

# Aspects Statistiques pour la dynamique des systèmes à N corps N petit ( $N < 10$ )

Jacques Laskar

CNRS, Observatoire de Paris

*Les Treilles, 22-27 Octobre 2012*



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N petit ( $N < 10$ )  
(Système solaire)

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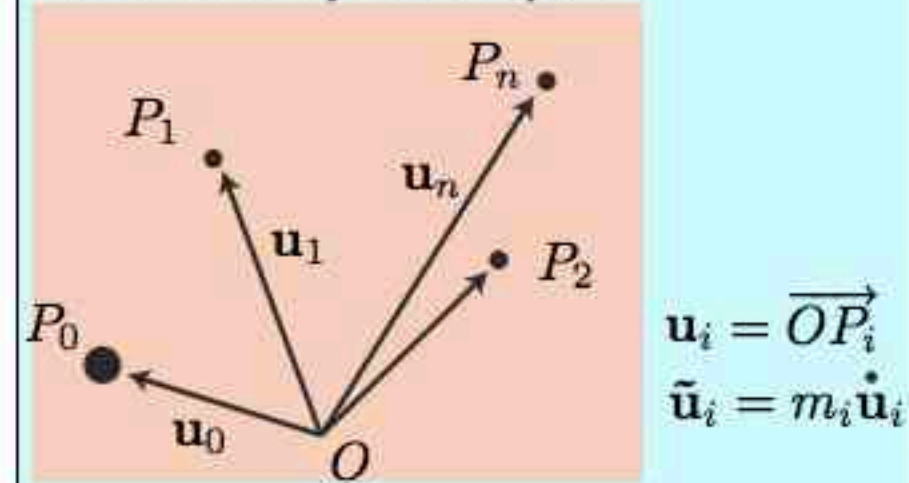
Les Treilles, 22-27 Octobre 2012



# Hamiltonien planétaire

$P_0, P_1, \dots, P_n$   
 $m_0, m_1, \dots, m_n$

Coord. Barycentriques

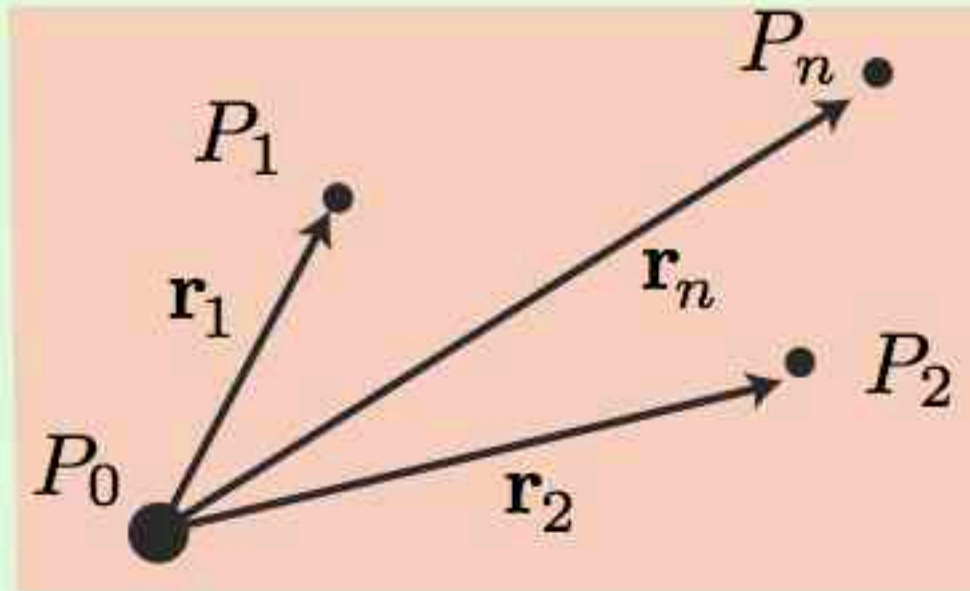


$$\mathbf{u}_i = \overrightarrow{OP_i}$$

$$\tilde{\mathbf{u}}_i = m_i \dot{\mathbf{u}}_i$$

$$H = \frac{1}{2} \sum_{i=0}^n \frac{\|\tilde{\mathbf{u}}_i\|^2}{m_i} - G \sum_{0 \leq i < j} \frac{m_i m_j}{\Delta_{ij}}$$

Coord. Héliocentriques canoniques (Poincaré, 1896)



$$\mathbf{r}_i = \mathbf{u}_i - \mathbf{u}_0 \quad i \neq 0$$

$$\tilde{\mathbf{r}}_i = \tilde{\mathbf{u}}_i$$

$$\mathbf{r}_0 = \mathbf{u}_0$$

$$\tilde{\mathbf{r}}_0 = \tilde{\mathbf{u}}_0 + \tilde{\mathbf{u}}_1 + \dots + \tilde{\mathbf{u}}_n = 0$$

$$H = H_0 + H_1$$

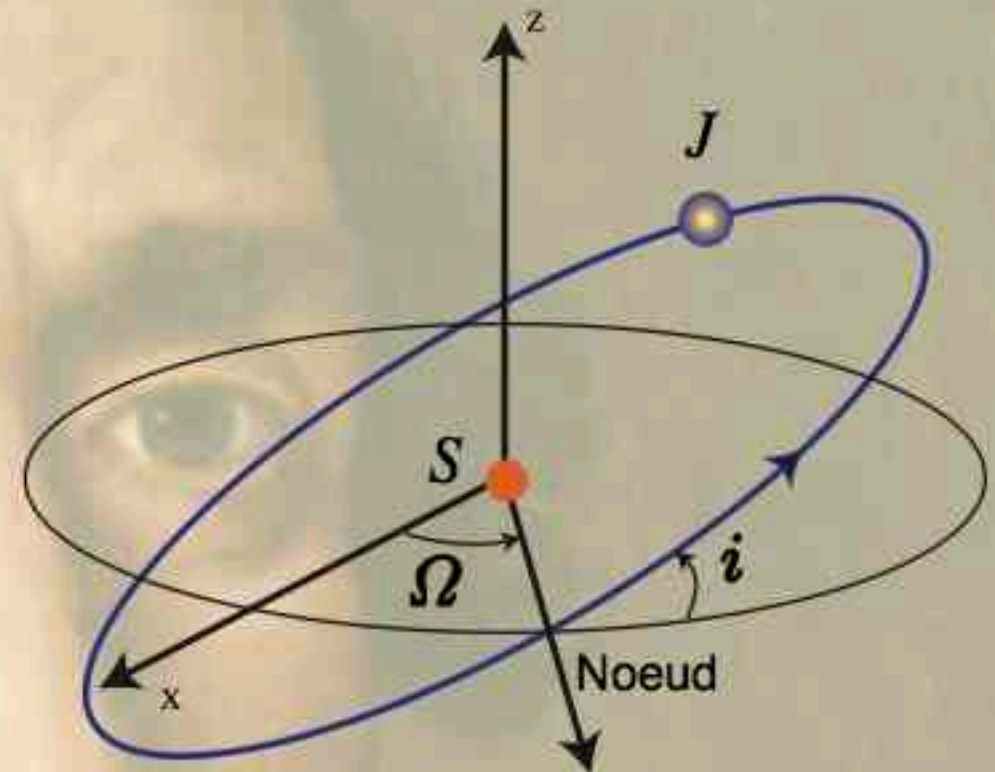
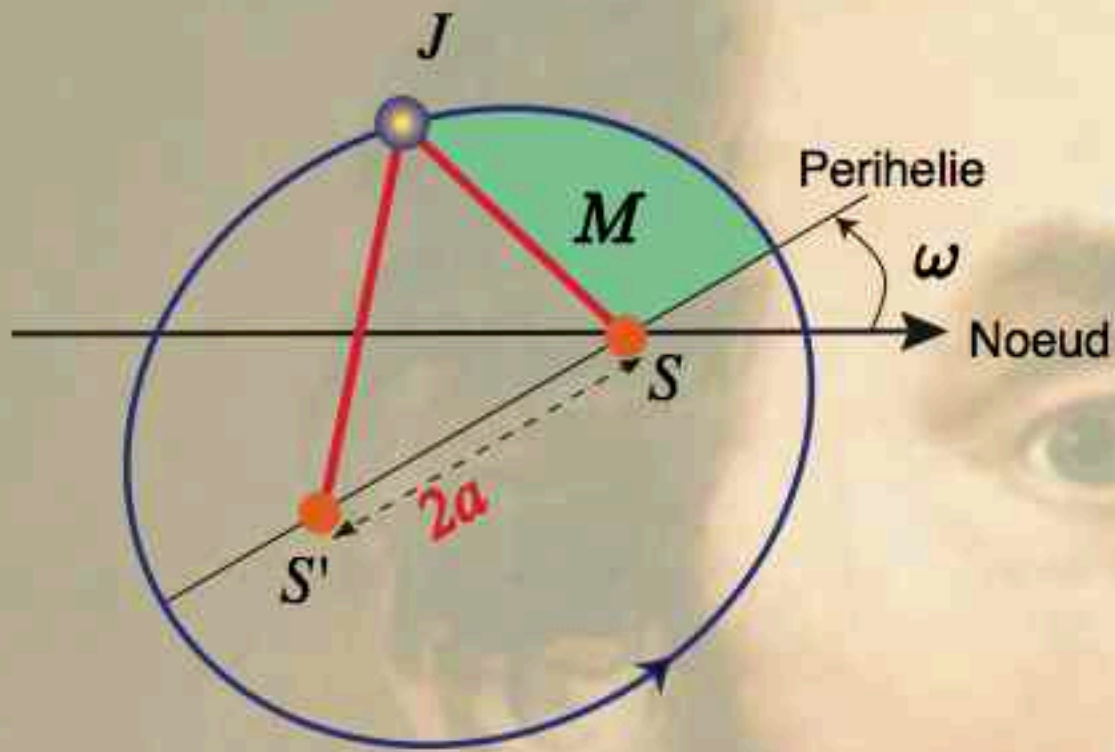
mouvements Képlériens

$$H_0 = \frac{1}{2} \sum_{i=1}^n \|\tilde{\mathbf{r}}_i\|^2 \left[ \frac{1}{m_i} + \frac{1}{m_0} \right] - G \sum_{i=1}^n \frac{m_0 m_i}{r_i}$$

interactions  
planétaires

$$H_1 = \sum_{0 < i < j} \frac{\tilde{\mathbf{r}}_i \cdot \tilde{\mathbf{r}}_j}{m_0} - G \sum_{0 < i < j} \frac{m_i m_j}{\Delta_{ij}}$$

# Kepler (1609)



$\omega$  : argument du perihelie

$e$  : excentricite

$M$  : anomalie moyenne

$a$  : demi-grand axe

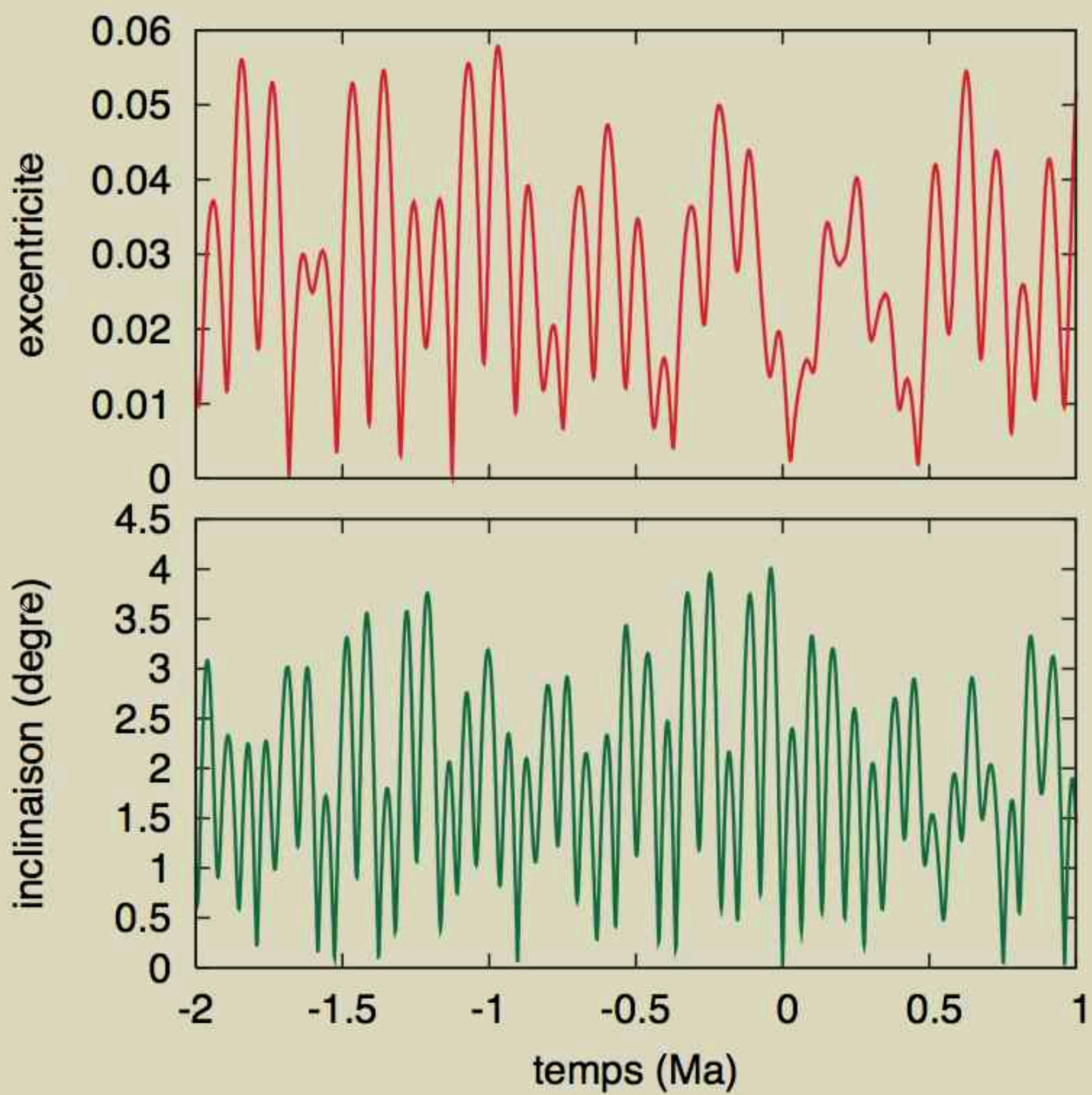
$\Omega$  : longitude du noeud

$i$  : inclinaison

~ actions (  $a, e, i,$   $M = nt$

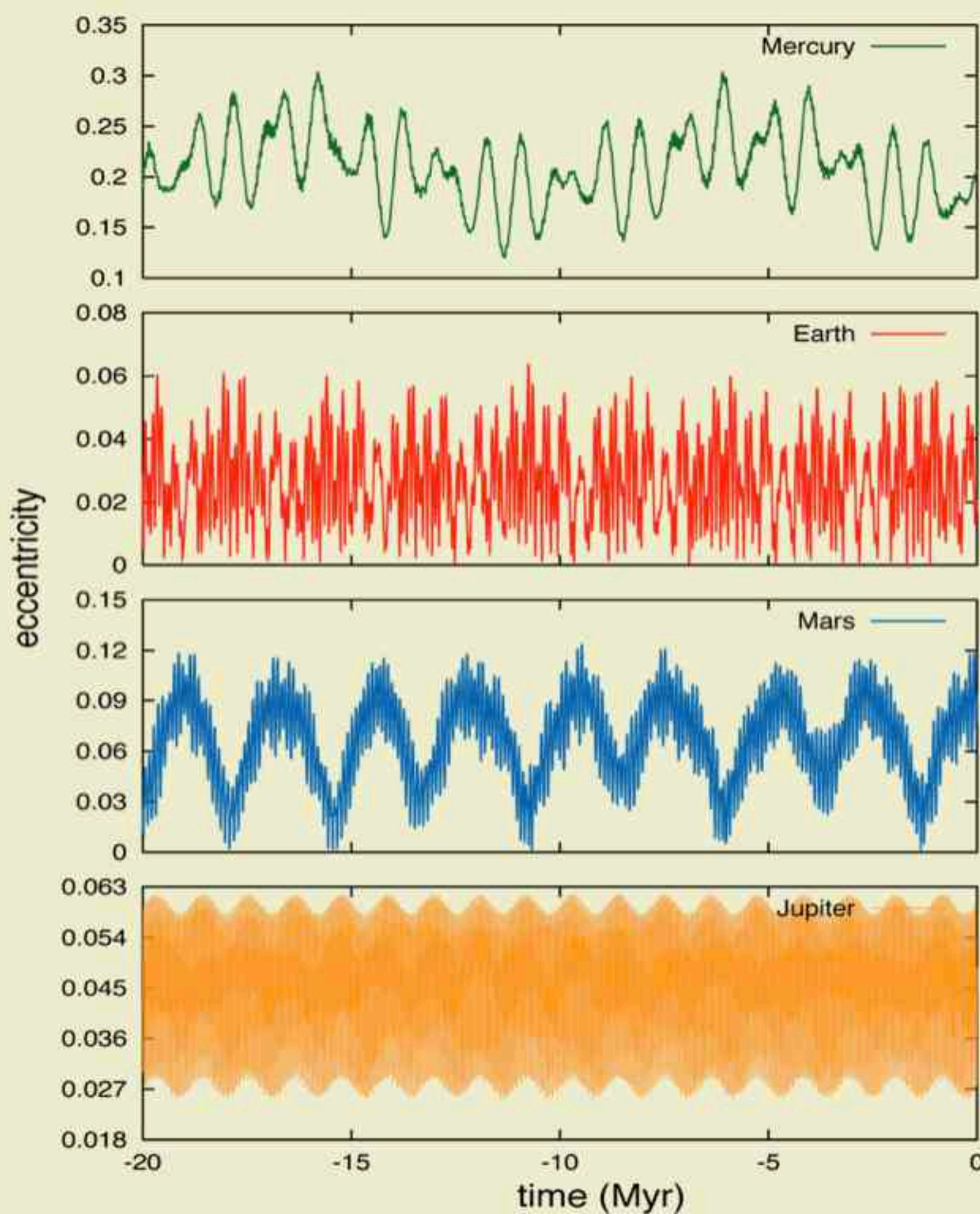
~ angles  $M, \omega, \Omega$  )







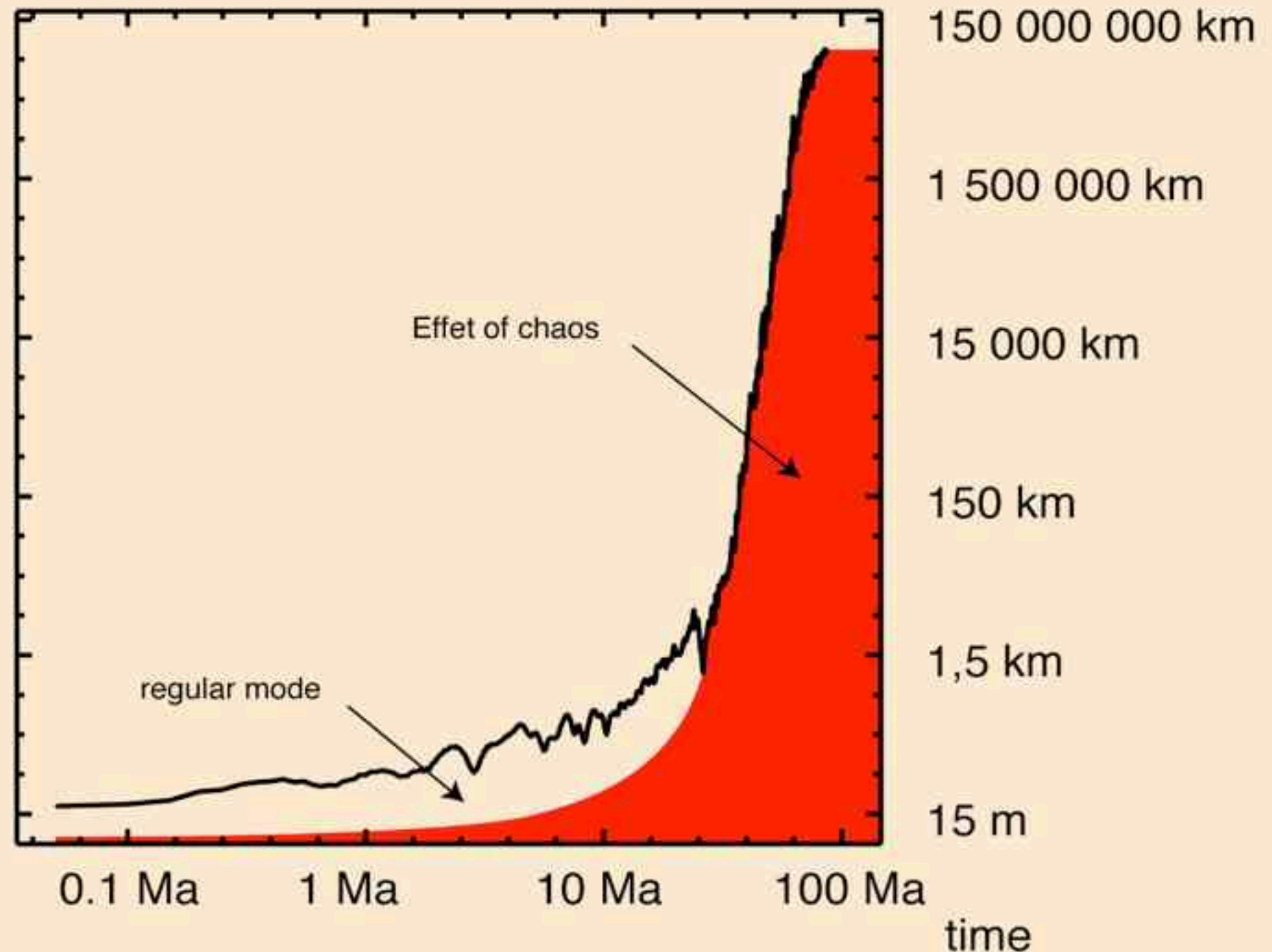
(Lagrange, 1774)



