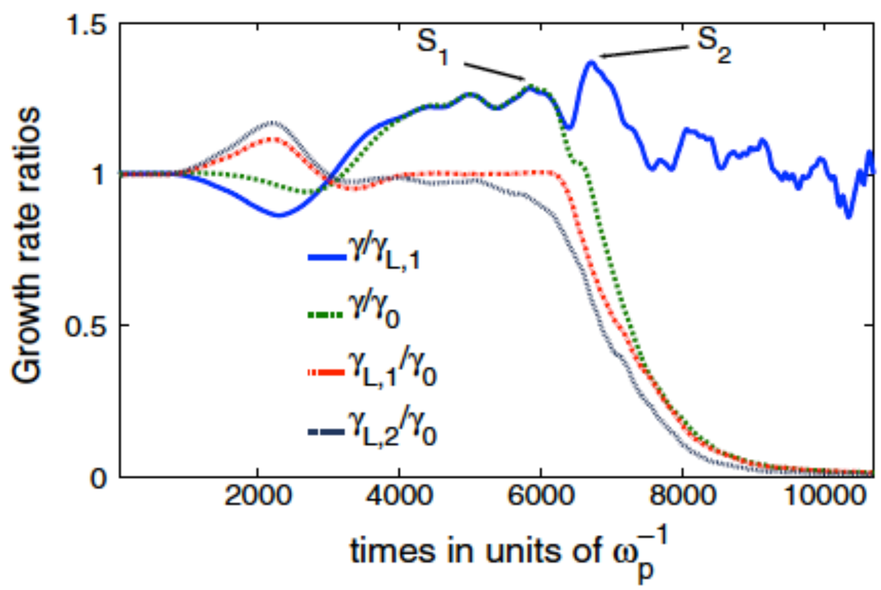


Figure 3. (Left) time evolution of nonlinearity parameters μ (continuous green line) and s_{ov} (dotted blue line). (Right) evolution of $\gamma/\gamma_{\mathrm{L}}$, $\gamma/\gamma_{\mathrm{L}}(0)$ and $\gamma_{\mathrm{L}}/\gamma_{\mathrm{L}}(0)$. Here the energy growth rate is $\gamma(t) = (2E_{\mathrm{w}}(t))^{-1}\mathrm{d}E_{\mathrm{w}}(t)/\mathrm{d}t$, where $E_{\mathrm{w}}(t) = \sum_{m=10}^{M} |\zeta(t, v_m)|^2$ (with $v_{10} \simeq 0.9$). Two average Landau growth rates are plotted, namely $\gamma_{\mathrm{L},1}(t) = (\sum_{m=10}^{M} \Delta v_m)^{-1} \sum_{m=10}^{M} \gamma_{\mathrm{L}}(t, v_m)\Delta v_m$ and $\gamma_{\mathrm{L},2}(t) = (\sum_{m=10}^{M} |\zeta(t, v_m)|^2)^{-1} \sum_{m=10}^{M} \gamma_{\mathrm{L}}(t, v_m)|\zeta(t, v_m)|^2$. The initial value is $\gamma_{\mathrm{L}}(0) = 10^{-3}$.

Conclusion : three stages

- Initial quasilinear : weak nonlinearity, QL is OK
- Intermediate nonlinear : ??? (Doxas-Cary)
- Chaotic plateau : strongly nonlinear, QL is OK

(conclusions robust with respect to initial *f*)

Thank you !

Questions ?

Conservation laws (global)

$$\int_{\Lambda} f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x = L,$$

the rescaled total momentum

$$\int_{\Lambda} v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x + L\sum_{m=1}^{M} k_m \frac{|\zeta_m|^2}{2} = L\mathfrak{P},$$

and the rescaled total energy

$$\int_{\Lambda}\left(\frac{v^2}{2} - \varepsilon\,\mathrm{Re}\left(\sum_{m=1}^{M}\frac{\beta_m}{k_m}\zeta_m \mathrm{e}^{\mathrm{i}(k_m x - \omega_m t)}\right)\right) f(t,x,v)\,\mathrm{d}v\mathrm{d}x + L\sum_{m=1}^{M}\omega_m\frac{|\zeta_m|^2}{2} = L\mathfrak{H}.$$

Momentum and energy exchange

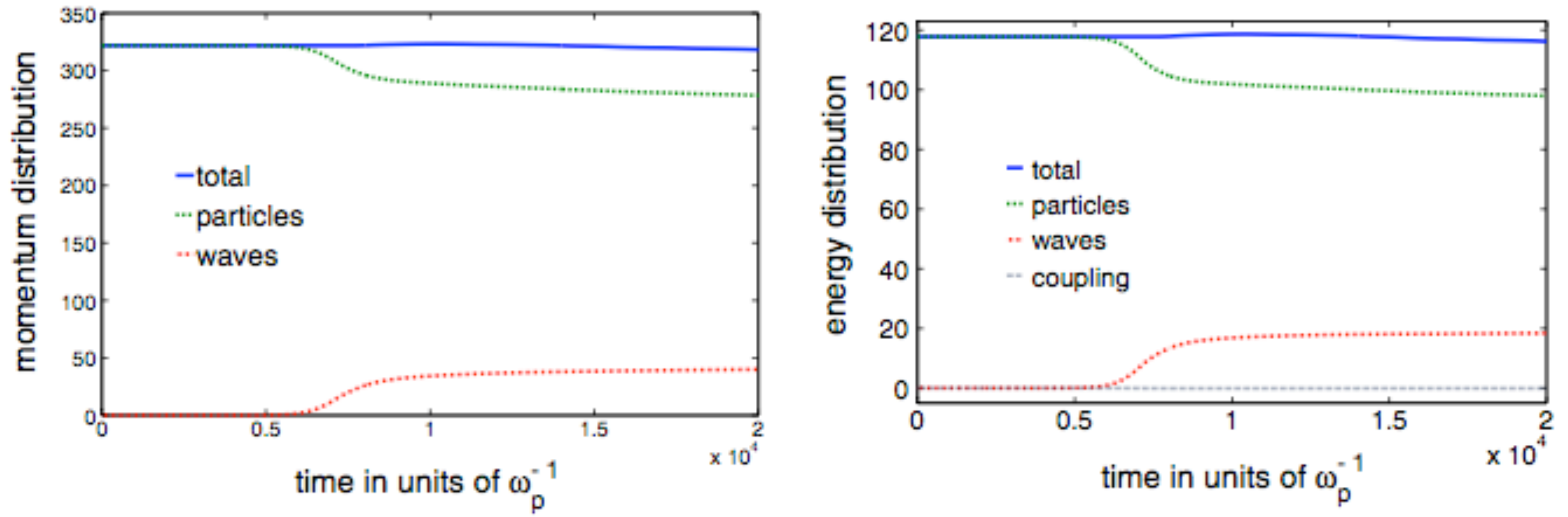


Figure 2. Sharing of momentum and energy between particles (green), waves (red) and coupling (light blue). (Colour online.)