# Second Species Orbits for the 3-body problem and for 2 charges in a magnetic field

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# Outline

1. Poincaré’s proposal
2. Circular restricted 3-body problem
3. Planar case
4. Non-planar case
5. 2 charges in magnetic field
6. Conclusion

# 1. Three-body Problem (3BP)

Motion of three bodies:

- "Sun", mass M
- "Jupiter", mass $\varepsilon$
- "Another", mass $\mu$

under their gravitational attraction:

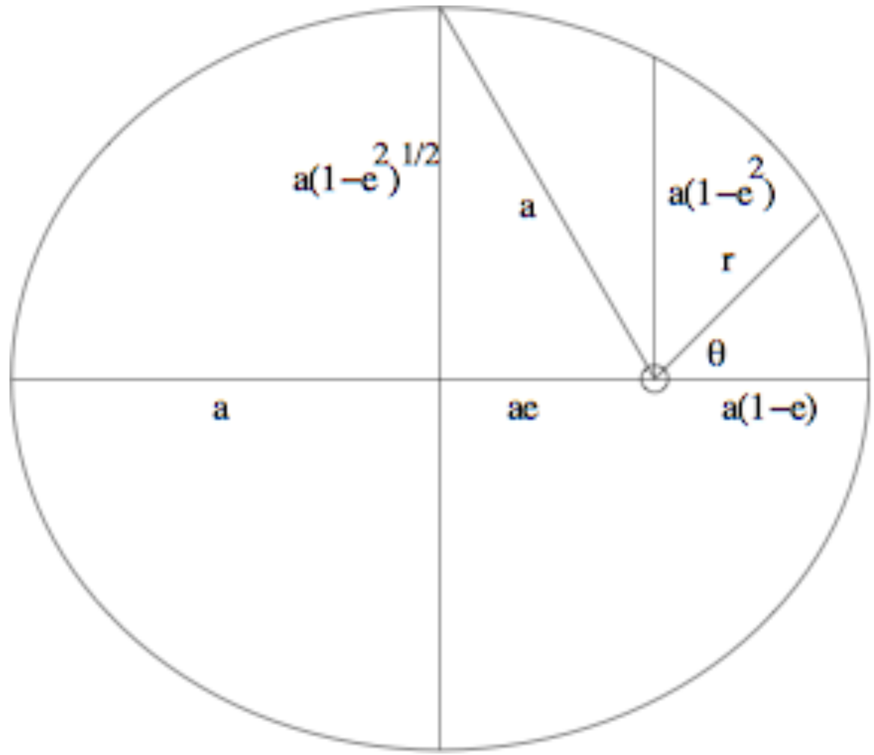
$$q_i'' = \sum_j m_j(q_j-q_i)/|q_j-q_i|^3, \quad q_i \text{ in } R^3$$

# Two-body problem

- The two-body problem was solved by Newton: Kepler orbits about centre of mass, e.g. Kepler ellipses

$r = a(1-e^2)/(1+e \cos \theta)$,

frequency $\Omega = a^{-3/2}$



# Periodic orbits

- Poincaré considered periodic orbits the key to understanding the 3BP.
- For $\varepsilon$, $\mu$ small, he proved existence of many, close to pairs of Kepler ellipses around S with rational frequency ratio.
- But he also proposed another type of periodic orbit in the 3BP.

# Second species orbits

Poincaré, 1892: “Les solutions périodiques dont il a été question jusqu’ici ne sont pas les seules dont il soit possible de démontrer l’existence. Ainsi le problème des trois Corps comporte des solutions périodiques de la nature suivante: les deux petits corps décrivent autour du grand des orbites très peu différentes de deux ellipses képlériennes E et E’; à un certain moment, ces deux petits corps passent très près l’un de l’autre et exercent l’un sur l’autre des perturbations considérables; puis ils s’éloignent de nouveau et décrivent alors des orbites qui se rapprochent beaucoup de deux nouvelles ellipses képlériennes $E_1$ et $E'_1$, très différentes de E et de E’. Les deux petits corps s’écartent très peu des ellipses $E_1$ et $E'_1$ jusqu’à ce qu’ils se trouvent encore une fois très près l’un de l’autre…”

# Poincaré, Ch 32, vol 3,1899

Pairs of segments of Kepler orbit joining at collisions at each end are either:

- Whole numbers of revolutions to the same point,
- Coplanar between distinct points, or
- Non-coplanar joining points on opposite sides of a line through S.

He claimed he could continue any periodic concatenation of such to small $\varepsilon,\mu$, subject only to conservation of energy and angular momentum at each collision.