# Mouvement chaotique du Système Solaire

Equations séculaires : 200 Ma : *J. Laskar (1989, 1990)*

Intégration directe : 100 Ma : *G. Sussman & J. Wisdom (1992)*



$$d(T) \approx d_0 10^{T/10}$$



(Laskar, Gastineau, Delisle, Farrès, Fienga ,  
A&A L, 15 July, 2011)



$1/22000$   
masse de la  
Terre

Vesta (DAWN/NASA)



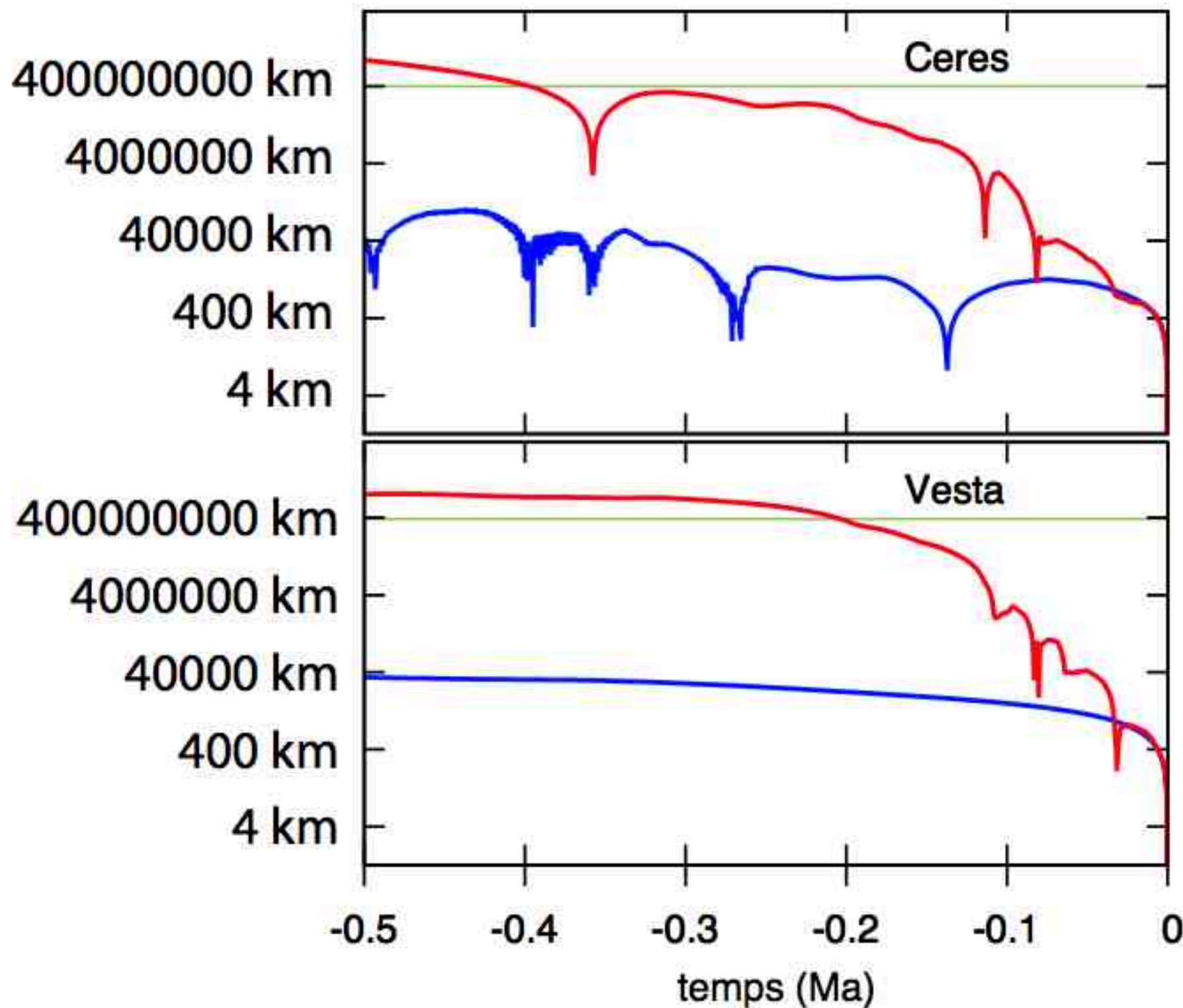
$1/6000$   
masse de la  
Terre

Ceres (Hubble)

**Instabilités dues aux  
rencontres proches des astéroïdes**



# Chaos de Ceres et Vesta

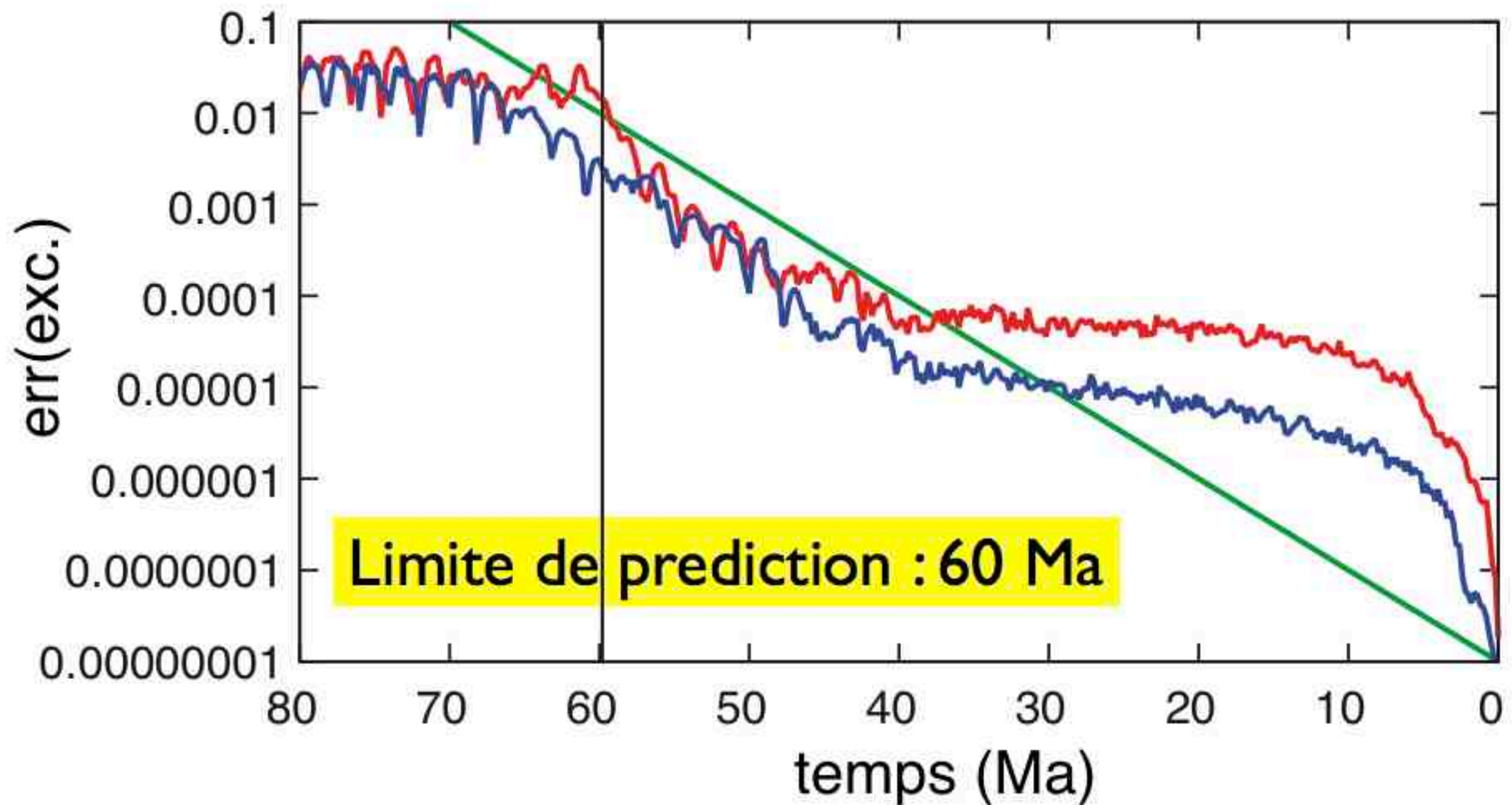


$d_0 = 40 \text{ m}$

perturbations  
mutuelles

Pas de  
perturbations  
mutuelles

## Excentricité de la Terre



Limite de prediction : 60 Ma

Interactions entre astéroïdes

Pas d'interactions entre astéroïdes

(20 cm)

(Laskar et al, A&A, 2011)

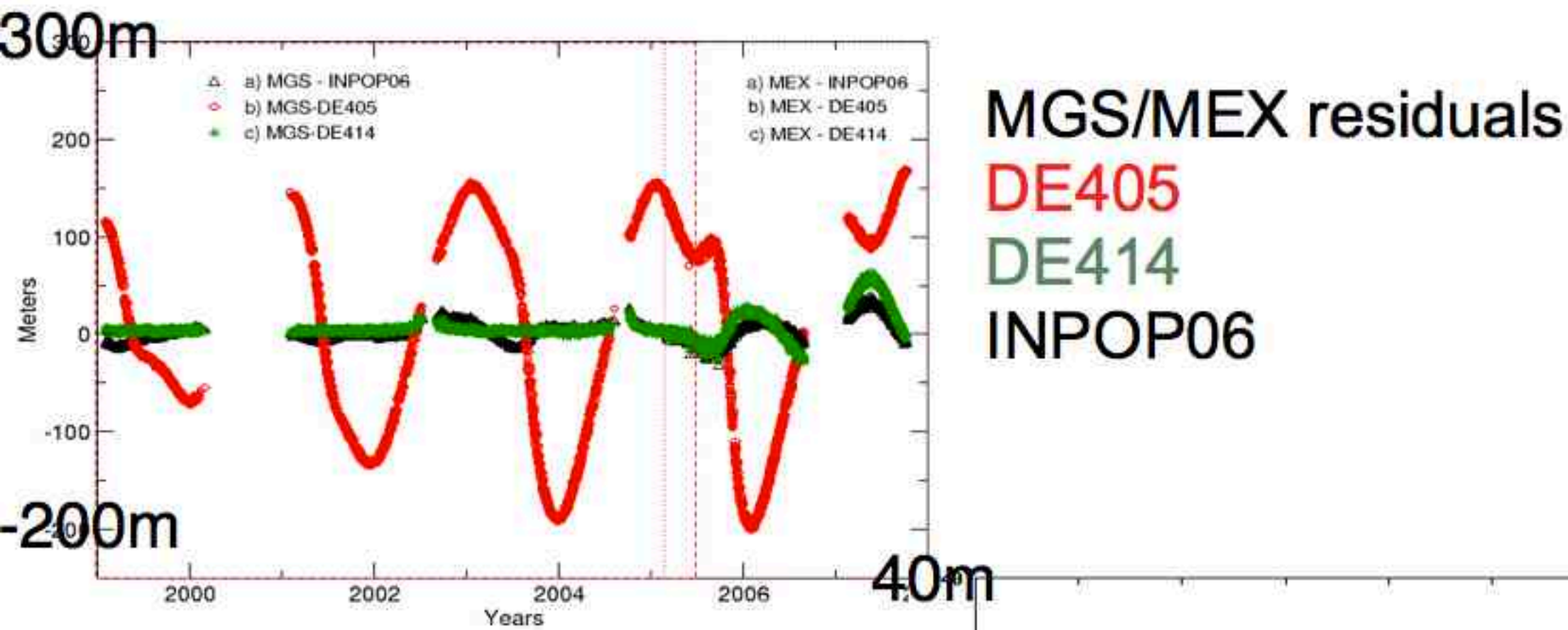


# Perturbation sur 100 ans

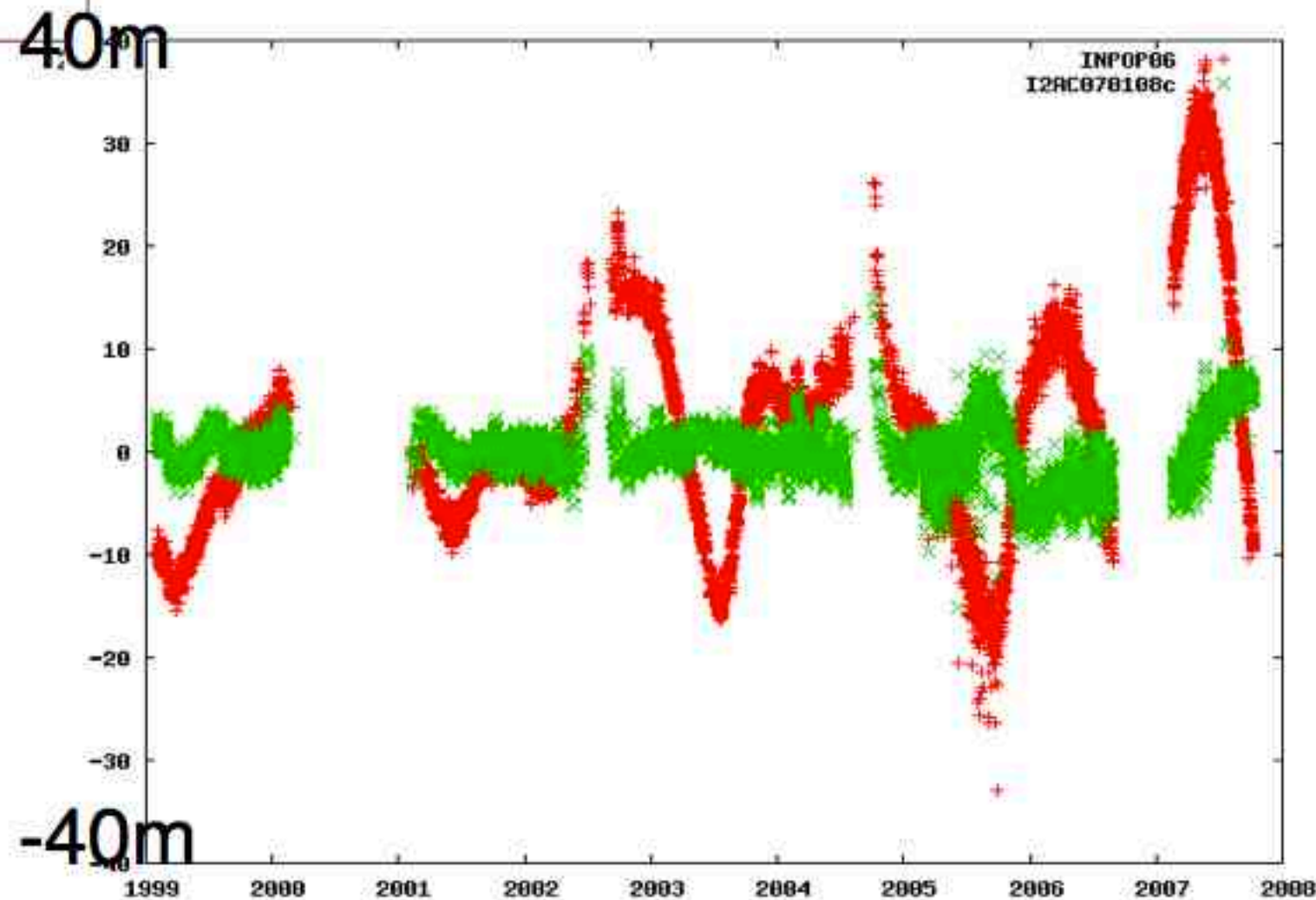
		diametre (km)	terre-mars (m)	terre-venus (m)
1	Ceres	848	7830	2657
2	Pallas	498	6612	977
4	Vesta	468	11455	164
10	Hygiea	407	807	132
6	Hebe	185	692	55
52	Europa	303	253	44
29	Amphitrite	212	107	43
16	Psyche	253	264	39
7	Iris	200	490	37
51	Nemausa	148	505	34
15	Eunomia	255	106	32
747	Winchester	172	222	28
532	Herculina	222	661	24
654	Zelinda	127	143	22
20	Massalia	146	286	21

(Kuchynka et al, A&A, 2010)

# Status of the INPOP Planetary Part: Mars



MGS/MEX residuals  
INPOP06  
INPOP07





# Premières Conclusions

Au delà de 60 Ma , le prix à payer

n'est pas

une amélioration d'un facteur 10 par 10 Ma,

mais

une amélioration d'un facteur 10 par 50 ka,

15 m -> 60.00 Ma

# Premières Conclusions

Au delà de 60 Ma , le prix à payer

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une amélioration d'un facteur 10 par 10 Ma,

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une amélioration d'un facteur 10 par 50 ka,