# Numerics, e.g. Arenstorf

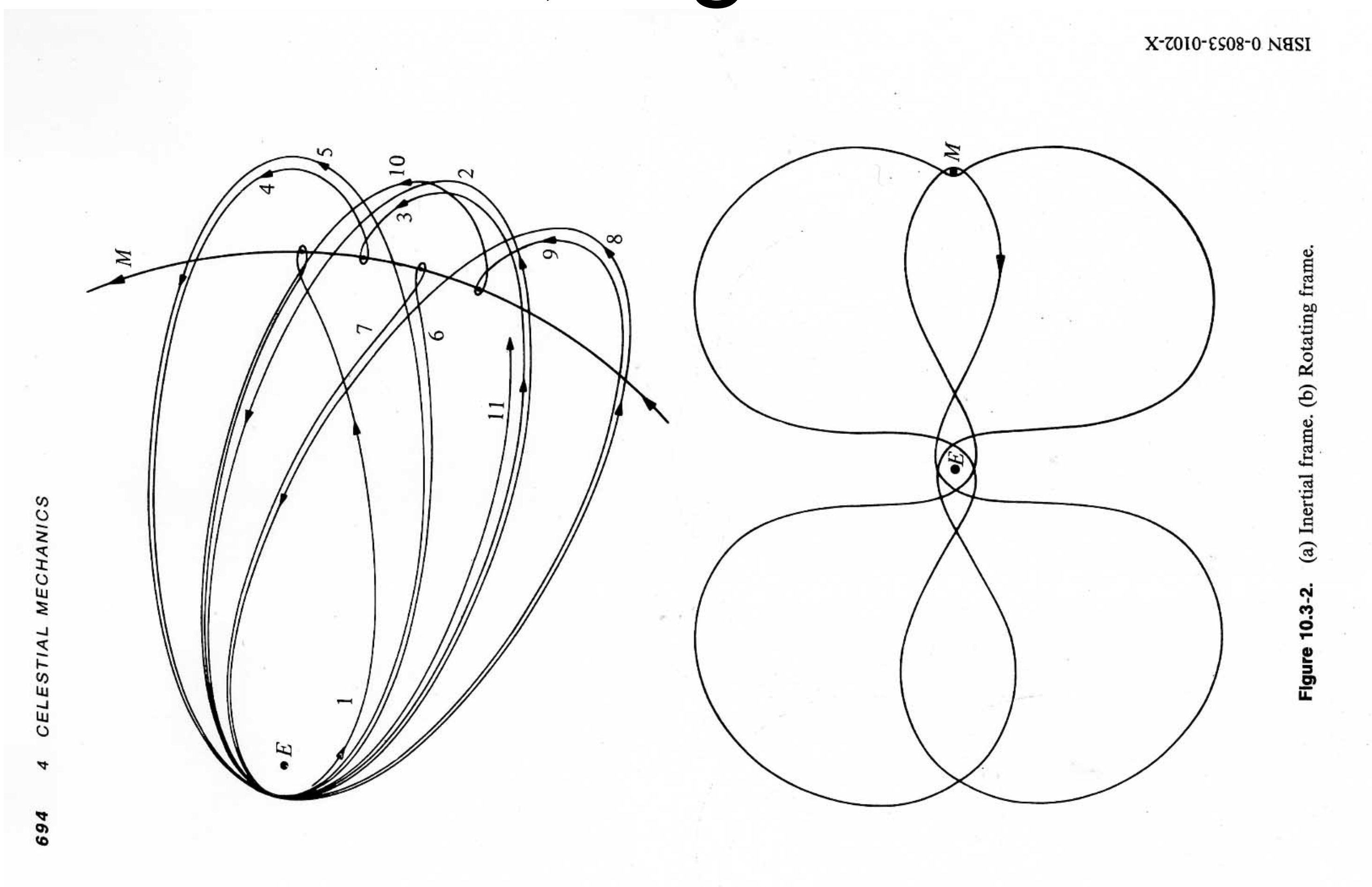

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Figure 10.3-2. (a) Inertial frame. (b) Rotating frame.

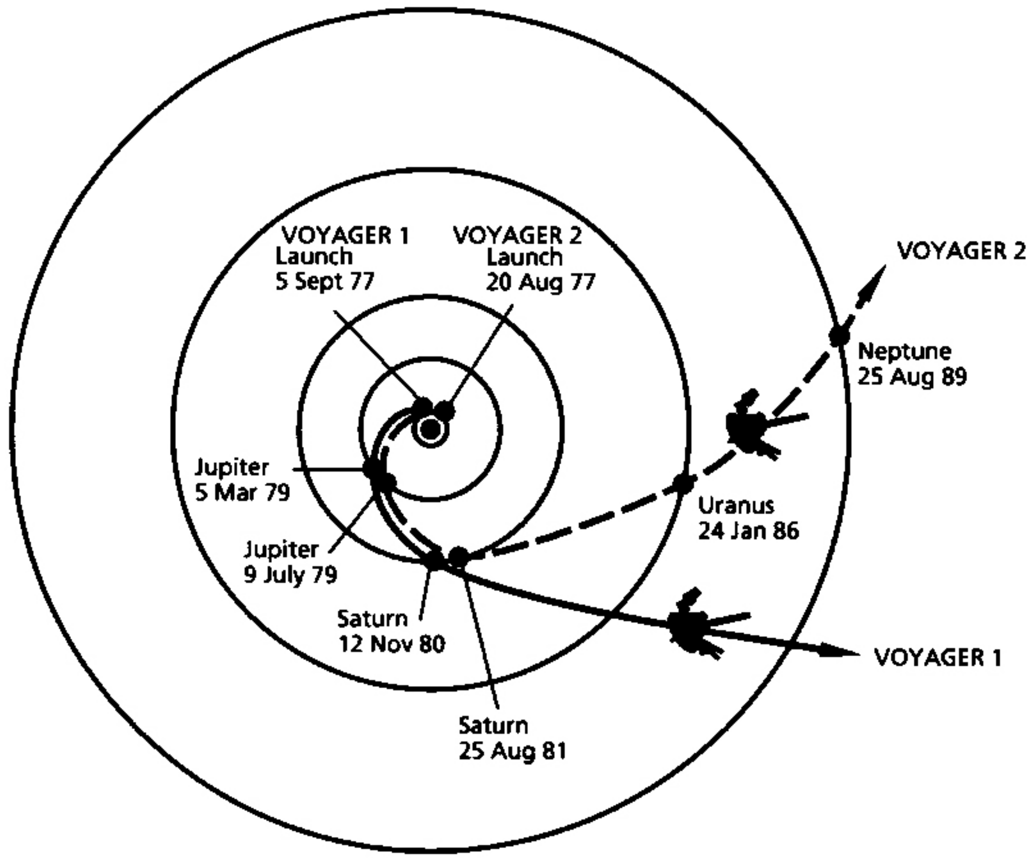
# Existence proof?

- Lévy, 1952: “L’existence des solutions périodiques de deuxième espèce du problème des trois corps n’est pas impossible, mais ne doit pas être considérée comme établie par l’analyse qu’en a faite Poincaré.”
- But Poincaré, 1899: “Je ne crois pas devoir insister davantage, car ses solutions s’écartent trop des orbites réellement parcourues par les corps célestes.”

# Yet important

- Hénon, 1997: “second species orbits play a major role in the problem: indeed they tend to dominate the picture.  Besides, second species orbits are of practical interest in space navigation, where close approaches to planets and other bodies are frequently used to change the velocity of a probe without expending energy (this technique is known as “flyby”, swingby”, or “gravity assist”).”

# Voyager Missions

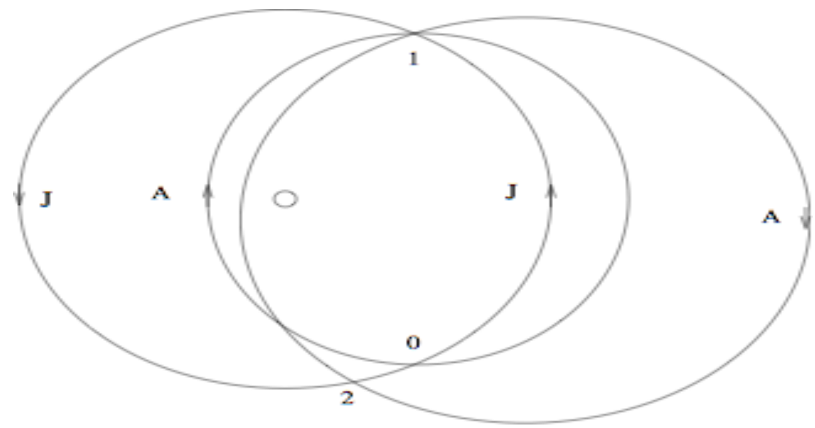


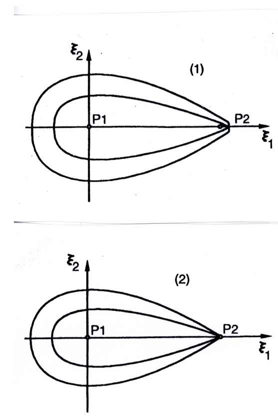
# Some other 2nd species missions

- Pioneer 10 (1973): Jupiter, escape
- Mariner 10 (1973-4): Venus, Mercury (x3), orbit round Sun
- Ulysses (1990-2): Jupiter, into polar orbit round Sun
- Cassini (1997-2004): Venus (x2), Earth, Jupiter, Saturn, moons of Saturn (x44) [show animations]

# Existence proofs again?

- Alexeyev 1970, Guillaume 1980, Perko 1981, Brjuno 1981, Gomez & Olle 1991…: matched asymptotics
- Marco & Niederman, 1995 (following Henrard, 1980): Symmetric periodic orbits of CR3BP with two near-approaches per period
- Bolotin&M: we allow arbitrarily many near-approaches, asymmetric and aperiodic orbits



# 2. Circular Restricted 3BP

- "Restricted" means $\mu$=0; so S and J follow Kepler orbits around their centre of mass O, unaffected by A ("asteroid").
- "Circular" means S and J are chosen to move in circles around O. Wlog, M = 1-$\varepsilon$, distance SJ = 1, and angular frequency = 1.
- "Planar" if A is chosen to move in the plane of S and J.