15 m -> 60.00 Ma

15 mm ->



# Premières Conclusions

Au delà de 60 Ma , le prix à payer

n'est pas

une amélioration d'un facteur 10 par 10 Ma,

mais

une amélioration d'un facteur 10 par 50 ka,

15 m -> 60.00 Ma

15 mm -> 60.15 Ma

# Premières Conclusions

Au delà de 60 Ma , le prix à payer

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mais

une amélioration d'un facteur 10 par 50 ka,

15 m -> 60.00 Ma

15 mm -> 60.15 Ma

15  $\mu$ m ->



# Premières Conclusions

Au delà de 60 Ma , le prix à payer

n'est pas

une amélioration d'un facteur 10 par 10 Ma,

mais

une amélioration d'un facteur 10 par 50 ka,

15 m -> 60.00 Ma

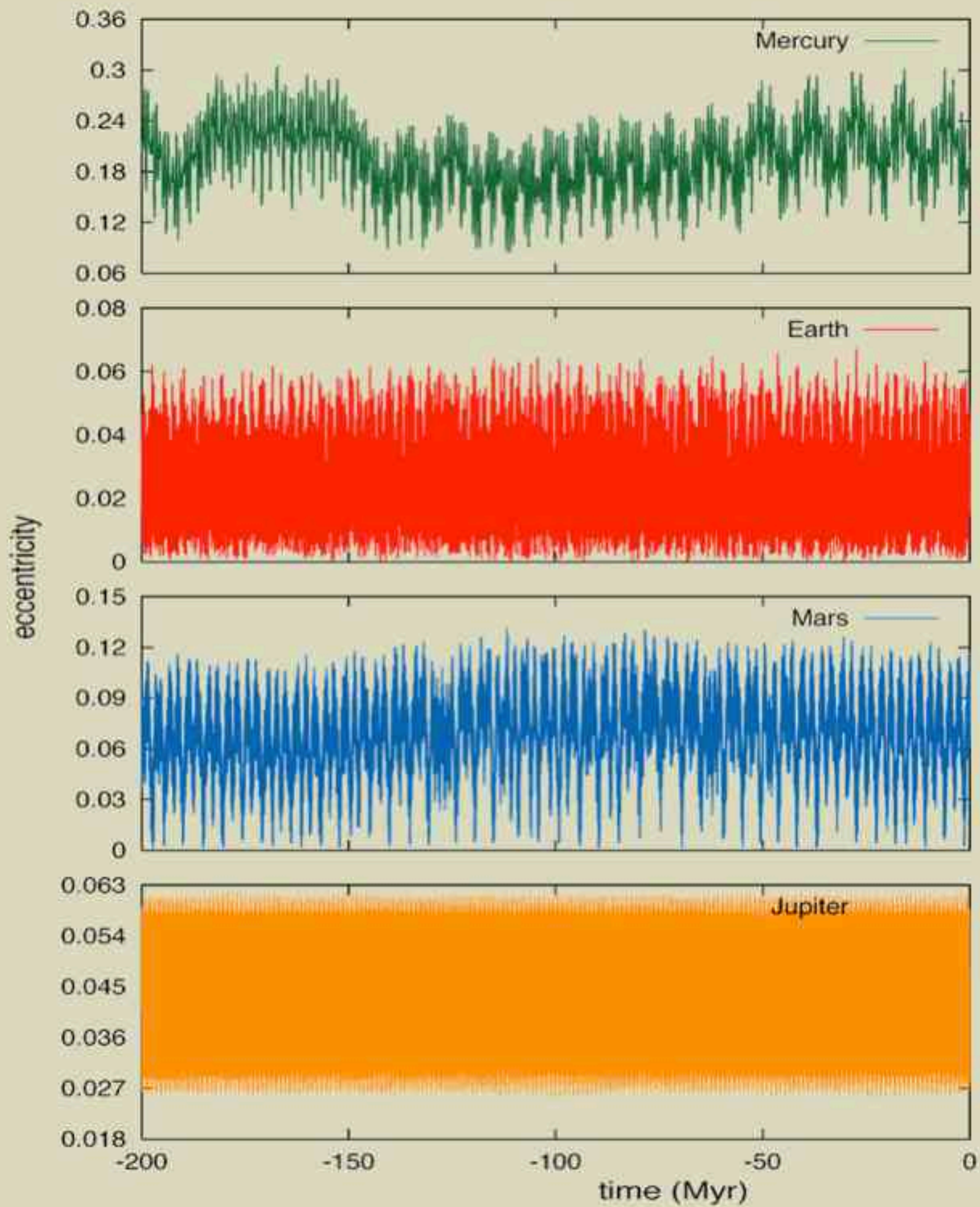
15 mm -> 60.15 Ma

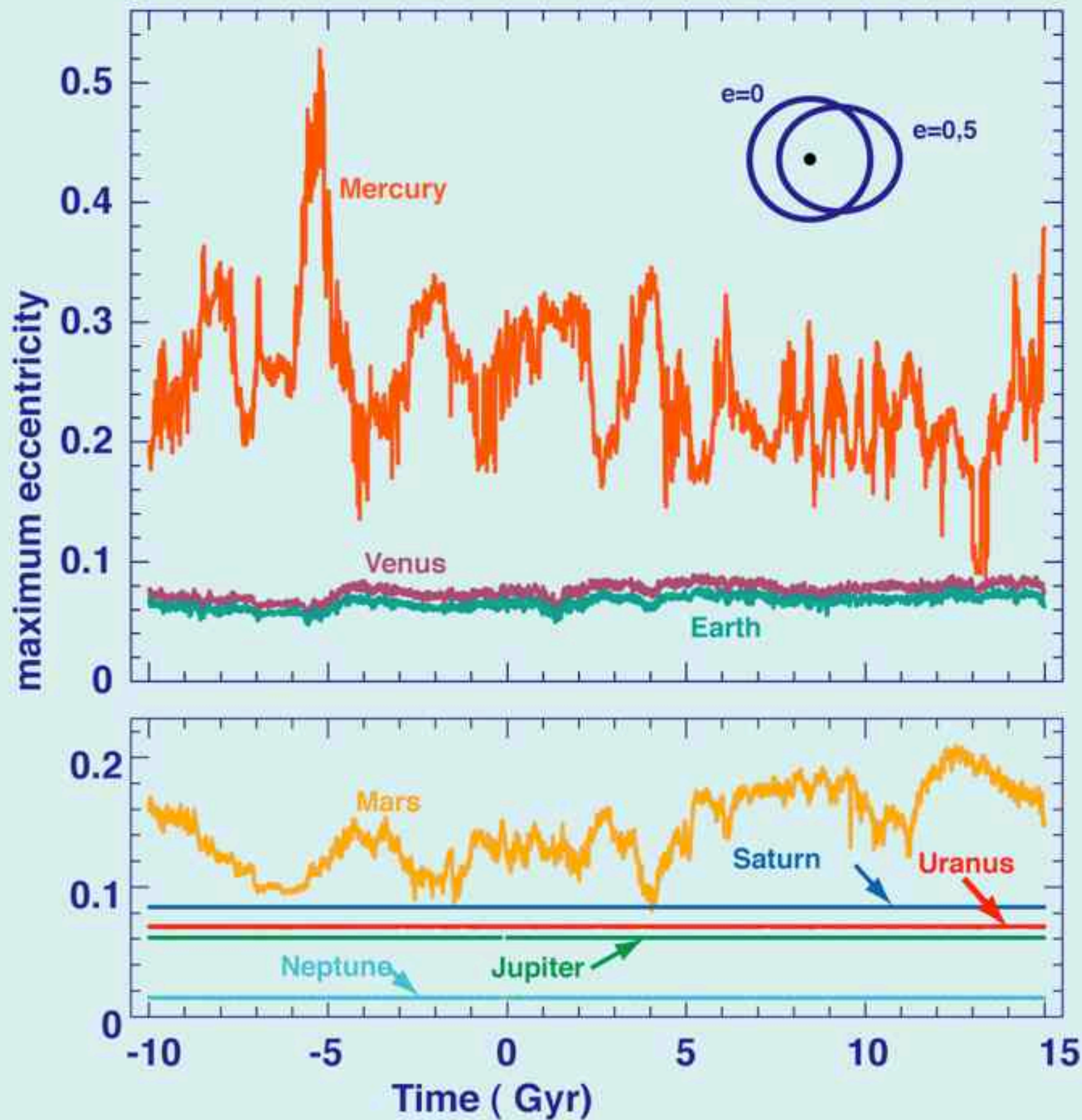
15  $\mu$ m -> 60.30 Ma

Very long times

$$T > 250 \text{ Myr}$$

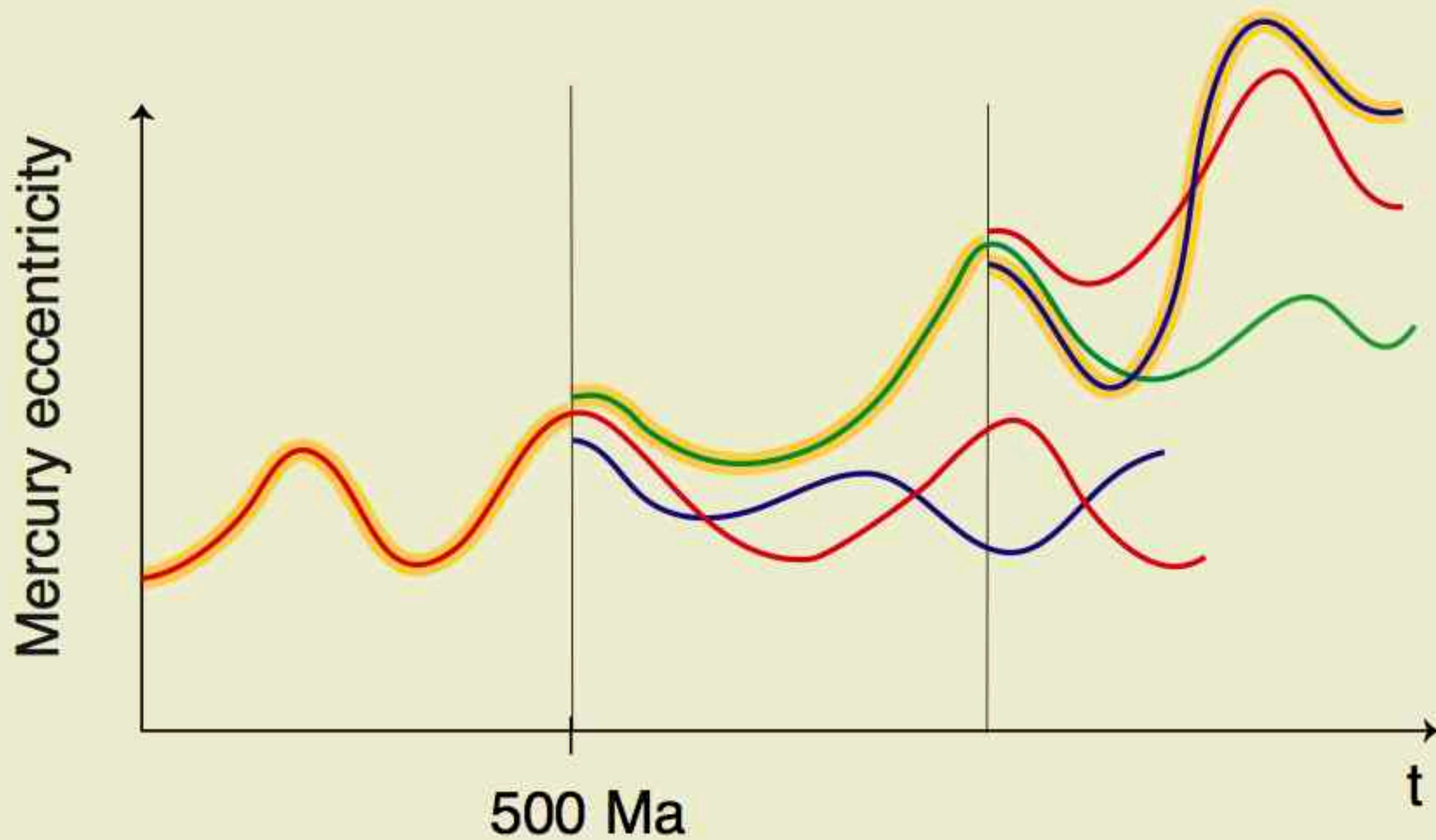






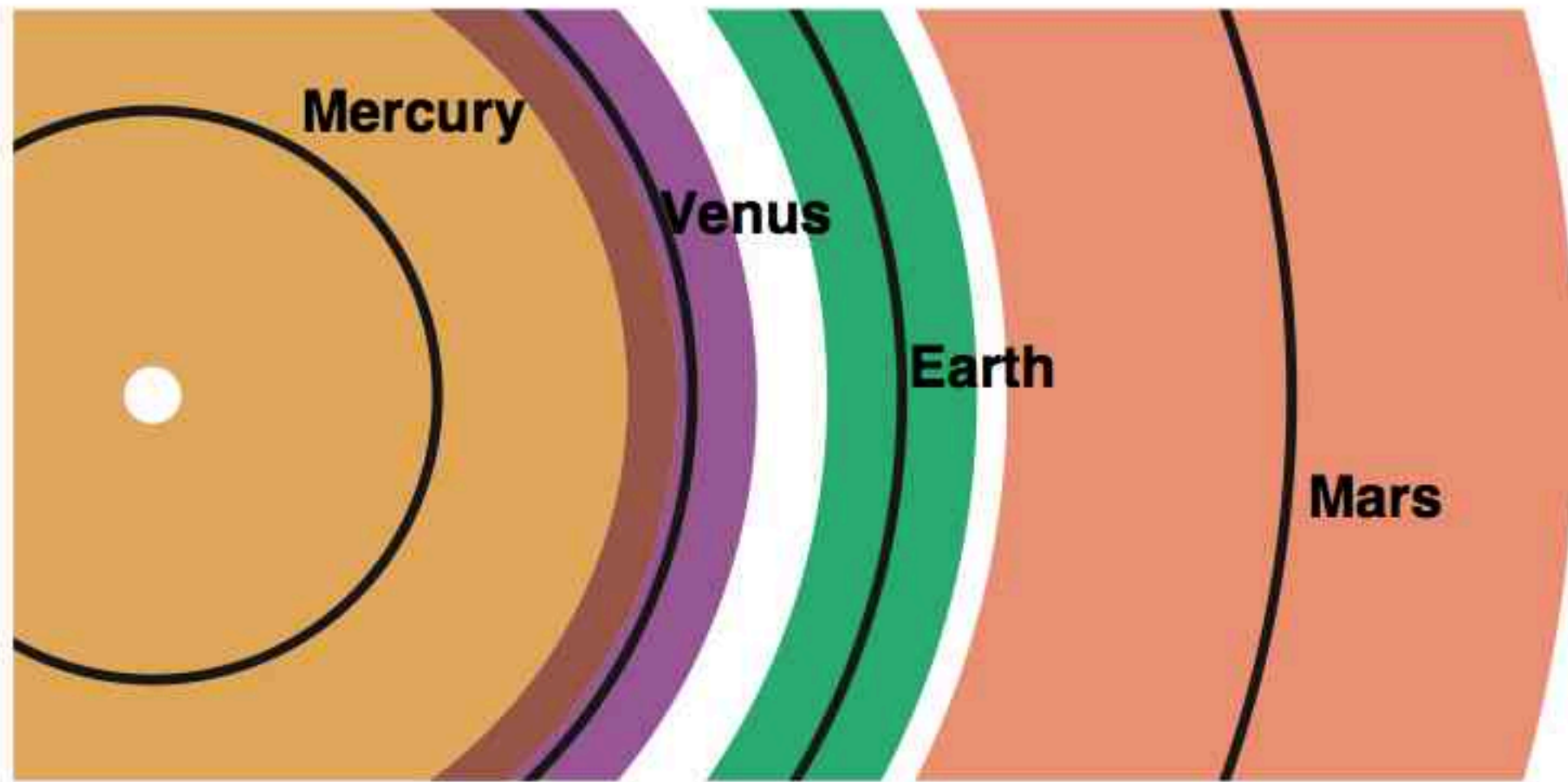


# shadow orbit for the collision of Mercury with Venus



$$d_0 = 15m \times 10^{-50}$$

(J. Laskar, A&A, 1994)





# Angular Momentum Deficit (J. Laskar, 1997, 2000)

$k$	$m_k \sqrt{a_k}$	$J_k$	$I_k$
1	6490	139	47
2	130804	3	230
3	191032	27	0
4	25029	109	13
5	136766856	160915	35338
6	55505910	85579	52280
7	12044023	12914	1095
8	17850190	721	8516
1-4	353355	278	290
1-8	222520335	260407	97519

$$\begin{cases} \Lambda = m\sqrt{a} & \lambda \\ J = \Lambda(1 - \sqrt{1 - e^2}) \approx \frac{1}{2}\Lambda e^2 & -\varpi \\ I = \Lambda\sqrt{1 - e^2}(1 - \cos i) \approx \frac{1}{2}\Lambda i^2 & -\Omega \end{cases}$$

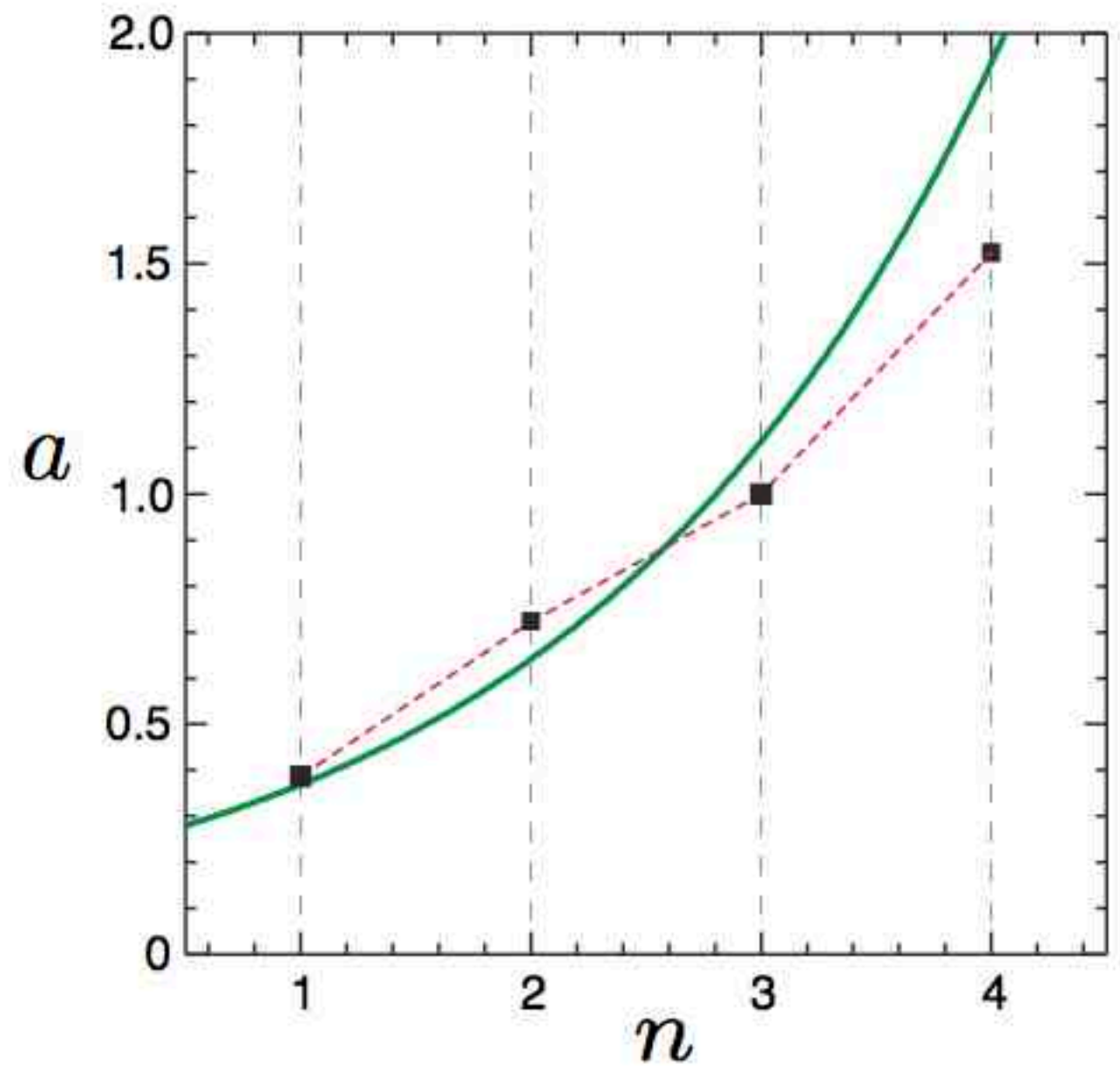
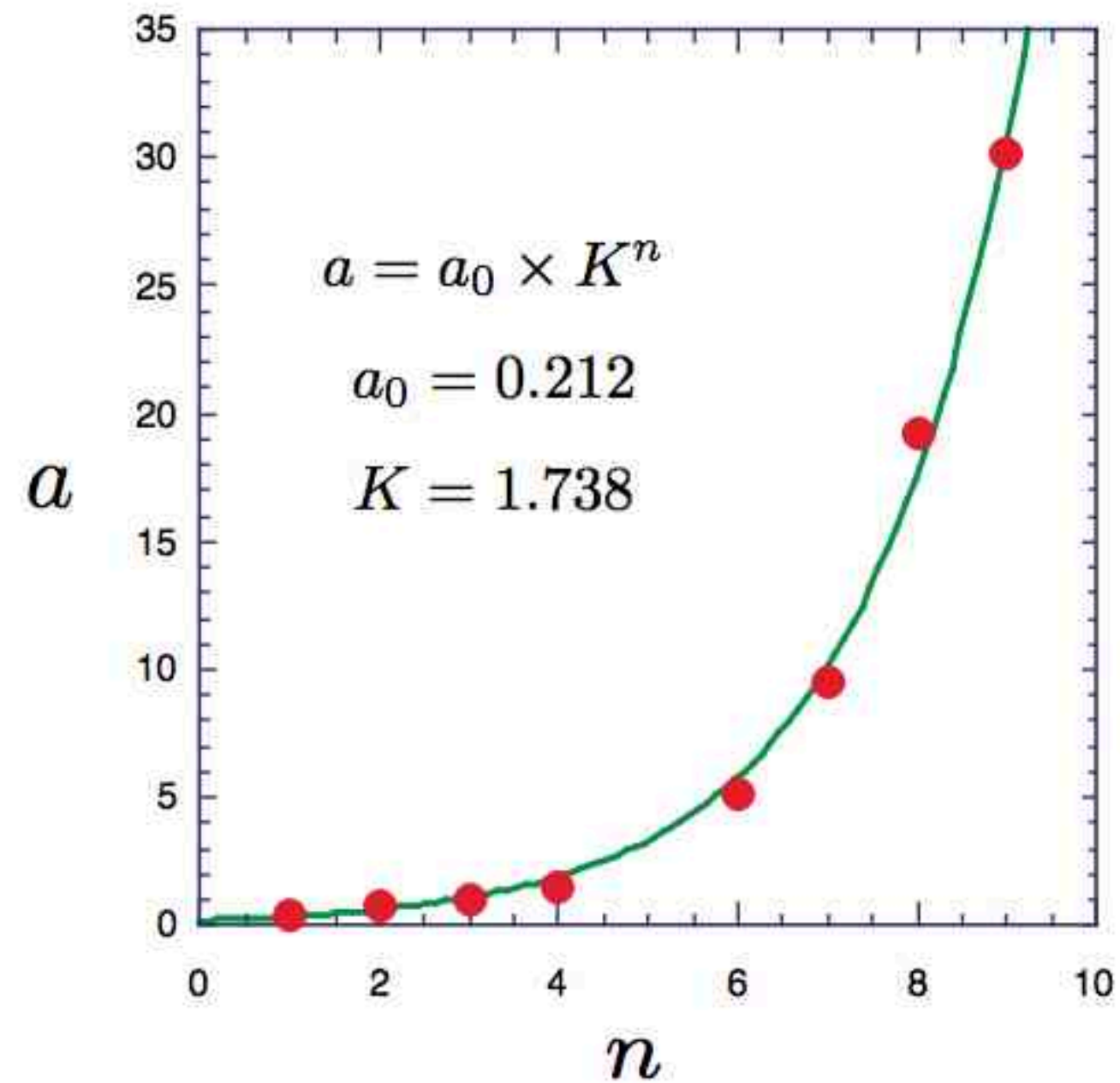
$$C = \sum_{k=1}^8 m_k \sqrt{a_k} - J_k - I_k$$

$$C' = \sum_{k=1}^8 J_k + I_k \approx \sum_{k=1}^8 m_k \sqrt{a_k} (e_k^2 + i_k^2)$$