# Jacobi constant

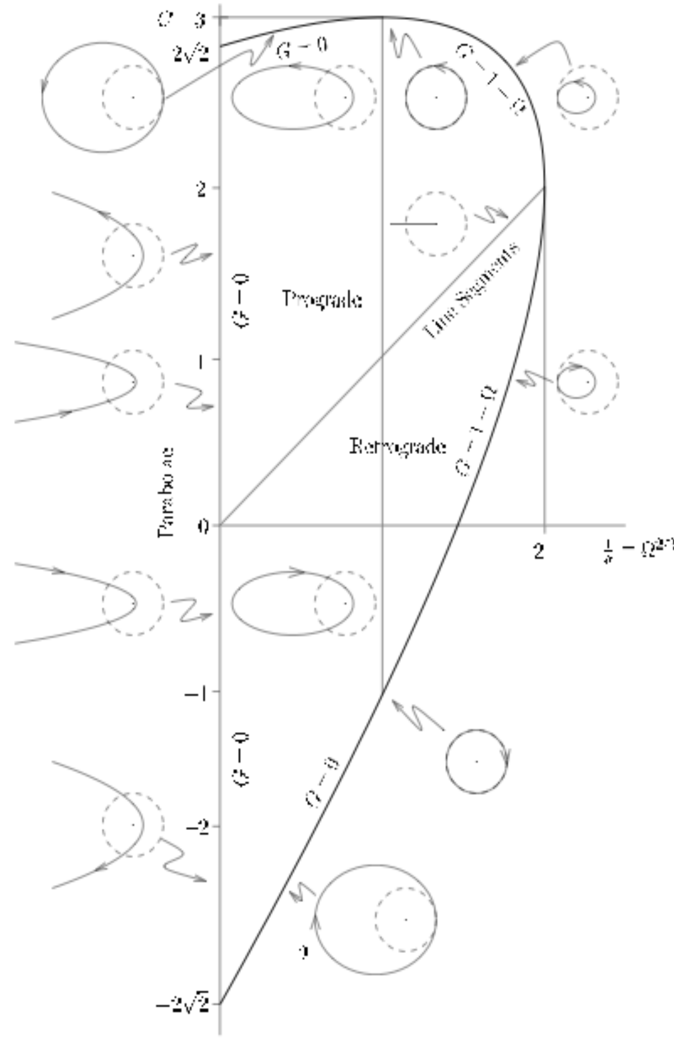
- In the CR3BP, $H = E - G_z$ is conserved (E and G are Energy and Angular Momentum of A about S per unit mass). Its value is usually denoted by $-C/2$ and C is called "Jacobi's constant".
- If $\varepsilon=0$, Kepler ellipse has

$$C = 1/a + 2\sqrt{a(1-e^2)} \cos i,$$

  where "inclination" i is the angle between the angular momentum vectors for A and J.

# Circle-crossing orbits

- For ε=0, which Kepler ellipses cross J's orbit?
- Planar case: allowed set A(C) of frequencies is 1 or 2 intervals for each C in (-√8,+3).



# 3. Strategy in Planar Case

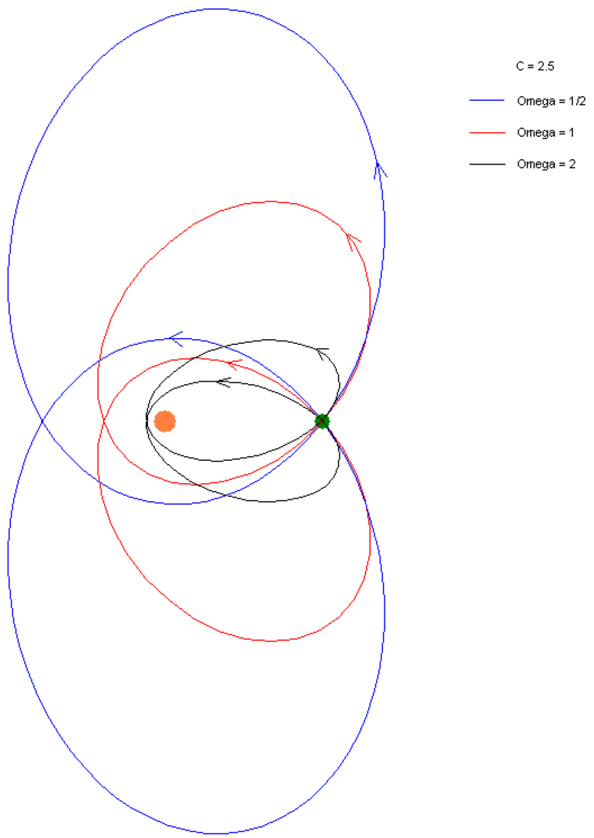
- Marco & Niederman's orbit was obtained by alternation between two arcs of planar Kepler orbits joining distinct points on J's orbit.
- In contrast, Bolotin & I chose to study concatenations of whole revolutions of planar ellipses.