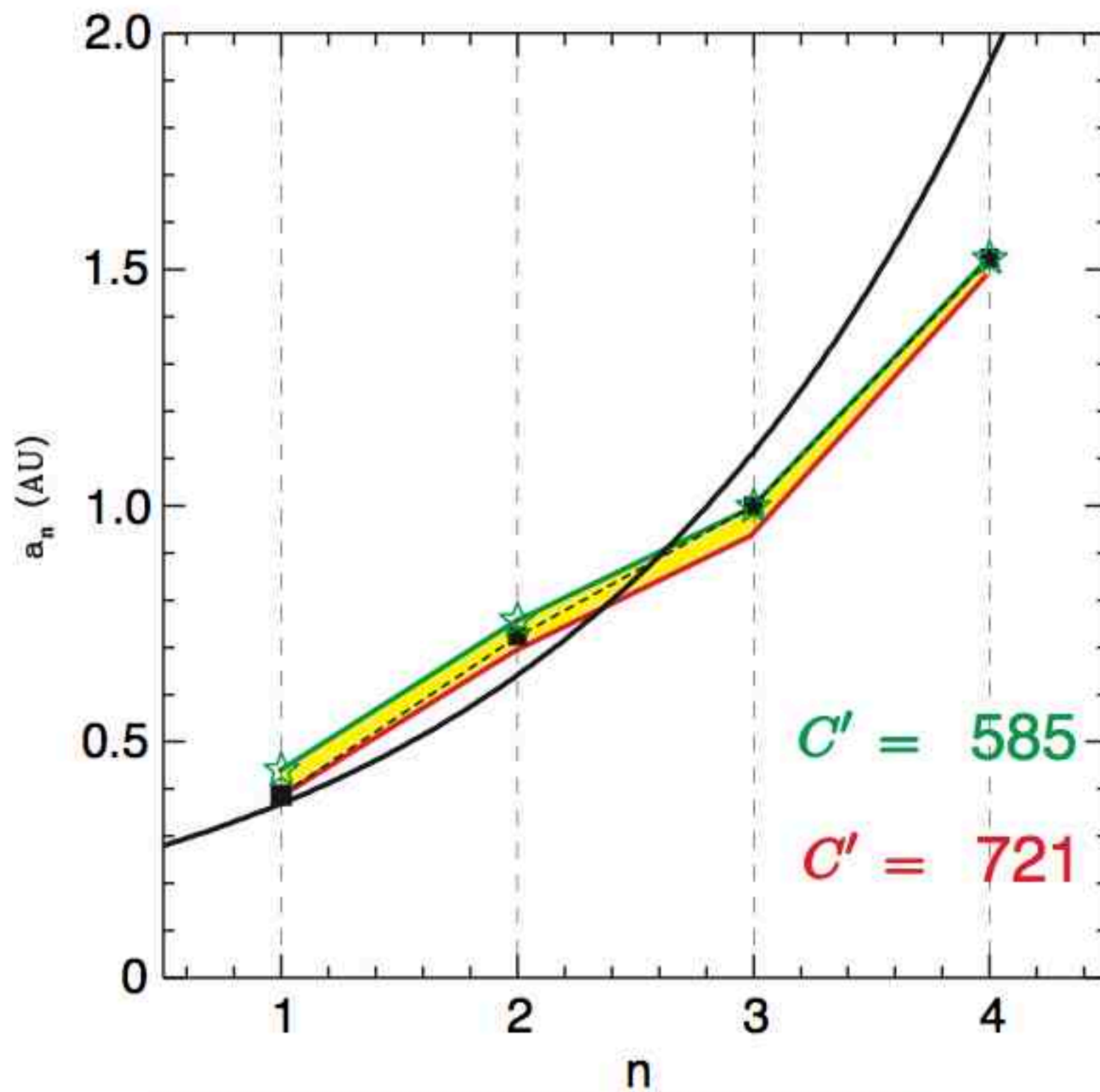
$$e_k \leq \sqrt{\frac{2C'}{m_k \sqrt{a_k}}}$$

(J. Laskar, A&A, 1997)

## Loi de type Titius - Bode



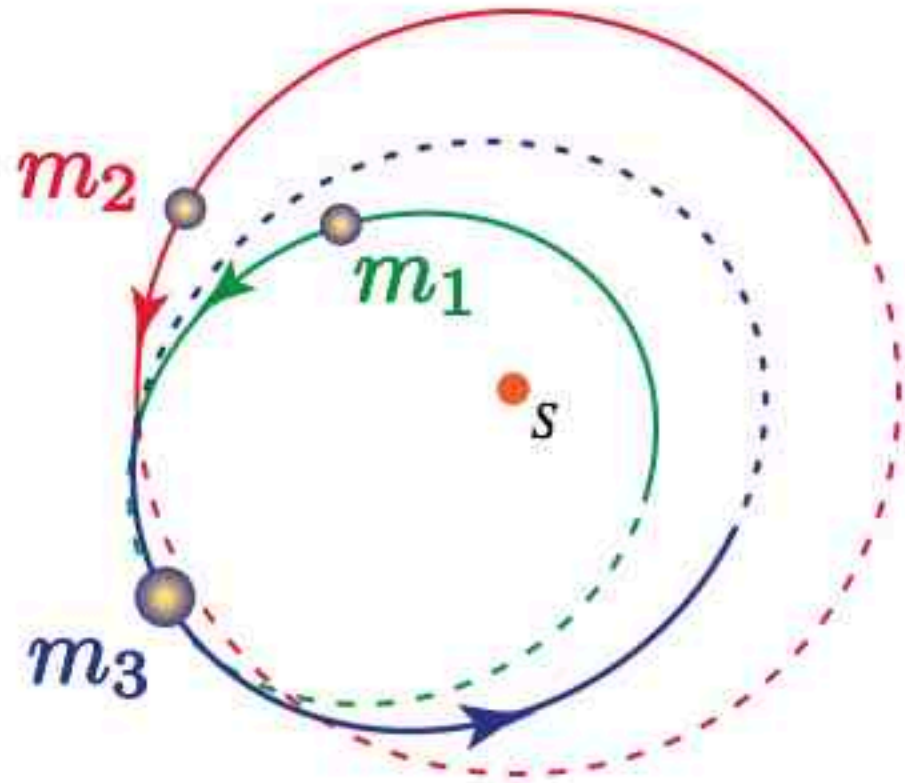




$$e_k \leq \sqrt{\frac{C'}{\Lambda_k}}$$

$$a_k(1 + e_k) \leq 0.9 a_{k+1}(1 - e_{k+1})$$

# Simple Planetary System (SPS)



$$m_3 = m_1 + m_2$$

$$m_3 \dot{\mathbf{r}}_3 = m_1 \dot{\mathbf{r}}_1 + m_2 \dot{\mathbf{r}}_2$$



$$m_3 \mathbf{r}_3 \wedge \dot{\mathbf{r}}_3 = m_1 \mathbf{r}_1 \wedge \dot{\mathbf{r}}_1 + m_2 \mathbf{r}_2 \wedge \dot{\mathbf{r}}_2$$



# Scaling laws

$$\rho(a) = \zeta a^p$$

$p$	$k$	$a$	$m(a)$
0	$\frac{1}{4}$	$\sqrt{a} = \sqrt{a_0} + (\tilde{C}/2\zeta)^{1/3} n$	$(4\tilde{C}\zeta^2)^{1/3} a^{1/2}$
$-\frac{1}{2}$	$-$	$a^{1/3} = a_0^{1/3} + (\tilde{C}/k\zeta)^{1/3} \frac{n}{3}$	$(\tilde{C}\zeta^2/k)^{1/3} a^{1/6}$
$-1$	$\frac{1}{2}$	$a^{1/6} = a_0^{1/6} + (2\tilde{C}\zeta)^{1/3} \frac{n}{6}$	$(2\tilde{C}\zeta^2)^{1/3} a^{-1/6}$
$-\frac{3}{2}$	$\frac{1}{2}$	$\log(a) = \log(a_0) + \left(\frac{2\tilde{C}}{\zeta}\right)^{1/3} n$	$(2\tilde{C}\zeta^2)^{1/3} a^{-1/2}$

TABLE I. Planetary distribution corresponding to different initial mass distribution. For  $p = -\frac{1}{2}$ , we have  $\frac{1}{4} < k < \frac{1}{2}$ .  $p = 0$  gives a law in  $n^2$  for  $a(n)$ , while  $p = -\frac{3}{2}$  gives a Bodes-like power law.

5000 SPS Simulations (n=10 000)

$$p = 0$$

n	4	5	6	7	8
$N$	451	1965	1939	591	52
$\sqrt{a}$	0.638	0.533	0.468	0.420	0.376
$m(10^{-7})$	6.884	5.189	4.161	3.521	2.415

Distribution of semi major axes (n = 6)

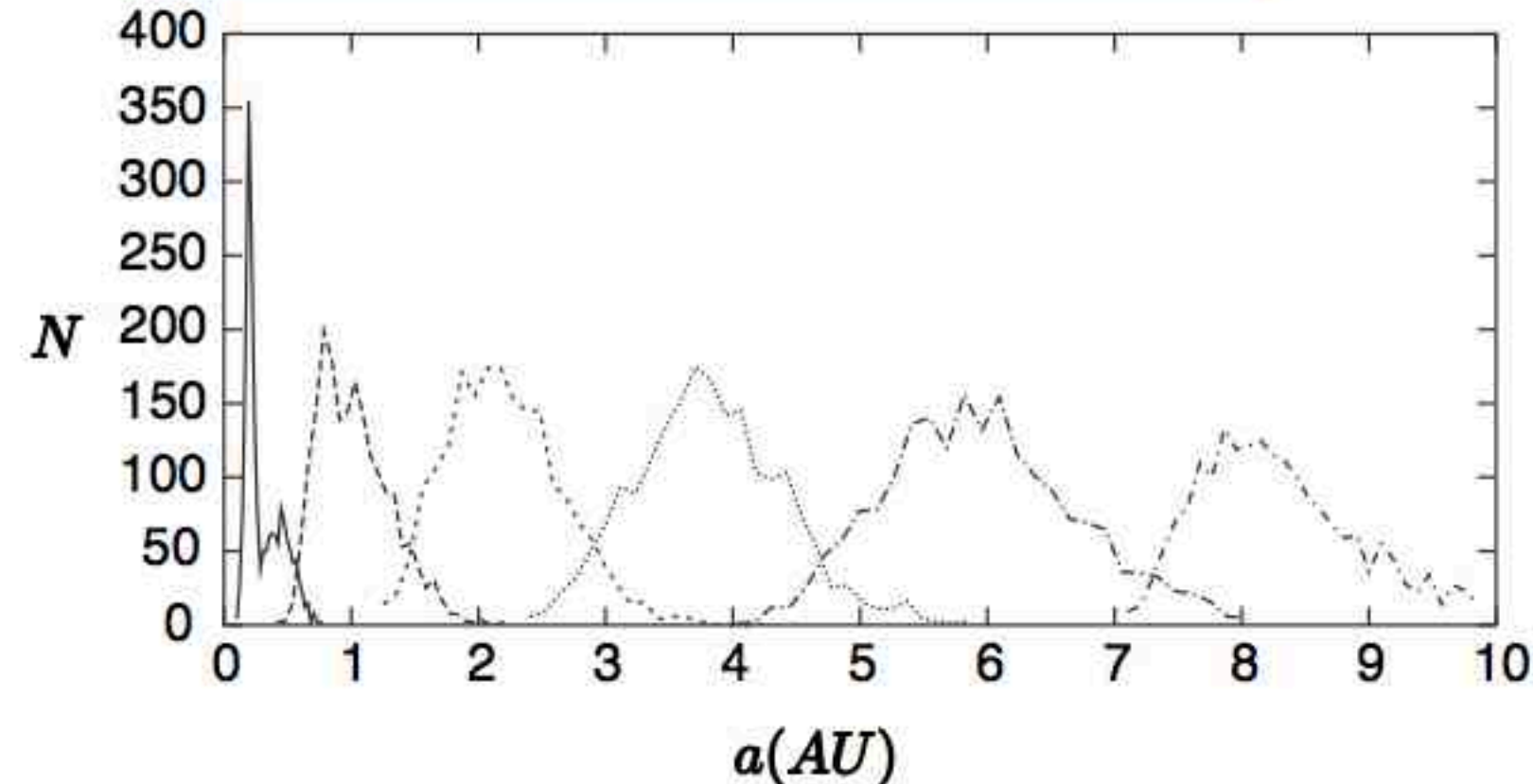


FIG. 2. Distribution of the semi-major axis of the 1939 system with 6 planets resulting from 5000 SPS simulations .

(J. Laskar, PRL, 2000)



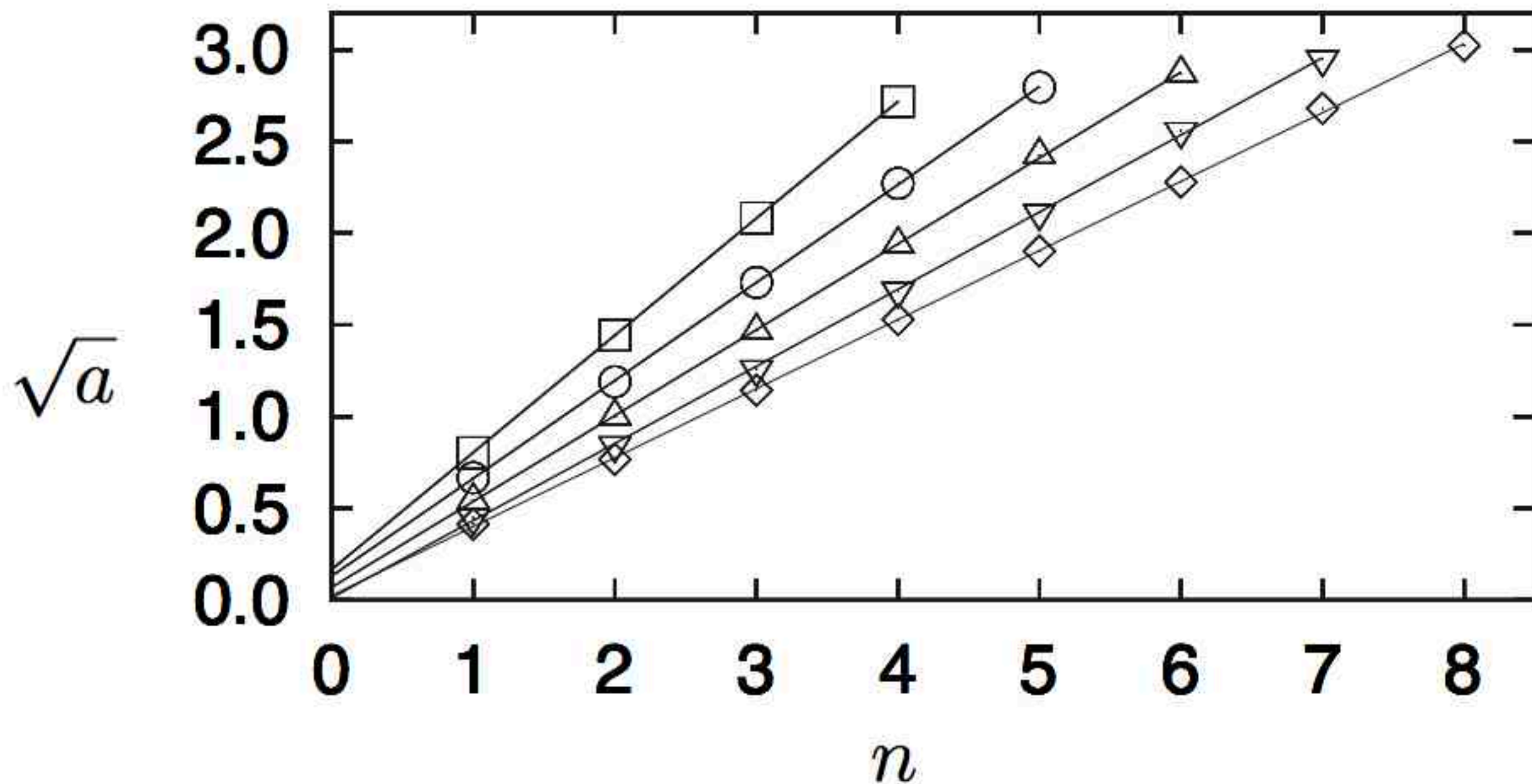


FIG. 3. Average values of  $\sqrt{a}$  versus  $n$  for the 5000 SPS simulations. The lines are straits lines which are least square fitted to the various solutions, depending of the final number (4, 5, 6, 7, 8) of planets.

(J. Laskar, PRL, 2000)

# Solar System

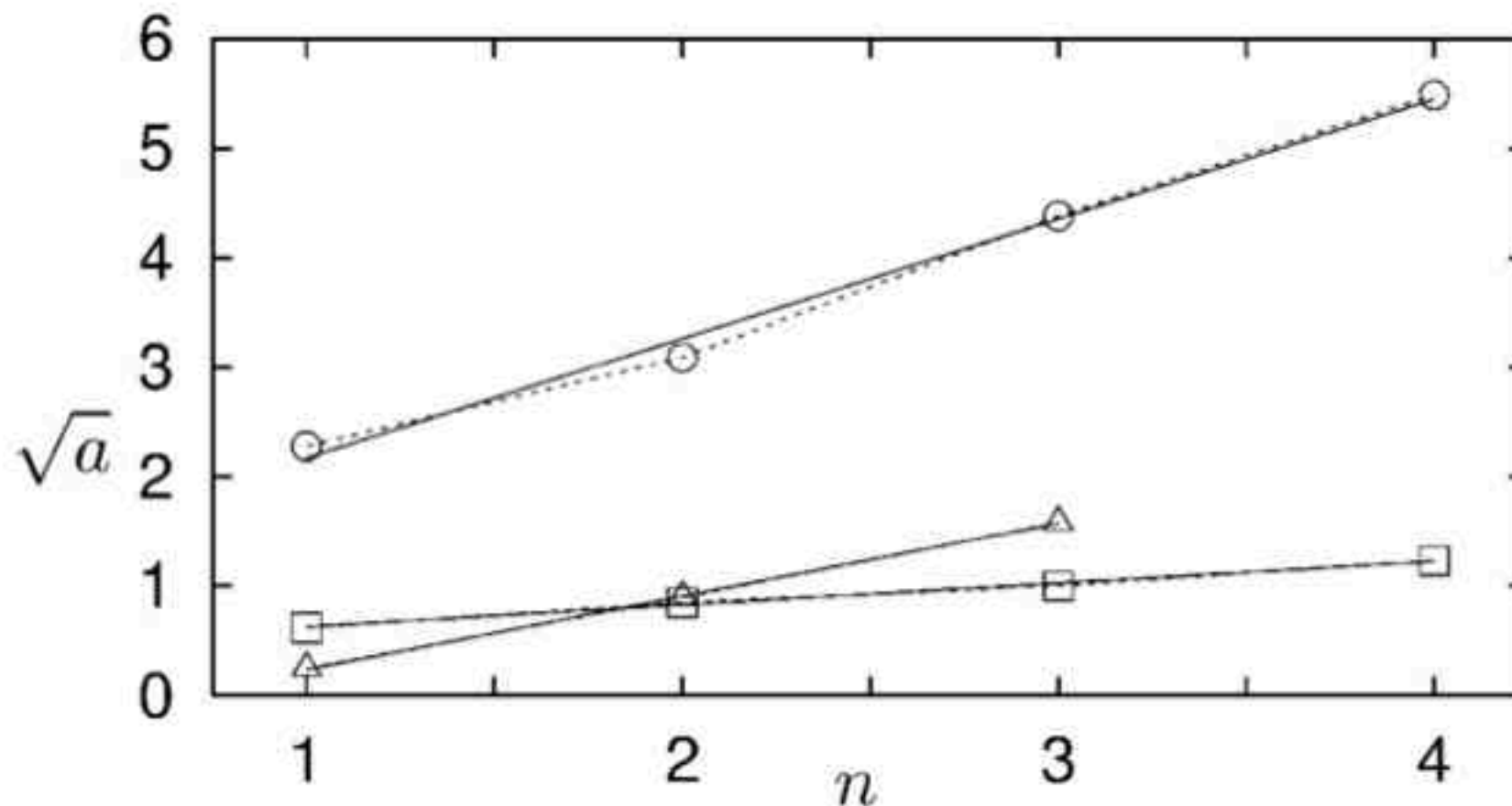


FIG. 4.  $\sqrt{a}$  versus  $n$  for the inner (squares) and outer planets (circles) of the Solar System, and for the known planets of Upsilon Andromedae (triangles).