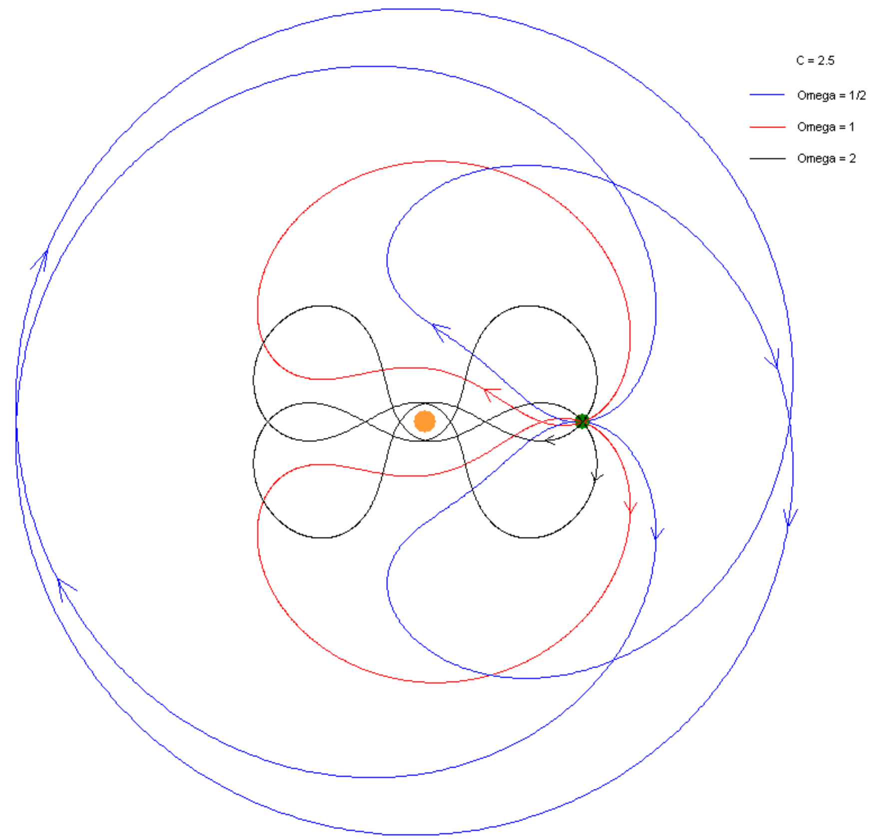
# Planar Theorem

- Bolotin & MacKay, 2000:

  For all C in $(-\sqrt{8},3)$ there is a dense subset S(C) of rationals in the set A(C) such that for all finite subsets T of S(C) there is $\varepsilon_0 > 0$ such that for any sequence $\Omega_n = m_n/k_n$ in T and $\varepsilon$ in $(0,\varepsilon_0)$ there is a unique trajectory of Jacobi constant C near to a chain of collision arcs consisting of $m_n$ revolutions of a Kepler ellipse during $k_n$ of J, modulo slow rotation.

# Example in Inertial & Rotating frames: Stephen Gin



# Topological Markov chain

- S(C) excludes ellipses which collide before a whole number of revolutions.
- Two ellipses through J for each $\Omega$ in S(C), label by $\sigma = \pm$.
- Transition rule: in frame rotating with J, must not leave J in the same or opposite direction.
- So we've made a lot of second species periodic orbits, given by choosing periodic sequences ($\Omega_n$, $\sigma_n$).
- & uncountably many aperiodic ones: "chaos".

# Instability

- The orbits are very unstable: their Lyapunov exponent $\lambda$ (= growth rate of typical infinitesimal displacements) ~ $\log(1/\varepsilon)$.
- For comparison, collinear Lagrange points have $\lambda = \frac{1}{2}\log(1+\sqrt{28})$ as $\varepsilon$ goes to 0.
- And Poincaré's chaos near first species periodic orbits has $\lambda \sim \exp(-\pi/\sqrt{2\varepsilon})$.
- So strong chaos and strong controllability.

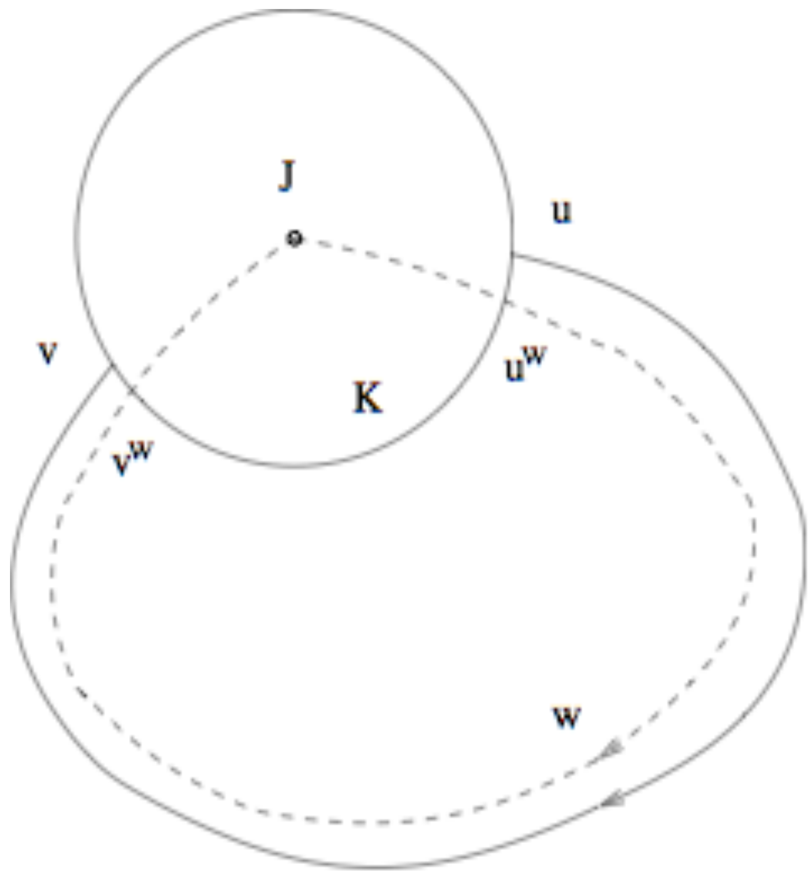
# Idea of the proof

- In the rotating frame, trajectories with given C between given points correspond to critical points of an "action" functional $S(\gamma) = \int^T (L-C/2)\, dt$,

$$L = |q'|^2/2+xy'-yx'+(x^2+y^2)/2+1/|q|+\varepsilon(1/|q|-1/|q-j|+x)$$

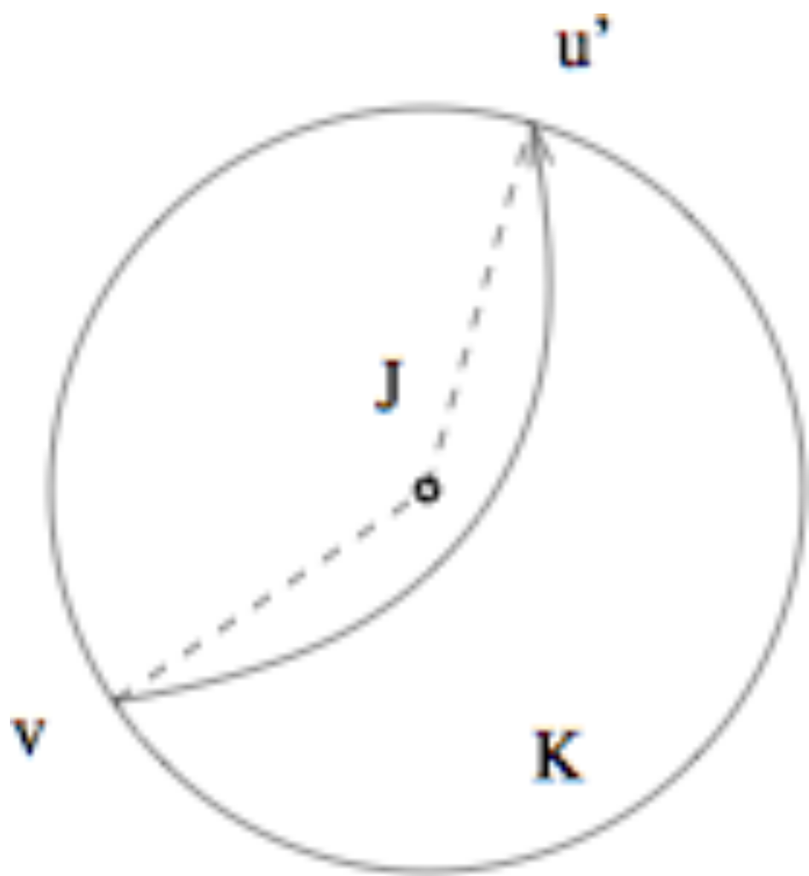
  in $H^1([0,1],R^3)xR^+$, $q=(x,y,z)$, $j=(1,0,0)$.
- Take a small circle K around J.