inner planets	: 0.14	AMD -> 0.20
outer planets	: 0.81	AMD -> 1.09



# Statistical view

(Laskar, Icarus, 2008)

## Numerical experiment

1000 orbits from 0 to -5 Gyr with close initial conditions :

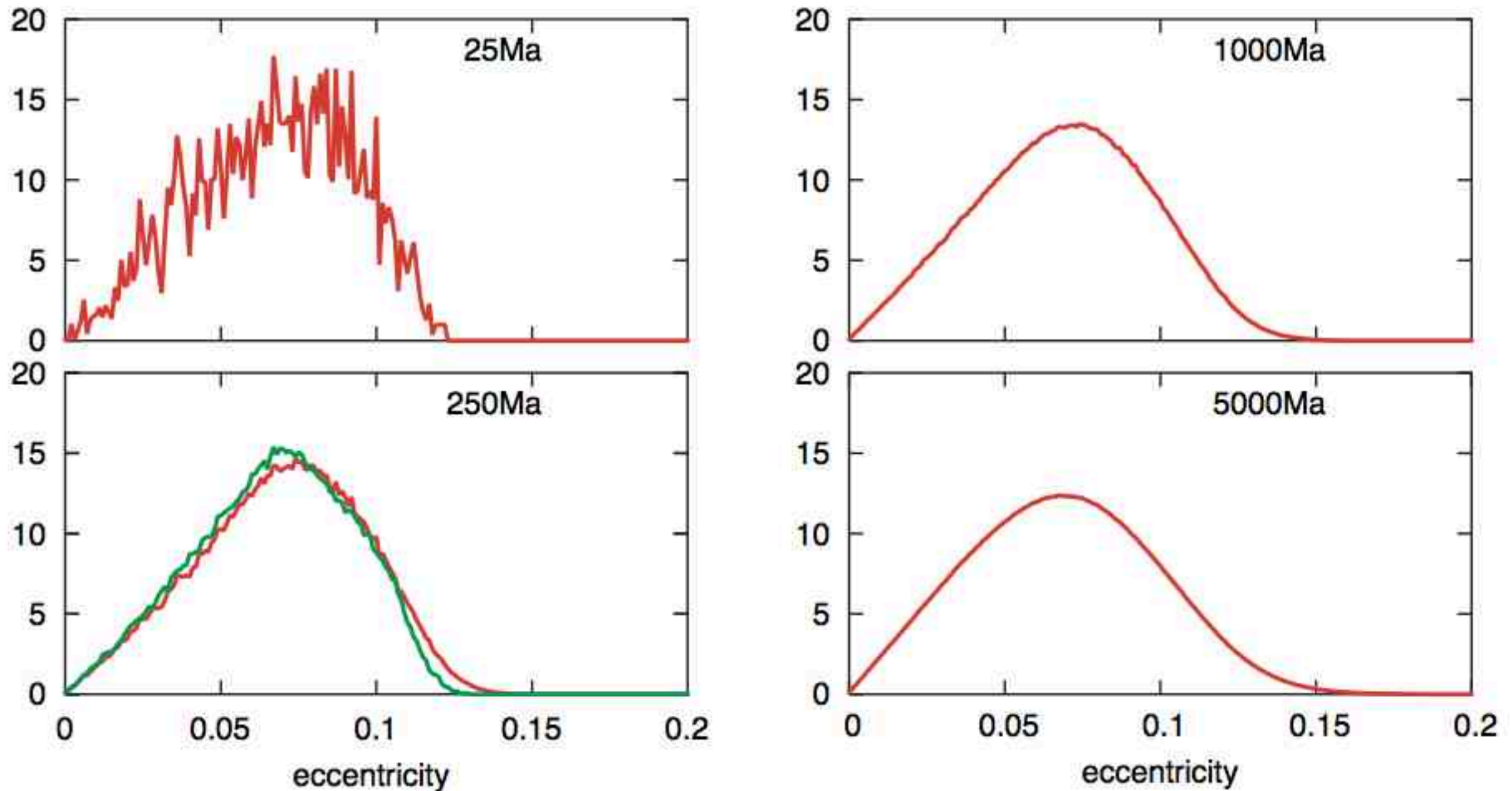
$$-500 \times 10^{-10} \longrightarrow +500 \times 10^{-10}$$

Secular equations (Laskar, 1989,2004)

# Probability Density Functions



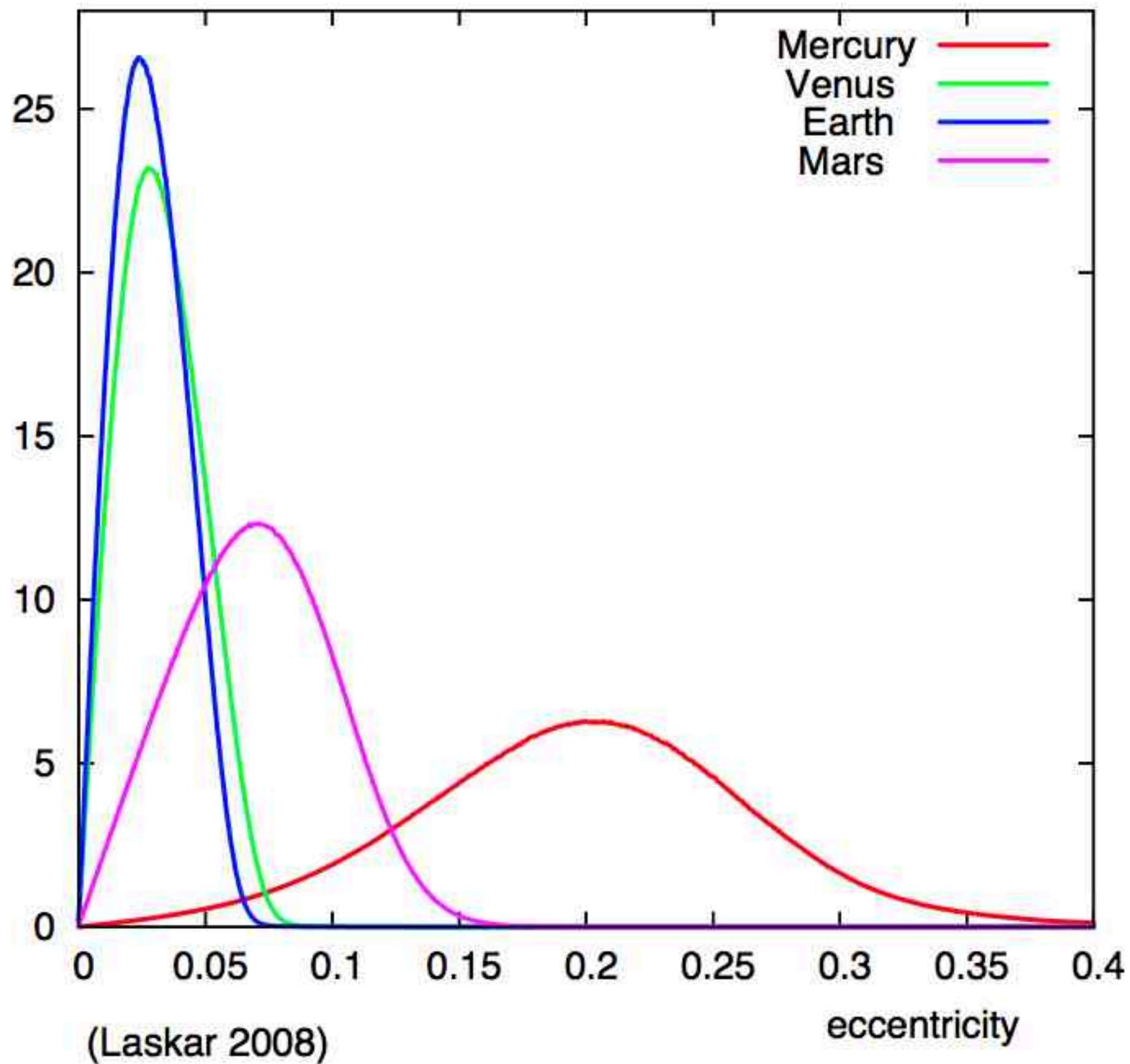
# Diffusion of Mars eccentricity



— 1 solution, non averaged equations  
— 1000 solutions, averaged equations

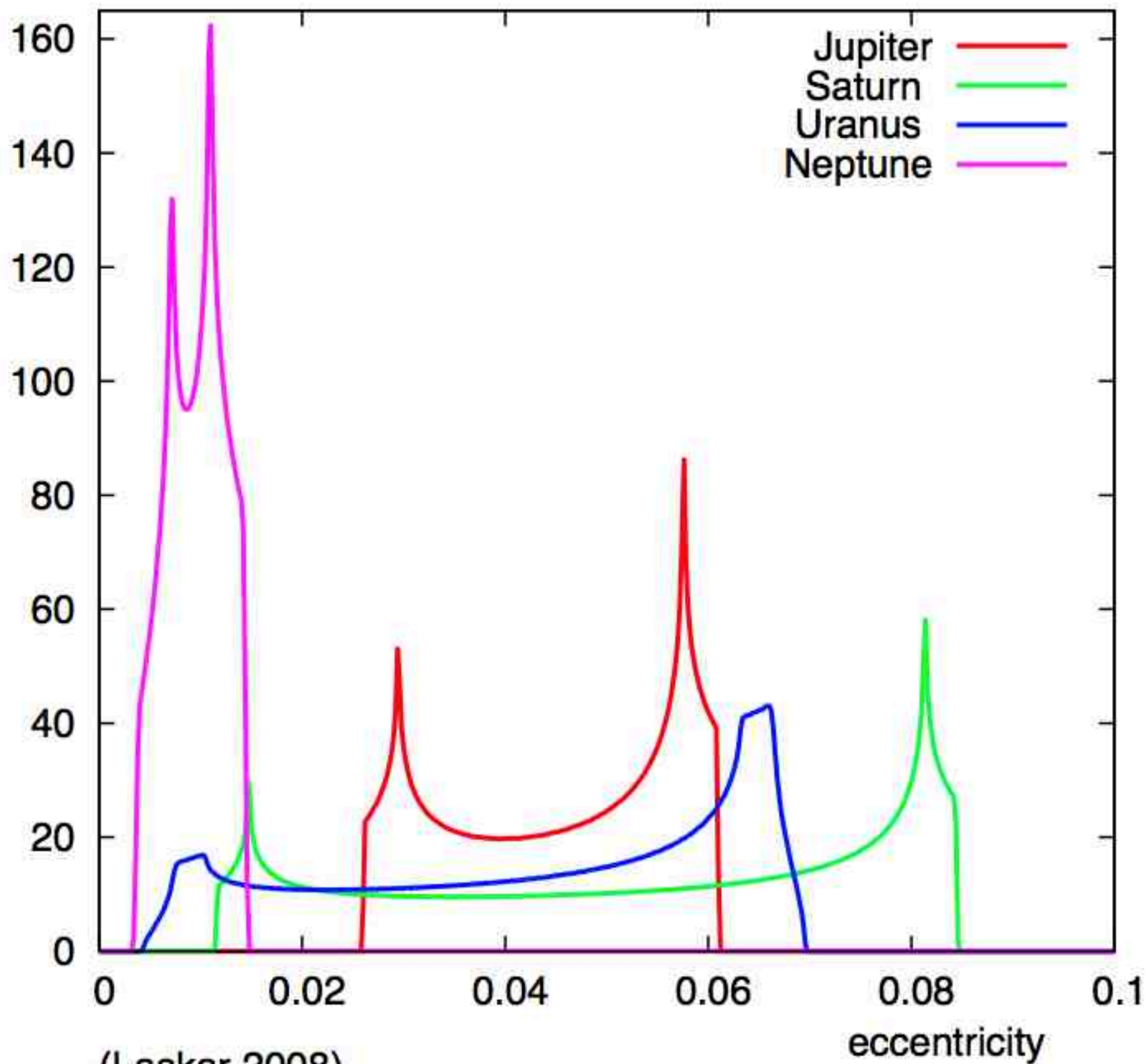
(Laskar et al, 2004)

# density function





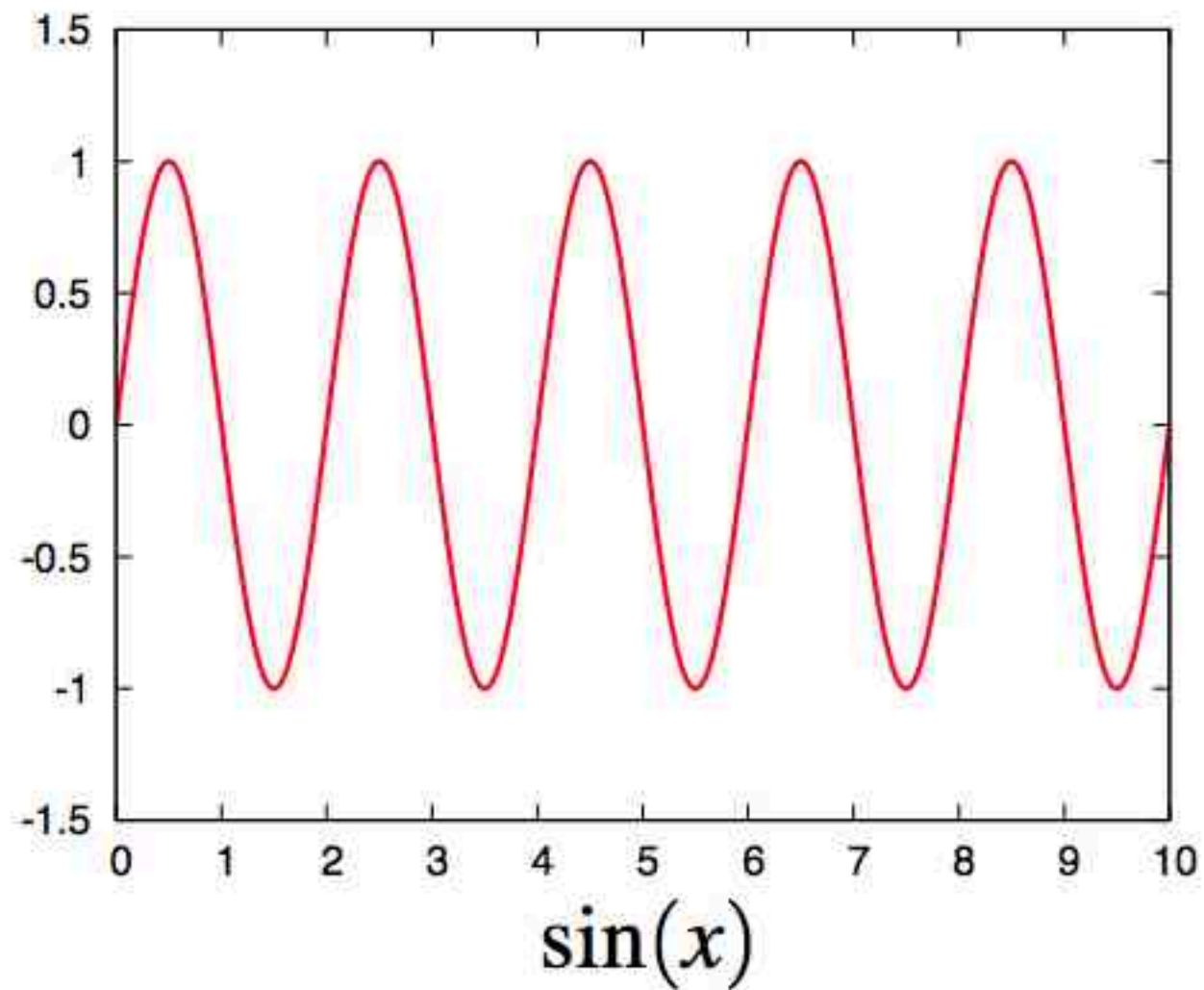
# density function



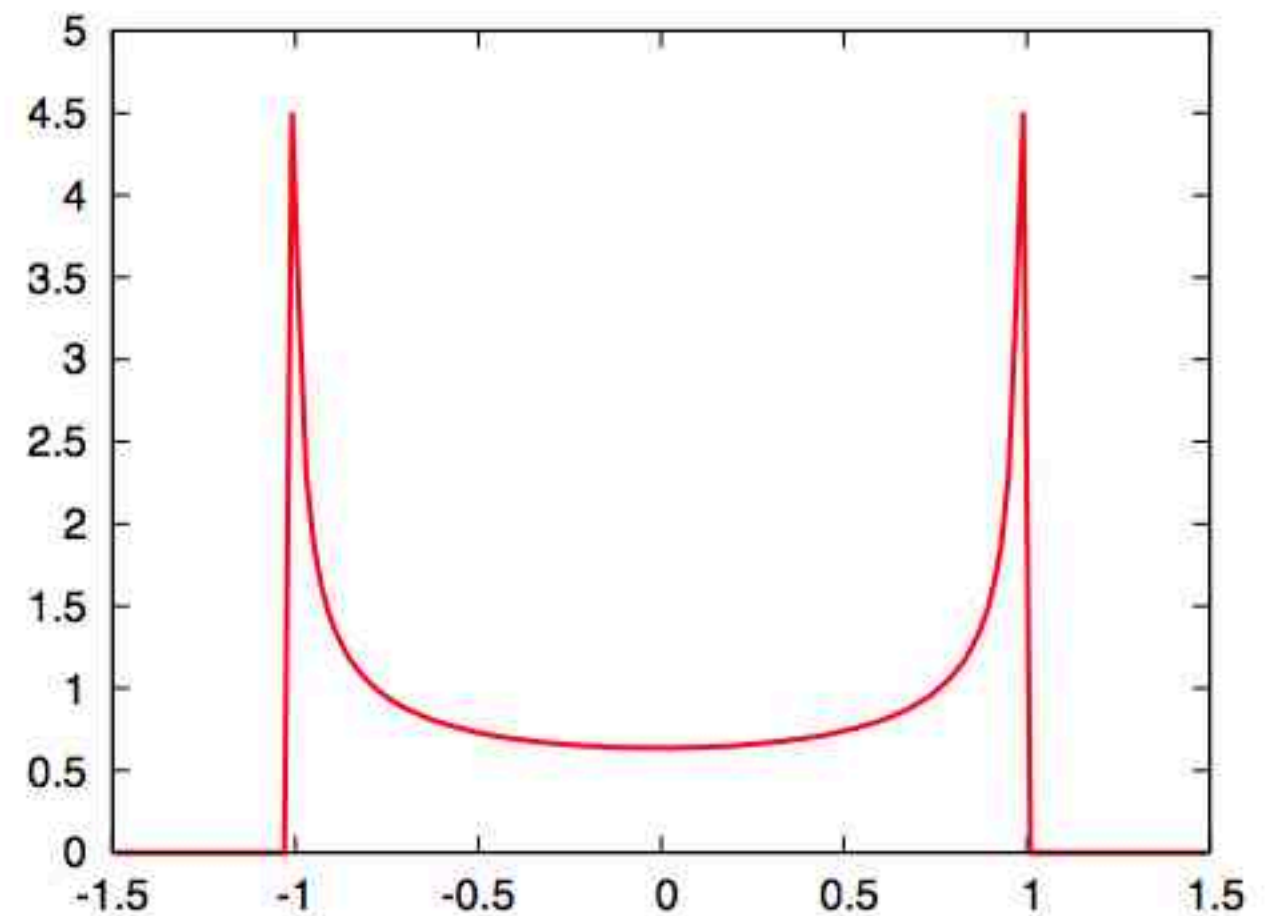
(Laskar 2008)

# Density function of $\sin(x)$

$\sin(x)$

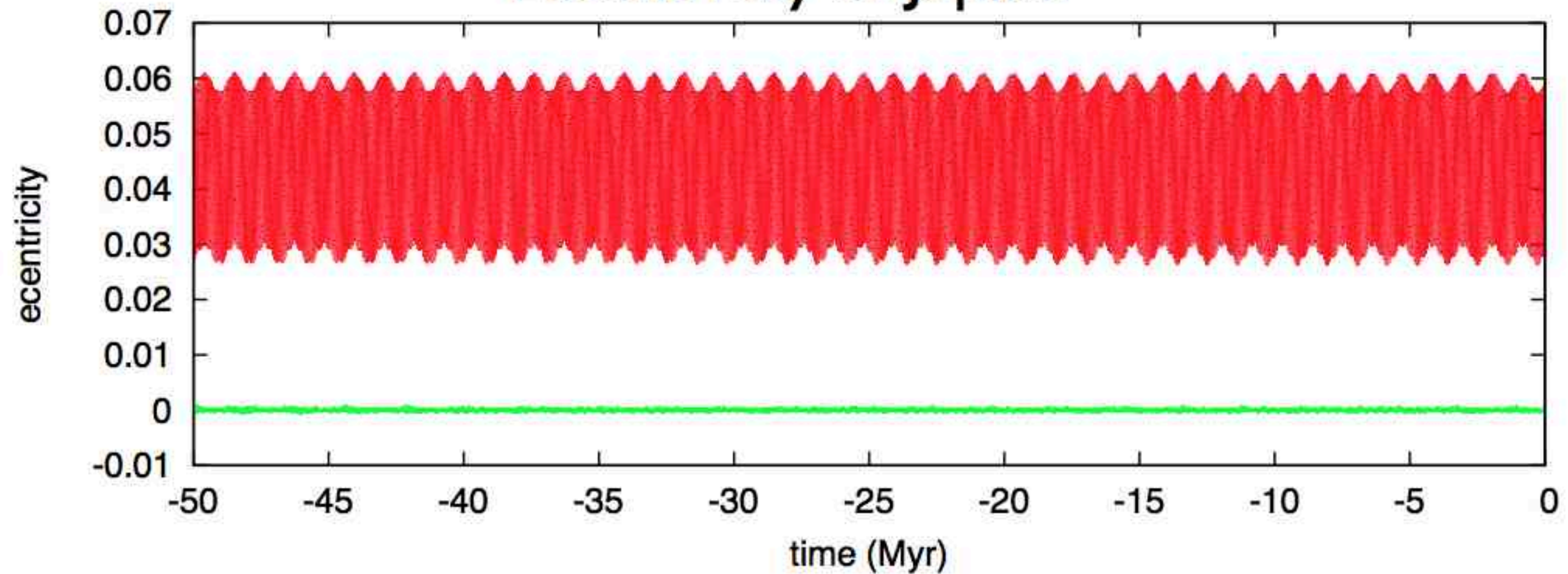


$$\frac{1}{\pi\sqrt{1-x^2}}$$





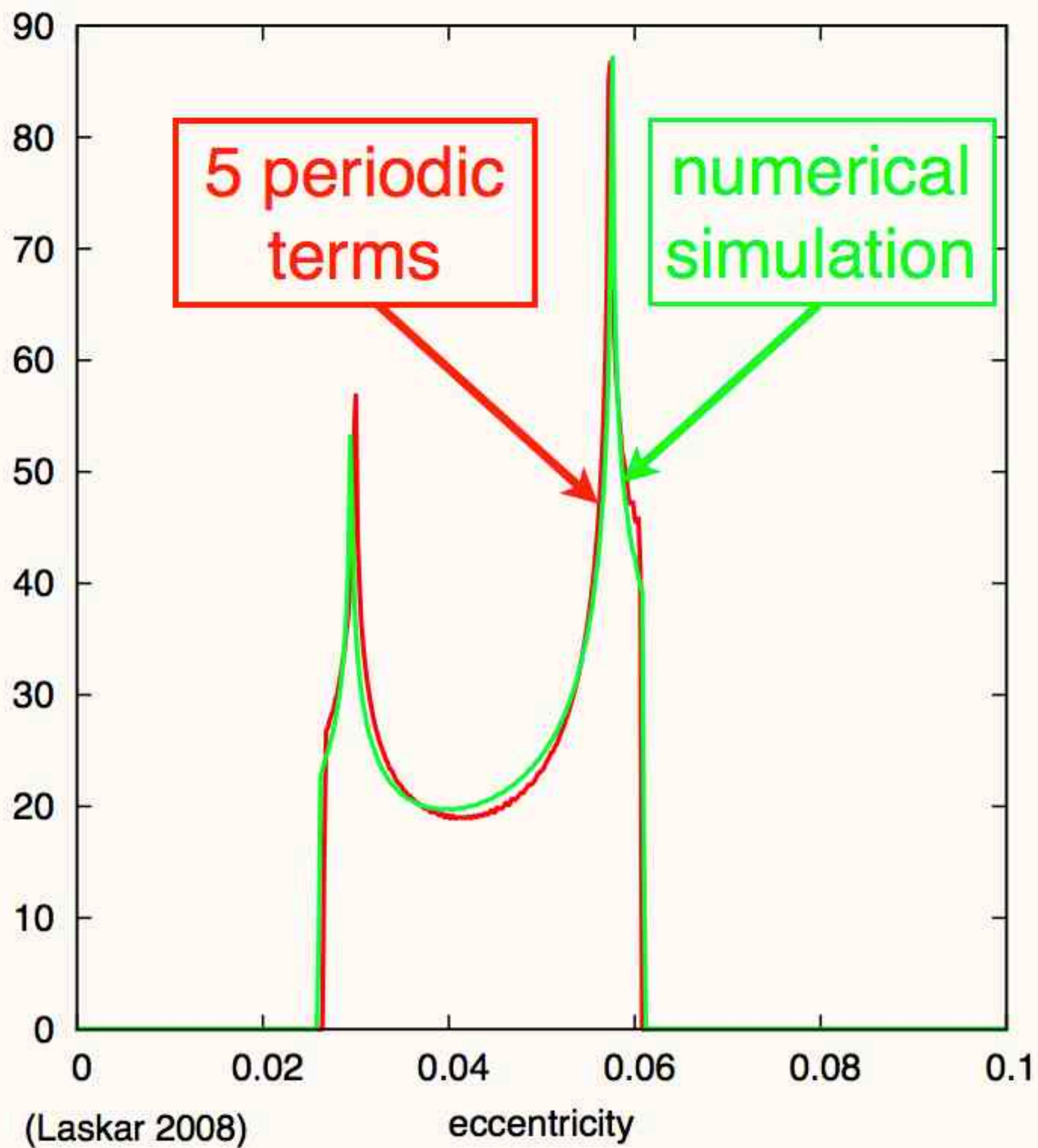
# Eccentricity of Jupiter



$$e = e_0 + \sum_{i=1}^5 a_i \cos(\nu_i t + \phi_i)$$

(Laskar 2008)

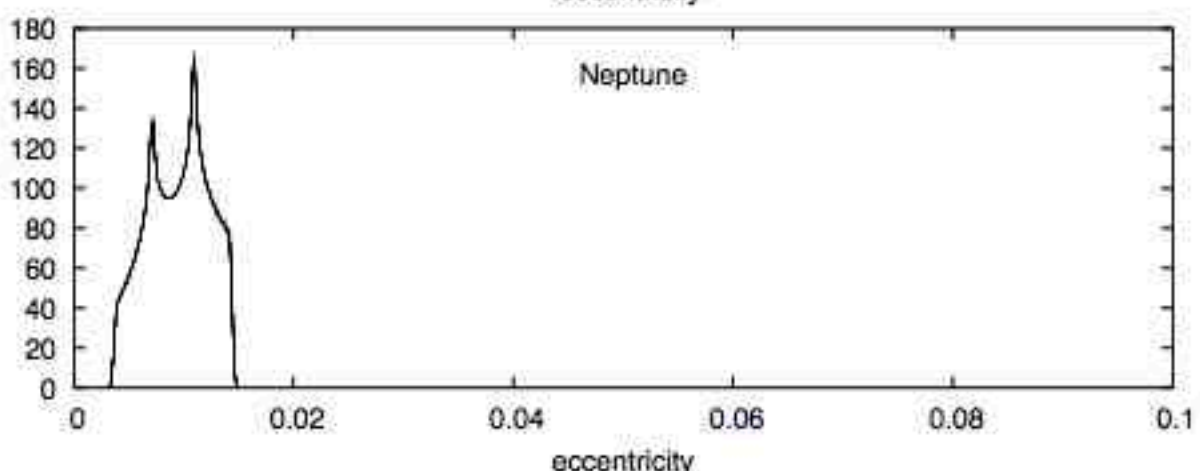
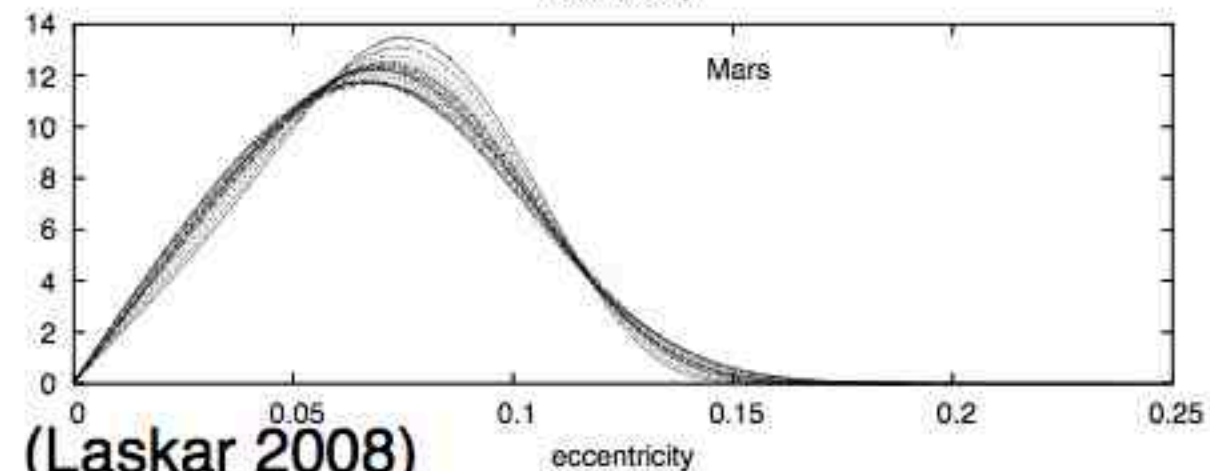
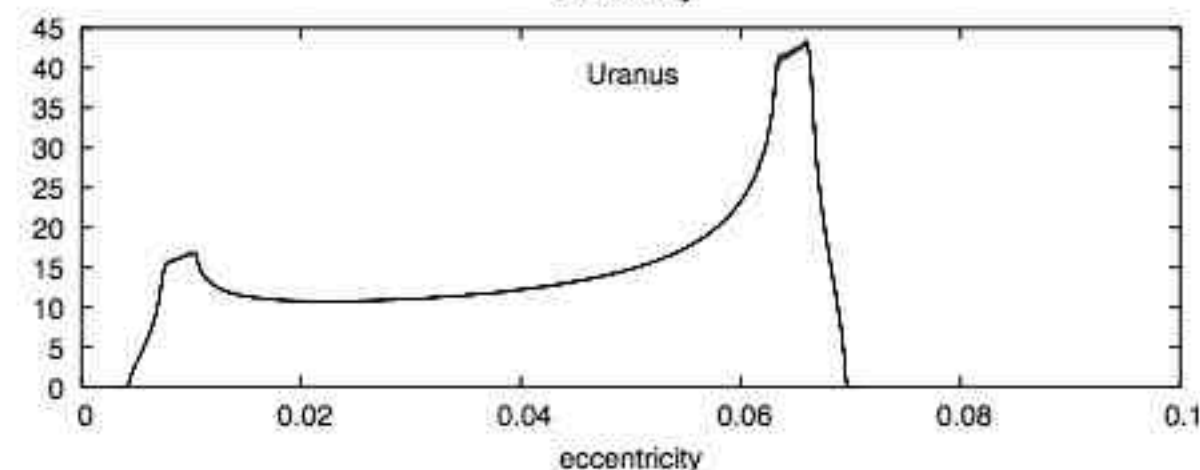
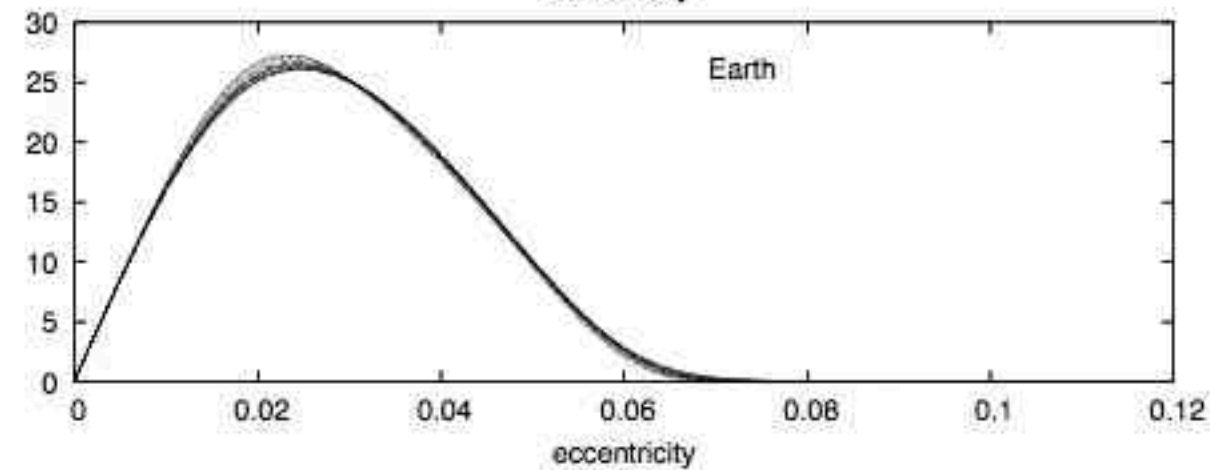
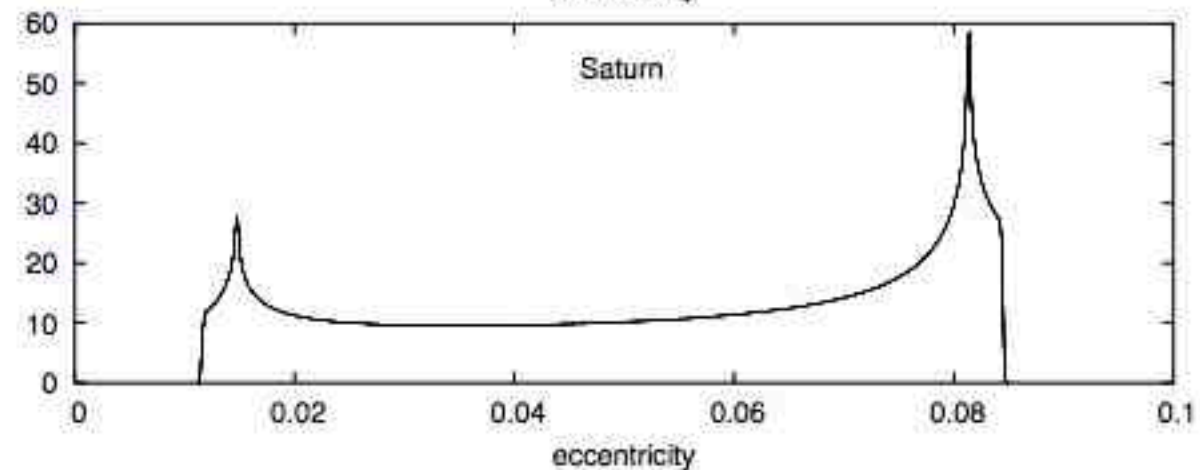
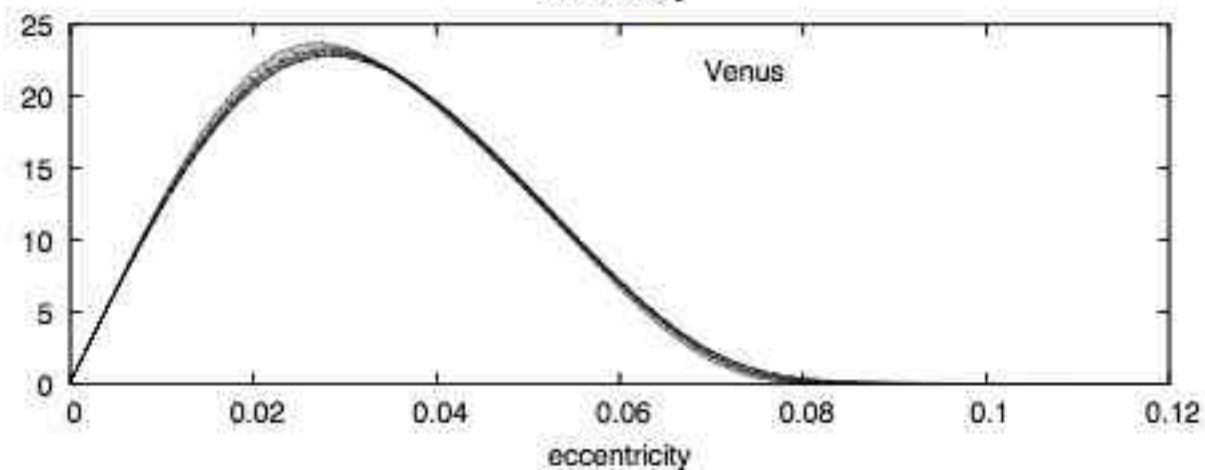
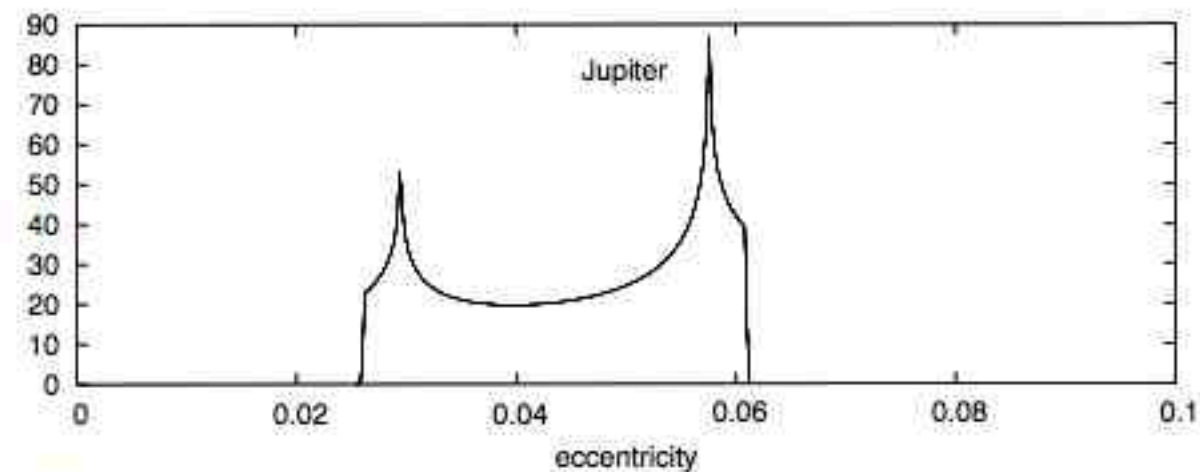
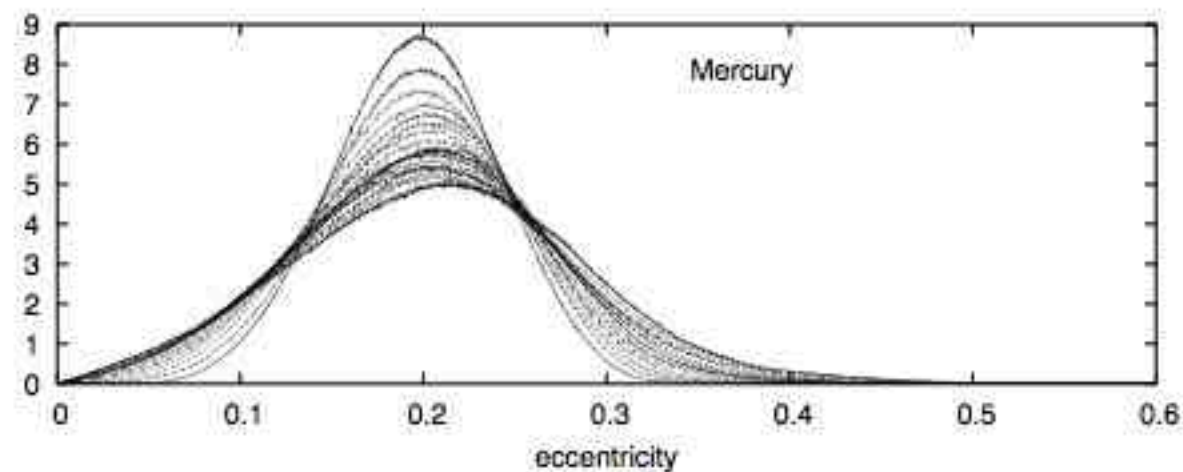
$\nu_i$	$a_i$
$g_6 - g_5$	0.015371
$2g_6 - 2g_5$	0.001848
$g_5 - g_7$	0.001679
$g_6 - g_7$	0.000502
$g_6 + g_7 - 2g_5$	0.000491