# Outside K




- For ε=0, the segment $[u^w,v^w]$ of any collision arc w outside K is a non-degenerate critical point of S.
- So it continues to a segment of trajectory for small ε and small displacements of the ends u and v on K, and its action depends $C^2$ on u,v.

# Inside K

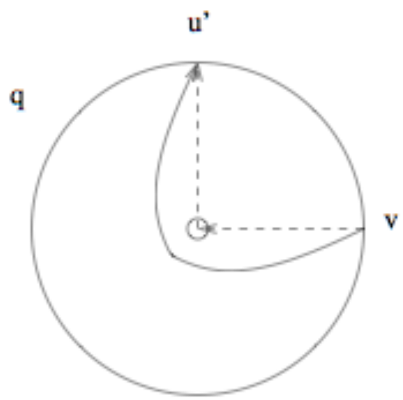


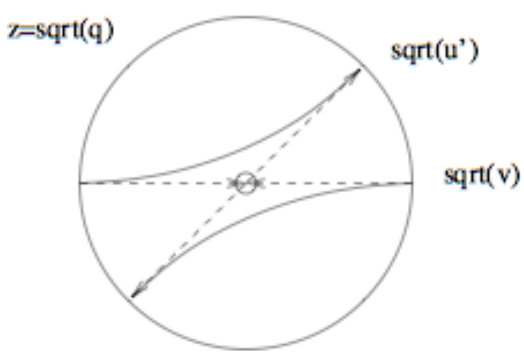

- For any pair of trajectories of $\varepsilon$=0 with given C, one from v on K to J and the other from J to u' on K, not tangent at J, there is a unique continuation of their concatenation for $\varepsilon$ small to a trajectory joining v to u'.
- Approximately a Kepler hyperbola (Rutherford scattering).

# Proof of “inside K”

- Use Levi-Civita regularisation to turn passage near the $\varepsilon/r$ singularity into passage at pseudo-energy $4\varepsilon$ near a hyperbolic equilibrium 0 of a smooth system in square-root coordinates with rescaled time.
- It turns the orbits vJ and Ju’ into orbits asymptotic to and from 0.

# Continued



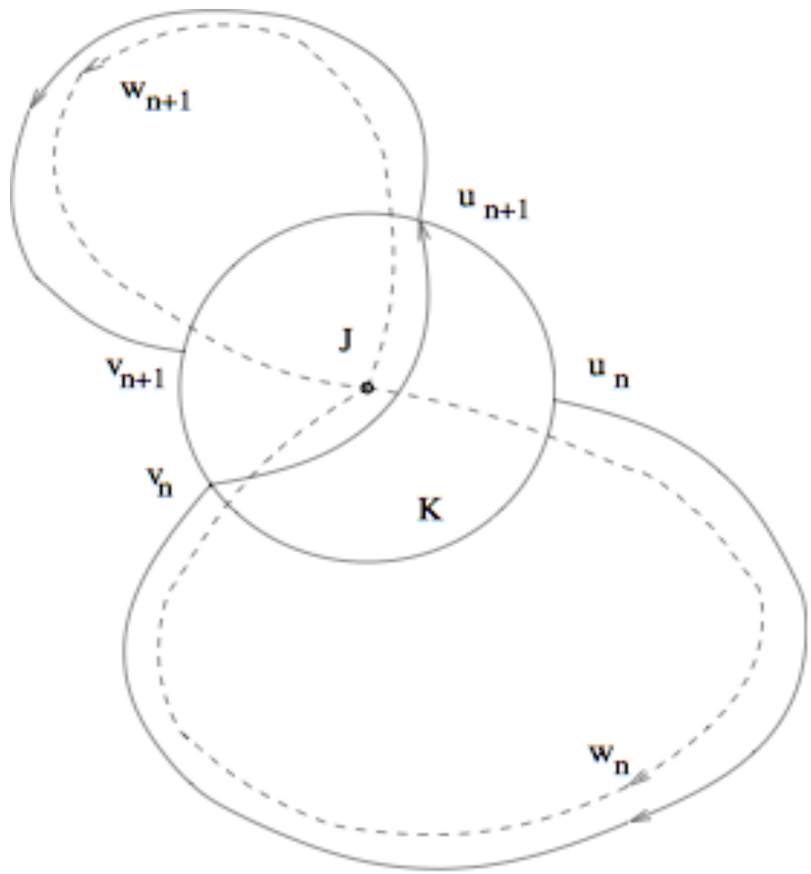




- For small ε > 0, there is an orbit from √v to √u' if the angle change from √{vJ} to √{Ju'} is less than π/2.
- Except for π/2, this can be achieved by choice of sign of one √.
- Squaring, there is an orbit from v to u' avoiding J if the angle change from vJ to Ju' is not 0 or π.

# Levi-Civita via Hamilton

- Start from Hamiltonian formulation:
  $q_j' = \partial H/\partial p_j$, $p_j' = -\partial H/\partial q_j$, $j$ = re, im, with
  $H(q,p) = |p-iq|^2/2 + W_\varepsilon(q) - \varepsilon/|q|$, $q,p$ in C.
- Canonical transformation: $q=z^2$, $p=w/2z^*$
  $H'(z,w) = |w/2z^*-iz^2|^2/2 + W(z^2) - \varepsilon/|z|^2$
- $K(z,w) = k + f(z,w)\,(H'(z,w)-E)$ on $K=k$ has same dynamics as $H$ on $H=E$, in variables $z,w$ and new time $ds/dt = f$
- $k=4\varepsilon$, $f=4|z|^2$ -> $K = |w|^2/2+4(W(0)-E)|z|^2+O(4)$
- $K$ is smooth and for $E>W(0)$ has a saddle ($E = -C/2$ and $W(0) = -3/2$, so saddle for $C<3$).