# Chaotic diffusion

# eccentricity : 20 slices of 250 Myr



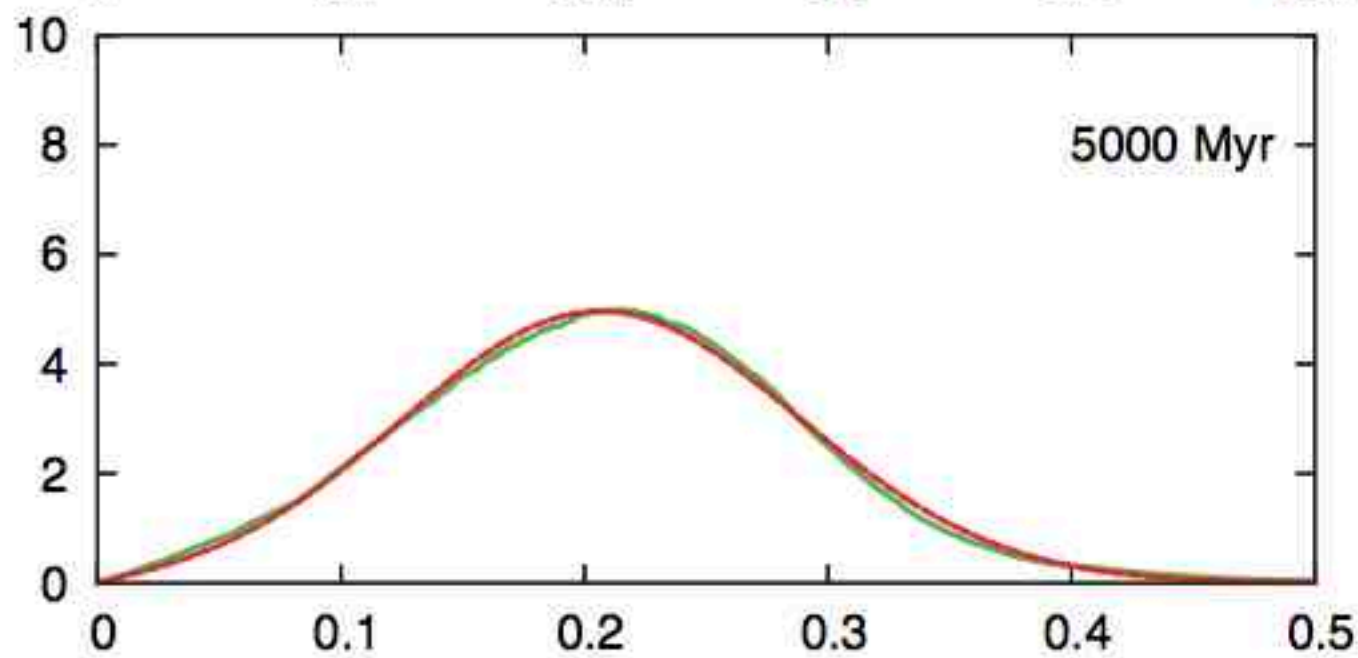
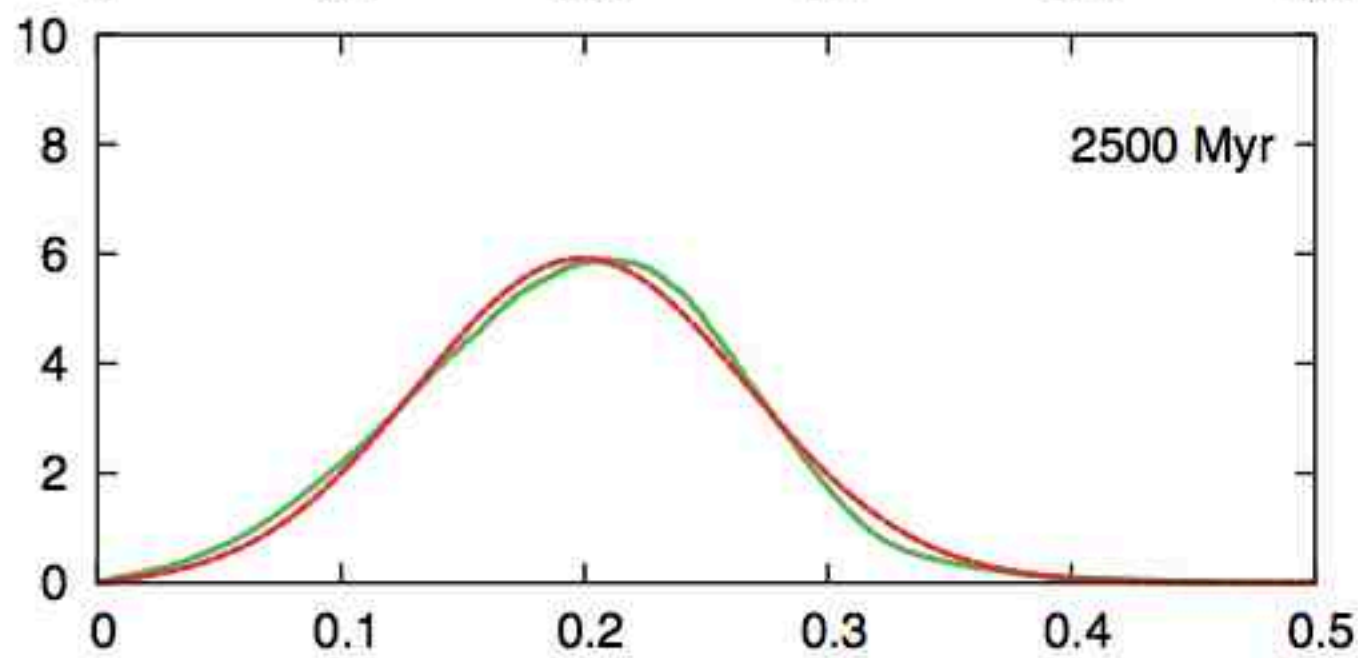
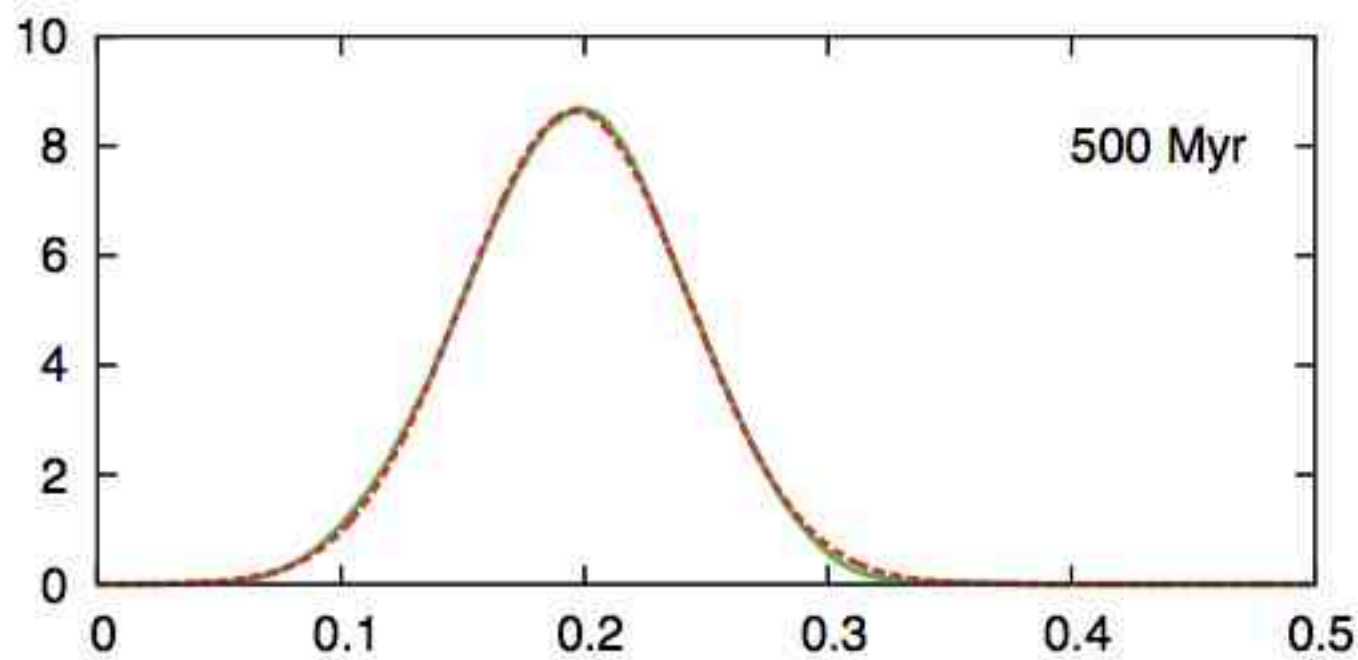
(Laskar 2008)



# Mercury

Rice density

$$f_{\sigma,m}(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2 + m^2}{2\sigma^2}\right) I_0\left(\frac{xm}{\sigma^2}\right)$$



eccentricity

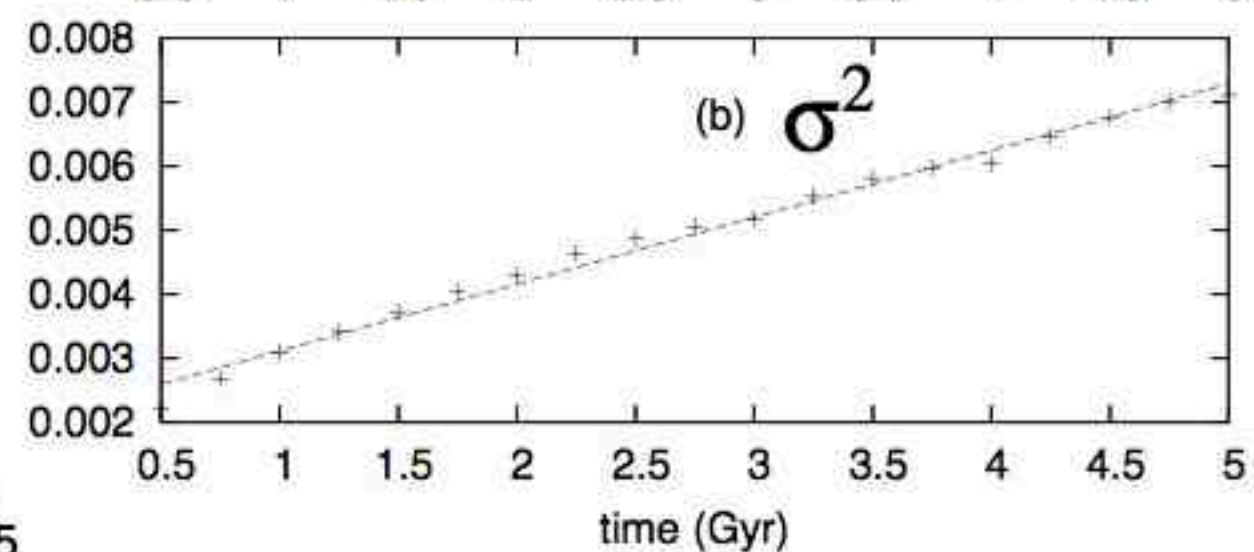
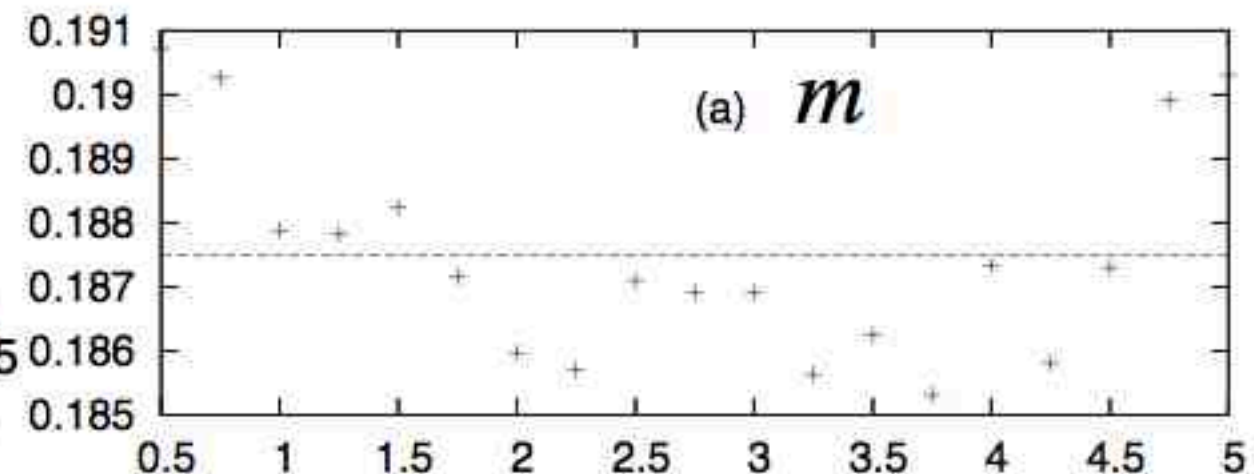
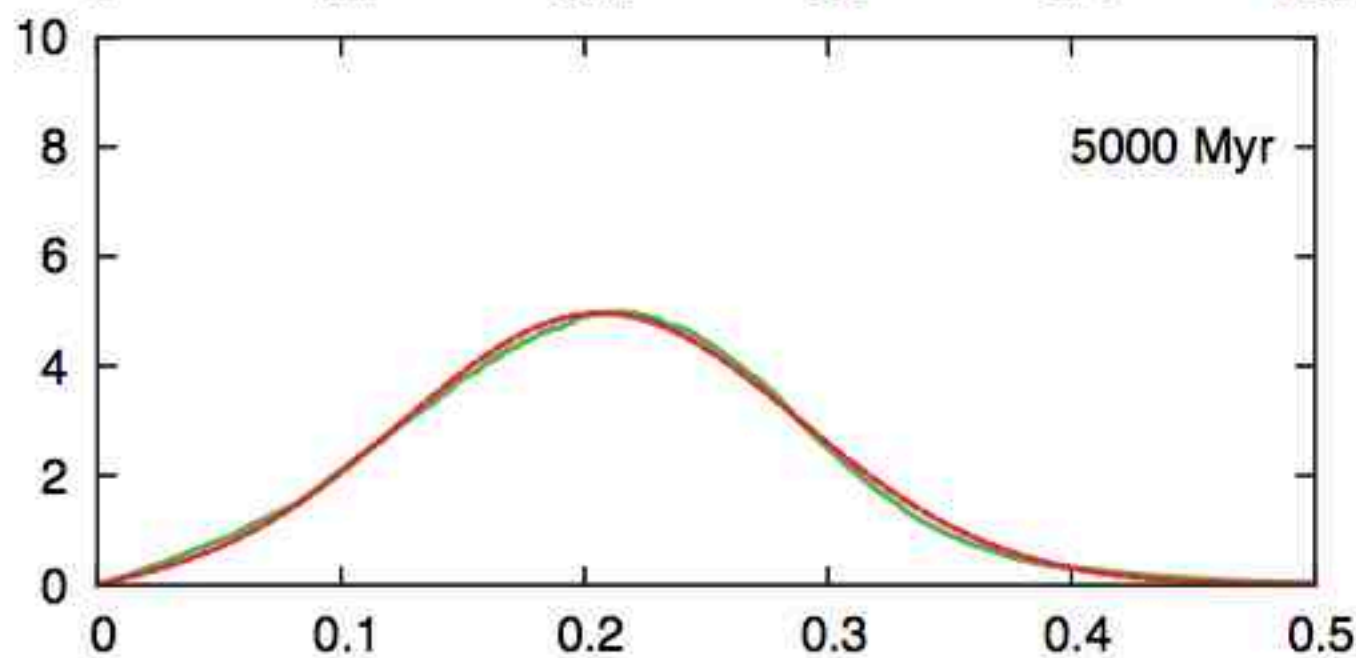
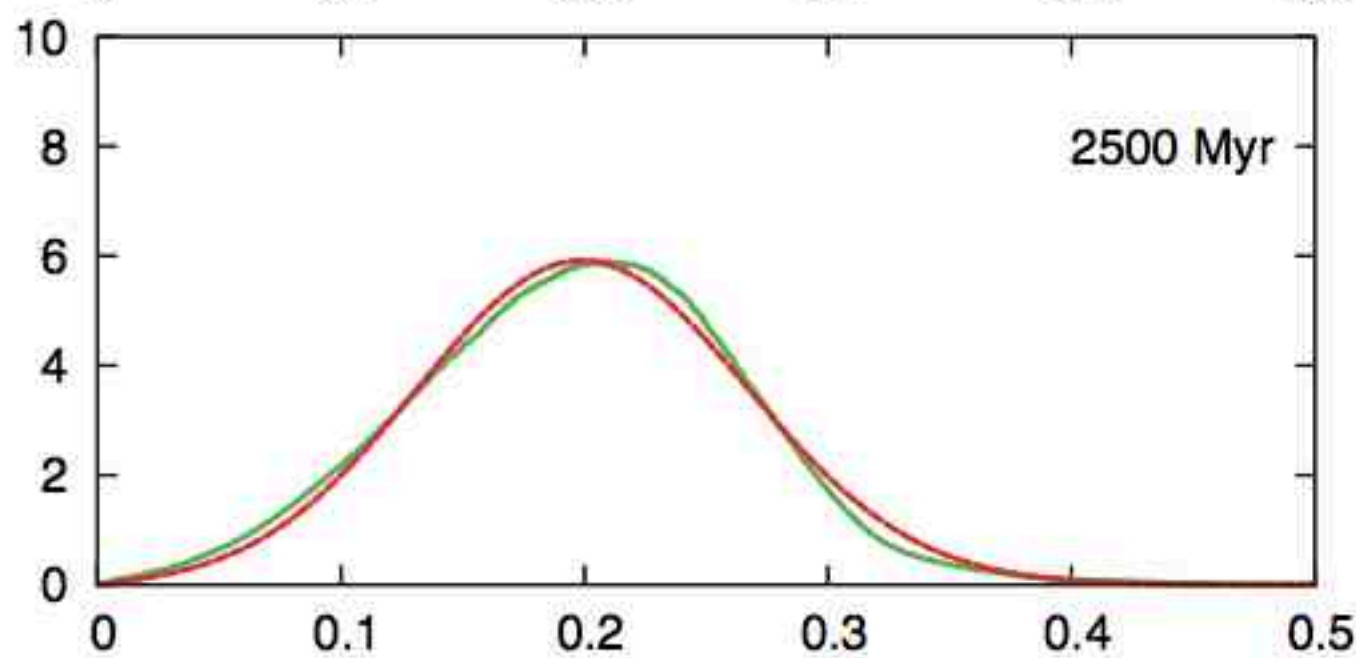
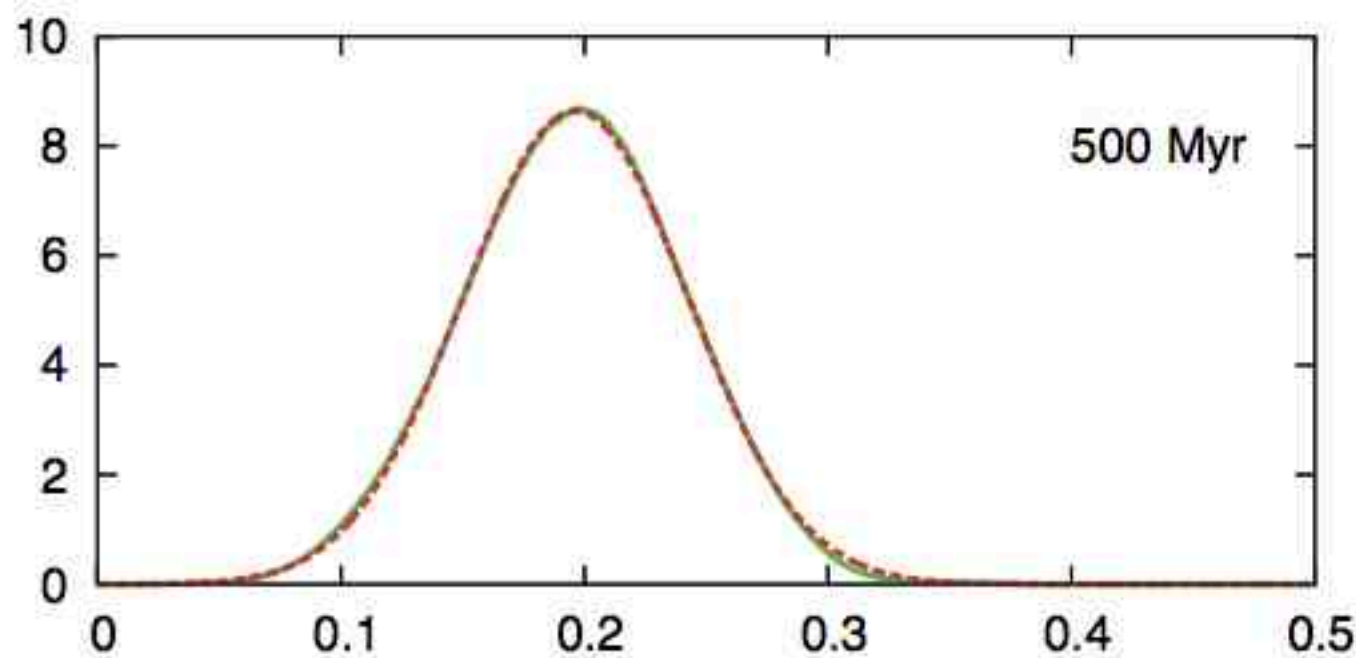
# Mercury

Rice density

$$f_{\sigma,m}(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2 + m^2}{2\sigma^2}\right) I_0\left(\frac{xm}{\sigma^2}\right)$$

$$m = 0.1875$$

$$\sigma^2 = 0.00207 + 0.00104 T$$





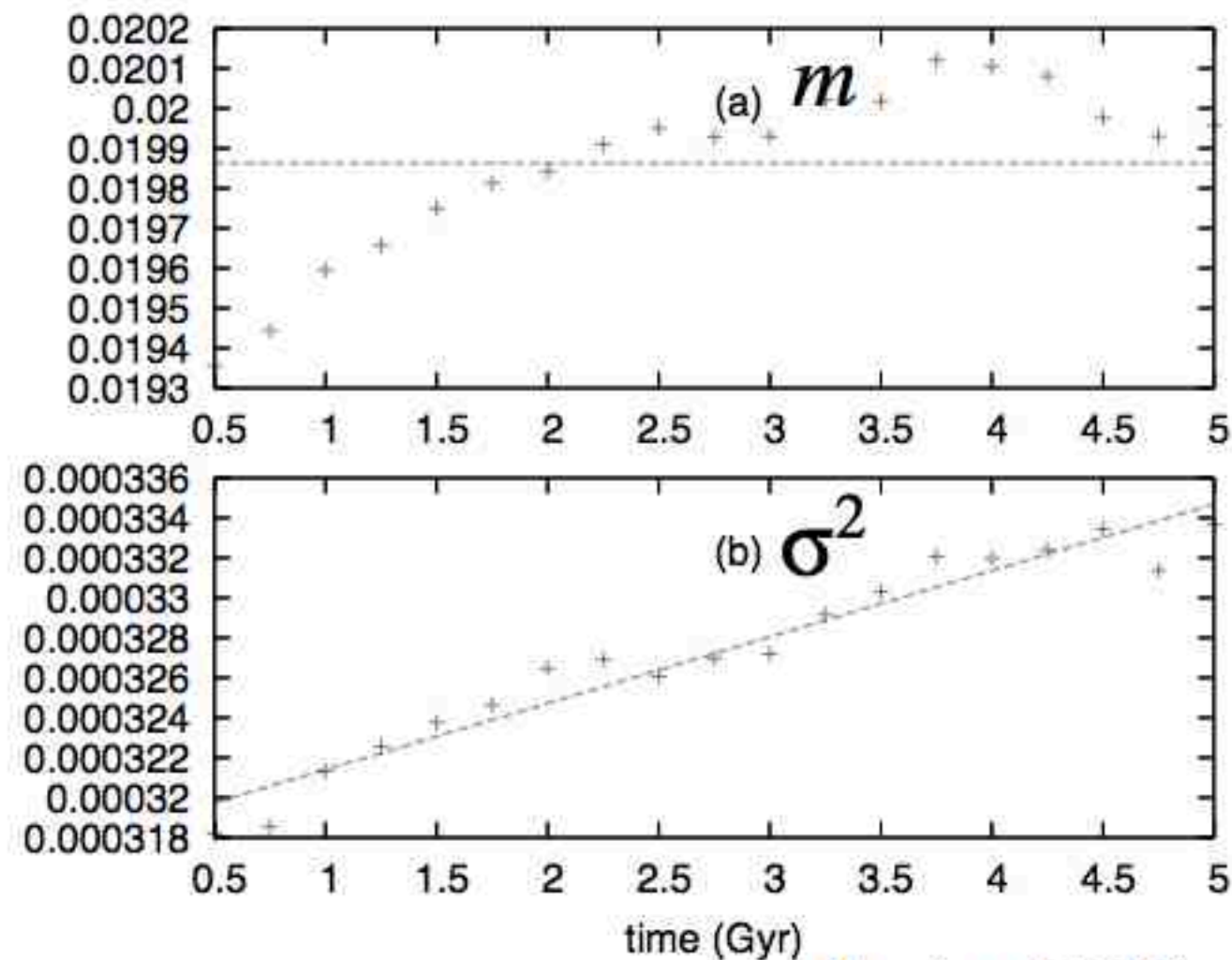
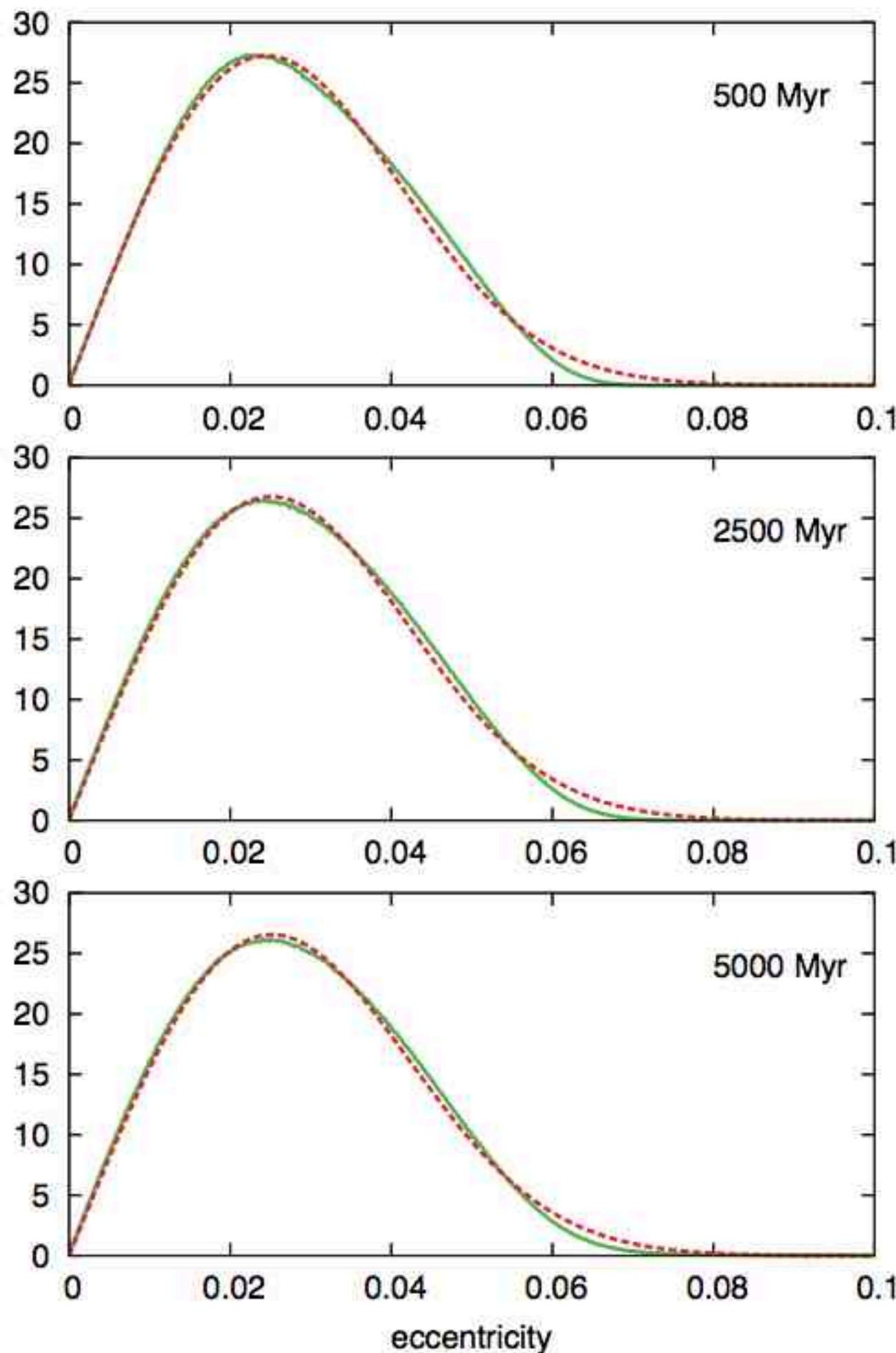
# The Earth

Rice density

$$f_{\sigma,m}(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2 + m^2}{2\sigma^2}\right) I_0\left(\frac{xm}{\sigma^2}\right)$$

$$m = 0.1875$$

$$\sigma^2 = 0.00032 + 0.0000006 T$$



(Laskar 2008)

Système complet

8 Planètes + Pluton

Relativité générale

Contribution de la Lune

Recherche d'une faible probabilité  $\sim 1\%$

2500 solutions sur 5 Ga

CI : diff. de 0.38 mm dans le grand axe de Mercure

6-7 millions d'heures de calcul

*(Laskar & Gastineau, Nature, 2009)*



# JADE



**1536 nœuds Intel E5472 : 12288 cœurs**

**147 Tflop/s**

**14ème du TOP500 (nov. 2008)**

**Période de test: Août-décembre 2008**

# Eccentricité de Mercure **AVEC** relativité

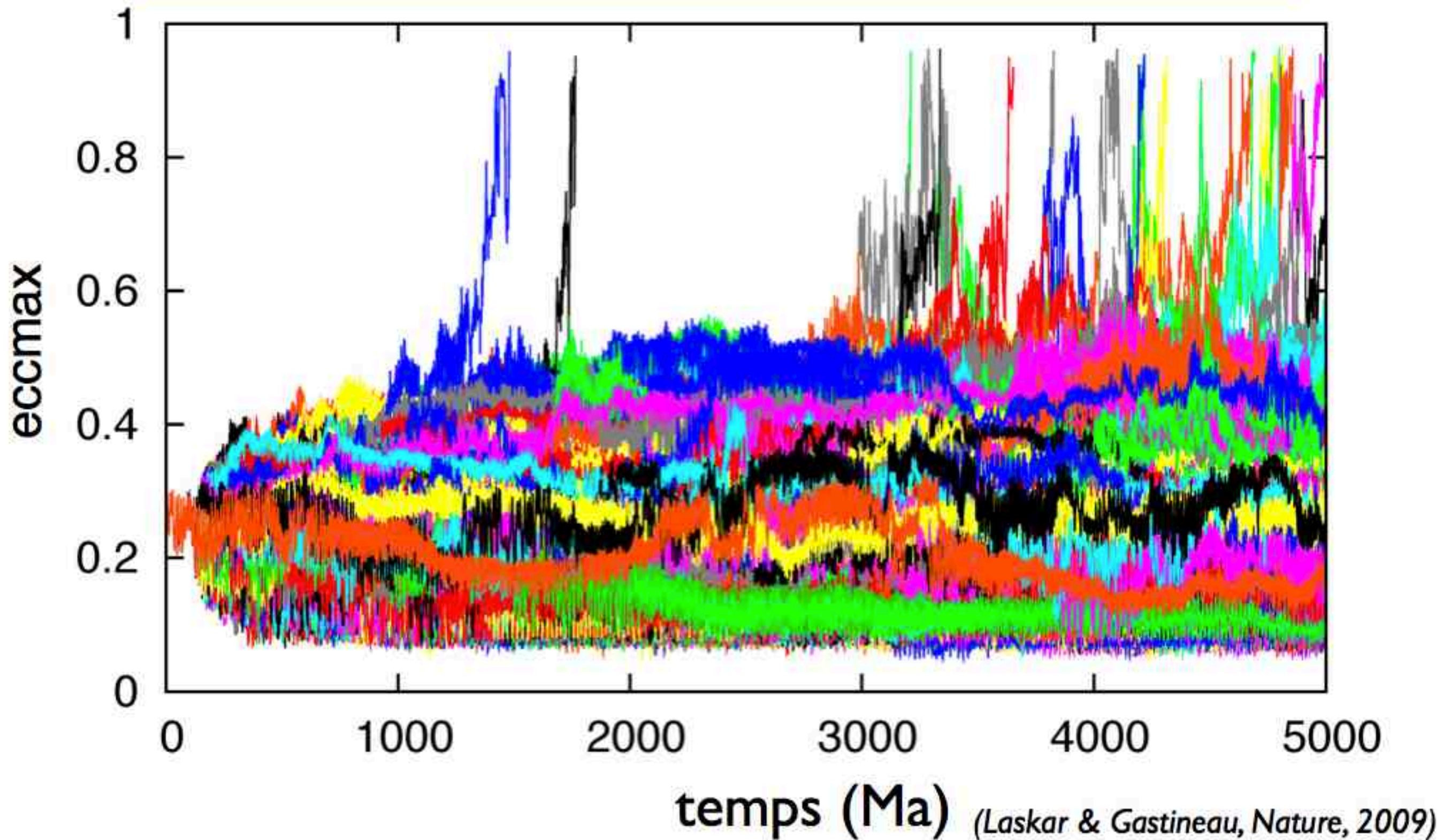
## équations directes (2501 sol)

### ramené à 1000

	500	1000	2000	3000	4000	5000
0.35	30	91	202	318	418	492
0.4	3	20	67	126	189	255
0.5	-	-	3	10	20	40
0.6	-	-	1	2	5	10
0.7	-	-	1	1	4	9
0.8	-	-	1	1	4	8
0.9	-	-	1	1	3	8



Eccentricité max de Mercure **AVEC** relativité  
équations directes (2501 sol)  
(0.38 mm)





## 21 solutions avec une excentricité $> 0.9$

6	Collision de Mercure avec Vénus
9	Collision de Mercure avec le Soleil
5	Atteignent 5 Ga sans collision (M-V 1800 km)
1	rencontre proche Mars-Terre (794km) puis collision de Mars avec le Soleil

# 201 solutions

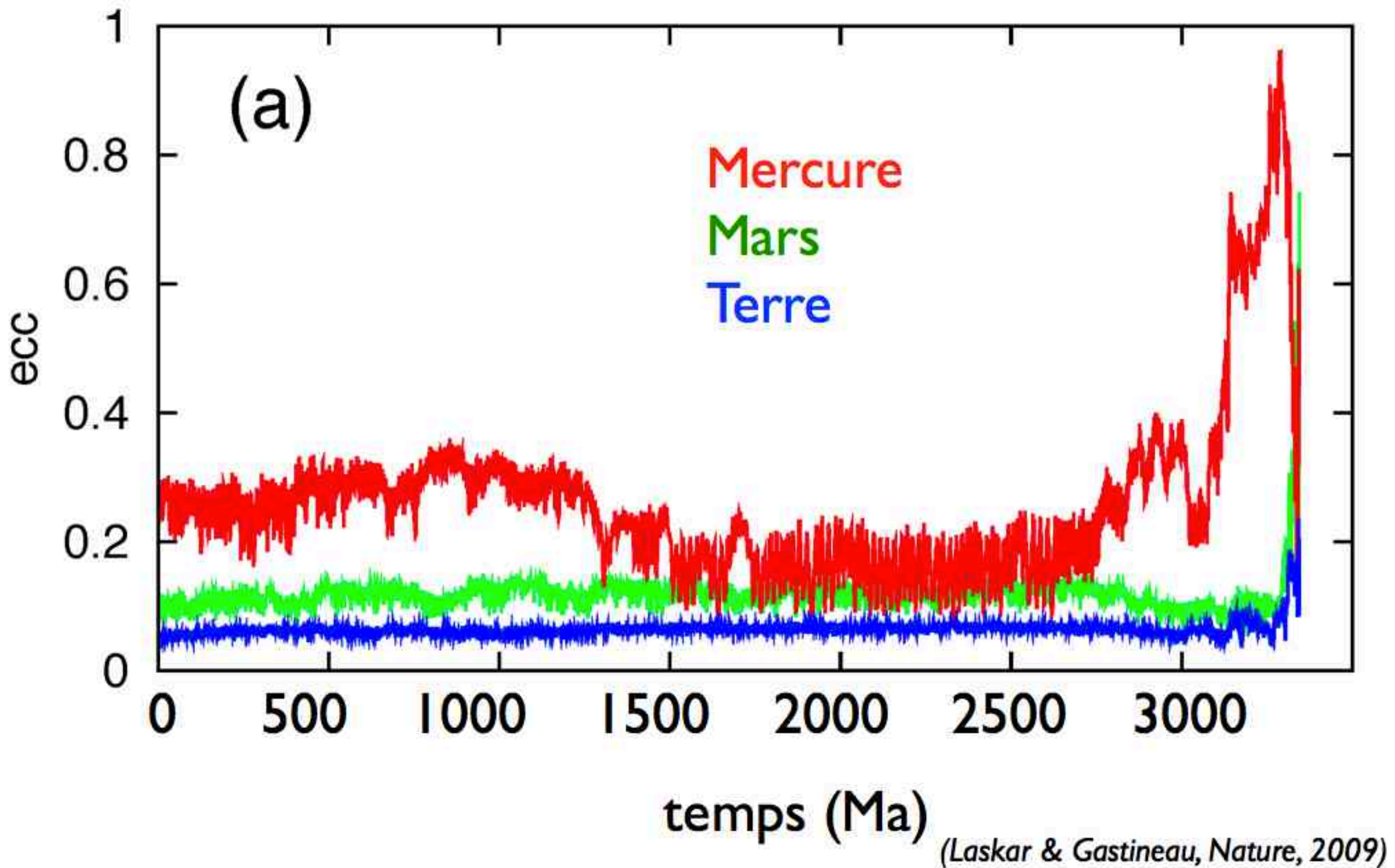
démarrées à 3.344298 Ga

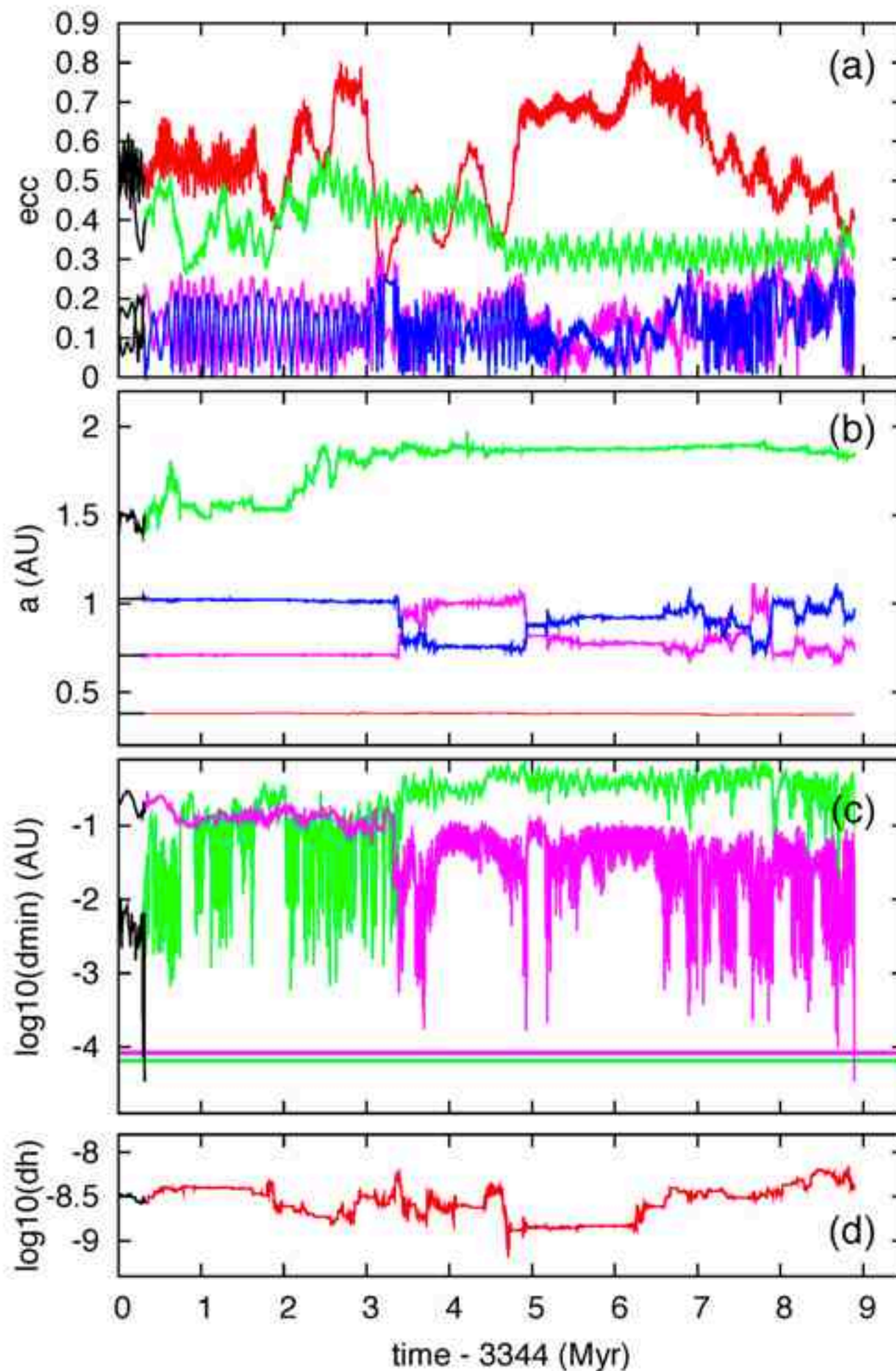
diff. de 0.15 mm dans le grand axe de Mars

33	Soleil-Mercure
48	Soleil-Mars
43	Mercure-Vénus
1	Mercure-Terre
1	Mercure-Mars
18	Vénus-Terre
23	Vénus-Mars
29	Terre-Mars
5	ejection de Mars



# Excentricité Max





Eccentricité.

**Mercure**, **Venus**, **Terre**,  
**Mars**.

Demi-grand axe.

**Mercure**, **Venus**, **Terre**,  
**Mars**.

distances.

**Venus-Terre**,  
**Mars-Terre**.

Conservation de l'énergie

(Laskar & Gastineau, Nature, 2009)