# Concatenation




- The action S for a concatenation of alternately exterior and interior arcs of trajectory from K to K has the form

  $\sum_n g_{w_n}(u_n,v_n) + s(v_n,u_{n+1};\varepsilon)$ ε

# Continuation from an "anti-integrable limit"

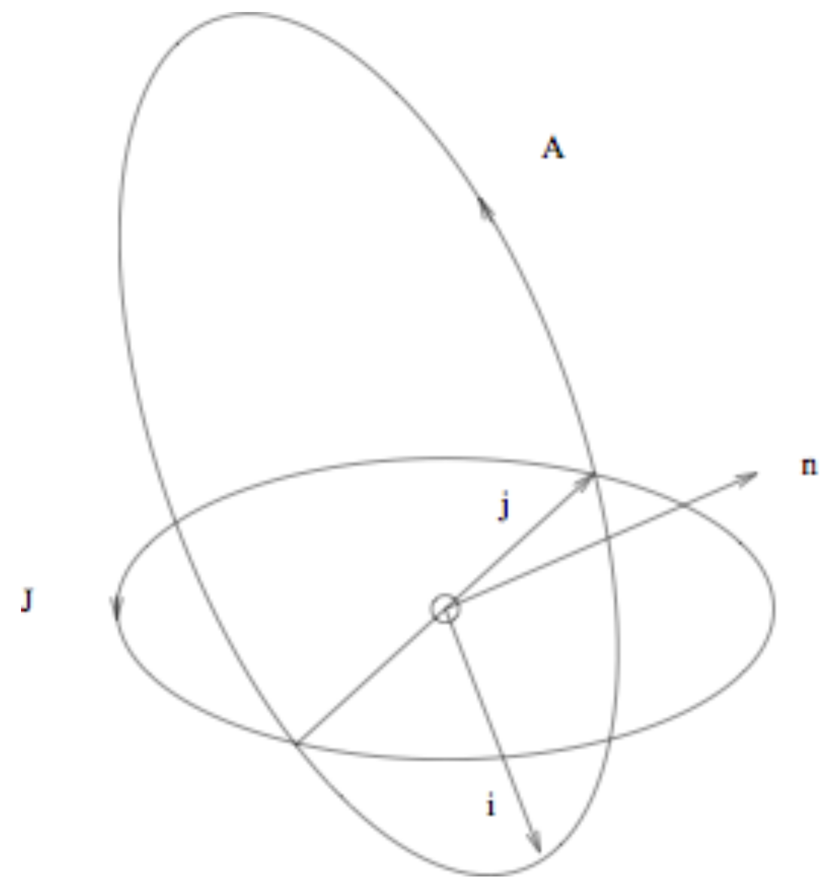
- The functions $g_w$ and s are $C^2$, and the $g_w$ have non-degenerate critical point at $(u^w,v^w)$ where w crosses K.
- Thus the sequence … $u^w{}_{n-1}, v^w{}_{n-1}$, $u^w{}_n, v^w{}_n$, $u^w{}_{n+1}$, $v^w{}_{n+1}$ … has a locally unique continuation to a critical point of S for small $\varepsilon$.
- The corresponding concatenation is a true trajectory.

# Proof of Instability

- The resulting trajectories are uniformly non-degenerate, meaning the spectrum of $D^2S$ is bounded away from 0.
- The coupling (off-diagonal terms of $D^2S$) between successive collision arcs is order $\varepsilon$.
- It follows that the invariant set is “uniformly hyperbolic” with expansion factor of order $1/\varepsilon$ per collision arc.
- Hence Lyapunov exponent of order $\log(1/\varepsilon)/T$, where $T$ = typical duration of collision arcs.

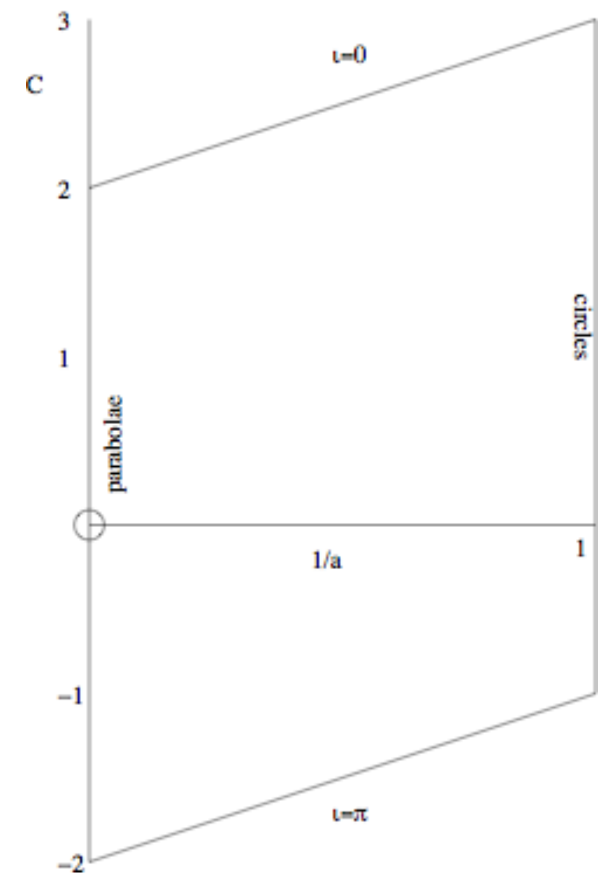
# 4. Nonplanar case

- We consider concatenation of segments of orbit of Kepler ellipse joining diametrically opposite points of Jupiter's orbit.



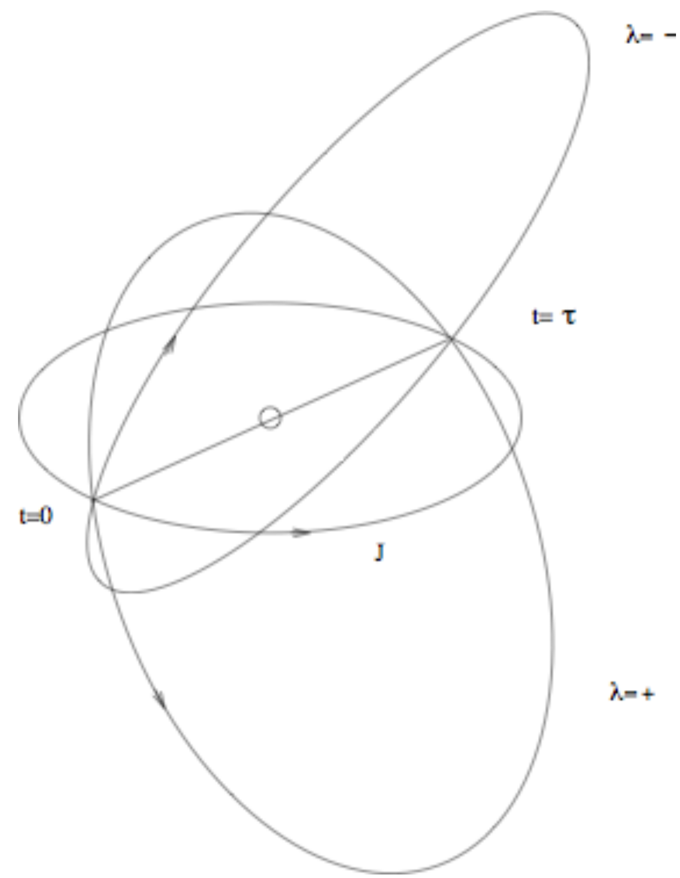
# Conditions

- $a(1-e^2) = 1$
- $1/a + 2 \cos i = C$
- So for each C in (-2,+3) there is an interval A'(C) of allowed frequencies $\Omega = a^{-3/2}$



# Two ellipses

- As in the planar case, there are two ellipses with the same parameters joining the same pair of points



# Nonplanar theorem

- Bolotin & MacKay, 2005:

  For all C in (-2,3) there is a dense subset S'(C) of A'(C) such that for all finite subsets T of S'(C) there is $\varepsilon_0 > 0$ such that for any sequence $\Omega_n$ in T and $\varepsilon$ in $(0,\varepsilon_0)$ there is a unique trajectory of Jacobi constant C near to a chain of collision arcs formed from nonplanar Kepler ellipses of frequencies $\Omega_n$, modulo slow rotation.

# Topological Markov chain

- Transition rule: in frame rotating with J, must not leave in the same or opposite direction.
- Constructs yet more periodic and aperiodic second species orbits.
- Same magnitude of Lyapunov exponent.

# Idea of proof

As in the planar case, but:

- dense set S’(C) is to obtain second collision
- replace circle K by a sphere around J
- use Kustaanheimo-Stiefel regularisation

# 5. Extensions?

- Include arcs between different points in the planar case: probably yes.
- Include whole revolutions in the nonplanar case: delicate.
- Mix planar and nonplanar arcs: delicate.
- Make unbounded orbits: delicate (Kaloshin?).
- Elliptic restricted 3BP: delicate (Bolotin, 2004)
- Unrestricted 3BP: delicate.
- N-body problem: delicate
- Can view Tel, Grebogi, Ott chaotic scattering from potential hills in same framework
- Interaction of charges in a magnetic field: yes (Pinheiro & M, Nonlinearity 19 (2006) 1713 & JNLS 18 (2008) 615)

# 5. Two charges in a magnetic field: (a) planar case

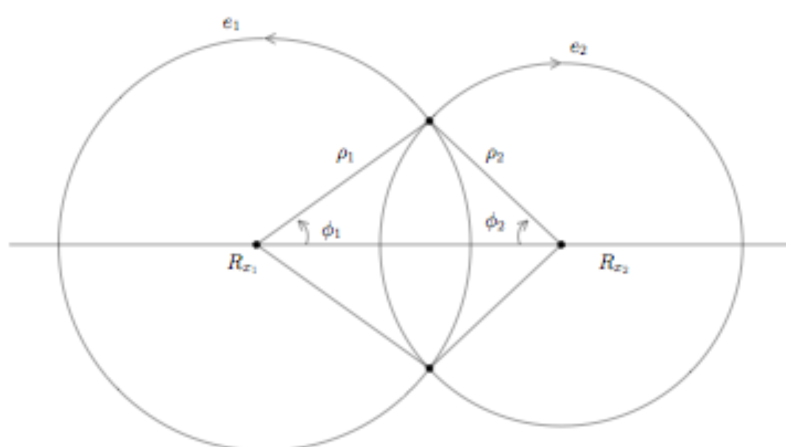
- Pinheiro & M, Nonlinearity 19 (2006) 1713-1745
- Suppose uniform magnetic field B, masses $m_1, m_2$ moving in perpendicular plane with charges $e_1, e_2$ of opposite sign, and $e_1/m_1+e_2/m_2 \neq 0$.
- Theorem: Every high enough energy level contains a covering of a suspension of a nontrivial topological Markov chain (TMC).

# More precisely

- There is a $\delta(e_2/e_1, m_2/m_1) > 0$ such that for $H^{3/2} > |e_1 - e_2|^3 B/(m_1+m_2)^{1/2} \varepsilon_0 \delta$ (H=energy) the reduction by linear and angular momentum possesses a suspension of a nontrivial TMC.
- Linear momentum $P = \sum p_j + k_j J q_j$, where $p_j = m_j dq_j/dt$, $k_j = -e_j B$, J=rotation by $\pi/2$
- Angular momentum $L = \sum q_j J p_j - k_j |q_j|^2/2$

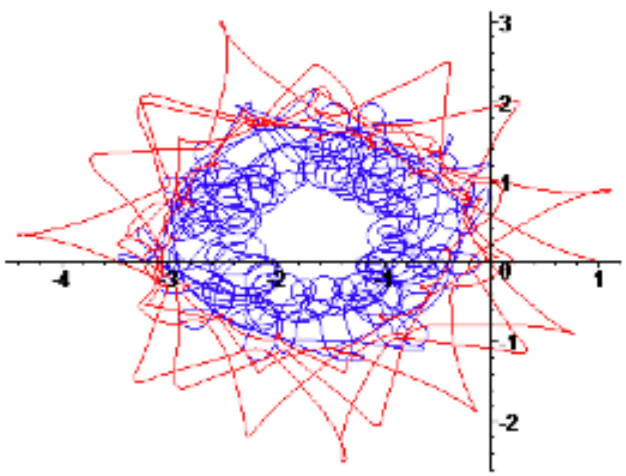
# Idea of proof

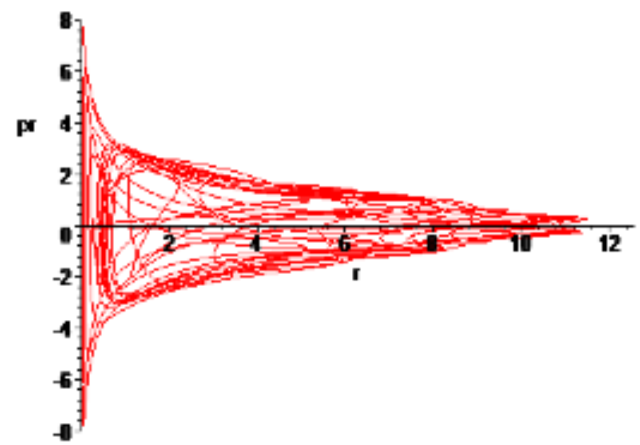
- Scaling symmetry: H large $\leftrightarrow$ $1/\varepsilon_0$ small
- Anti-integrable (AI) limit $1/\varepsilon_0=0$: declare its “traject-ories” to consist of sequences of segments of pairs of gyro-orbit joined at collisions, conserving H,P,L and with angle change of relative velocity not $0,\pi$



# continued

- When $e_1 e_2 < 0$, $e_1/m_1 + e_2/m_2 \neq 0$, the set of AI trajectories contains a suspension of a non-trivial TMC, and for $1/\varepsilon_0 < \delta$ they persist to a set of true trajectories (cf. second species chaos in celestial mechanics, Bolotin&M, 2000, 2006; also Grebogi et al scattering from three hills).



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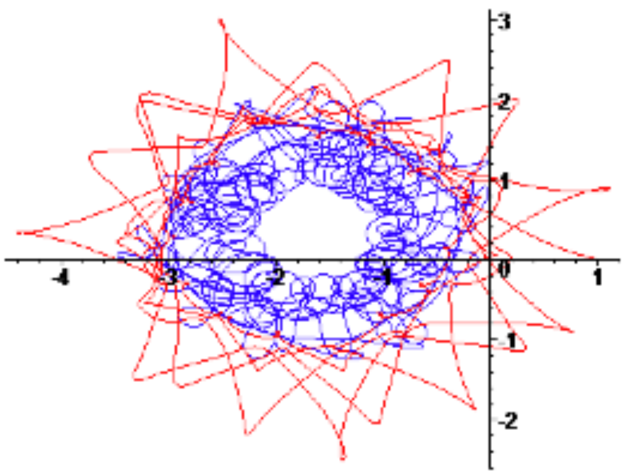
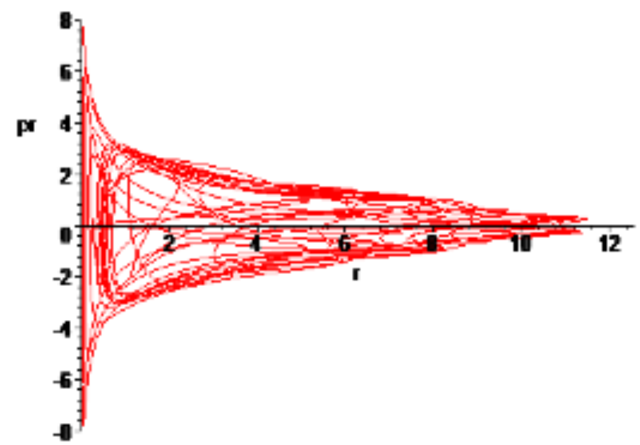
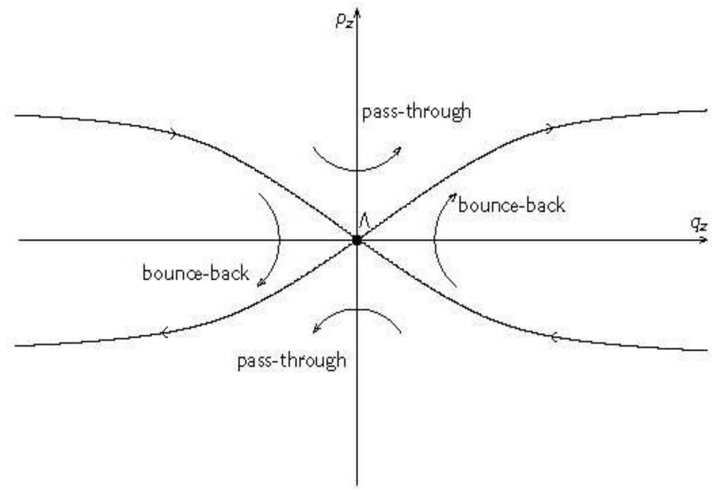
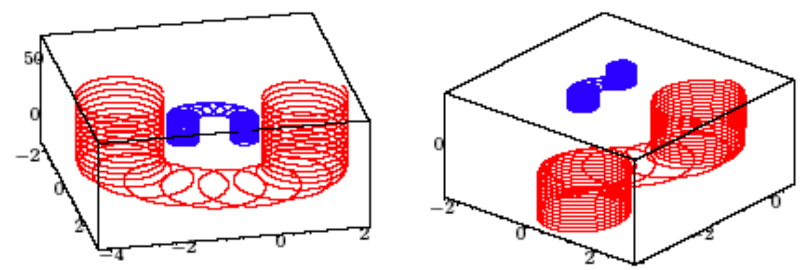


# (b) 3D case

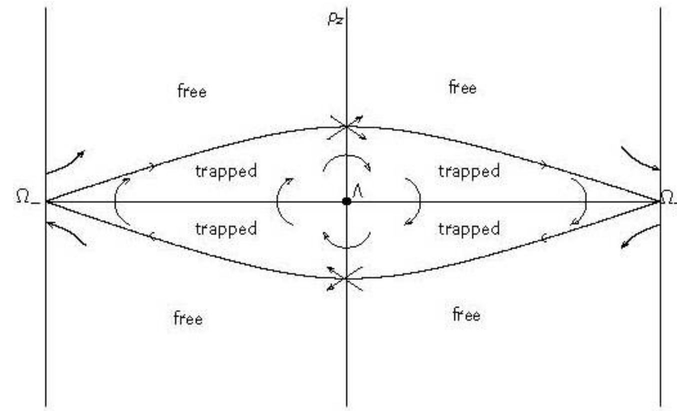
- For charges of same sign, the subspace of planar motions is essentially normally hyperbolic. Get regular scattering, partitioned into “bounce back” and “pass through”, plus formula for the flux.

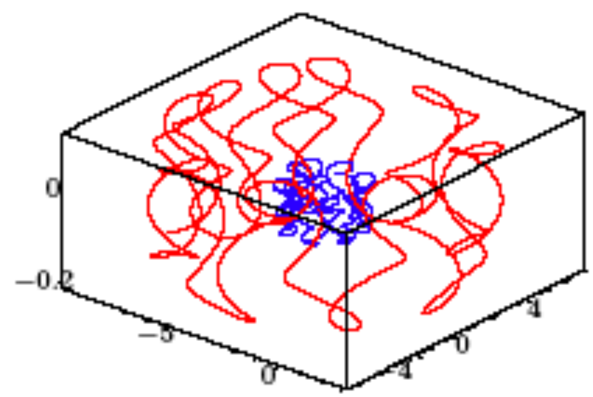


# Opposite signs

- Put $q = x/(1-x^2)^2$ and $ds/dt = 1/(1+q^2)$. Then $q=\infty$, $p=0$ becomes essentially normally hyperbolic and its invariant manifolds separate state space into regions of trapped and free motion with possible transitions at $q=0$, giving atoms and (numerical) chaotic scattering.



# Transport effects

- In general, scattering results in transfer of parallel momentum and perpendicular kinetic energy and changes in guiding centre lines.
- In particular, can obtain large perpendicular energy transfers for equal gyrofrequencies, because of “resonant interaction” rather than “collision”. This seems to be missed by classical magnetised plasma transport theory!

# 6. Conclusions

- Poincaré had a good idea
- But didn’t prove it
- And missed the occasion to discover much stronger chaos than his exponentially weak form
- Solar system missions have proved it experimentally
- So far only the simplest cases are proved mathematically
- But the idea has been extended to other contexts
- And I hope the insight will lead to better understanding.