Dégustation de quatre sujets

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Topics

- Space-time phases
- Derivation of Langevin equation
- Poincaré's second species orbits
- Gamma ray bursts

Space-time phases for spatially extended nonlinear dynamics

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Happy Birthday, Tassos!

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Complexity Science: my view

- The study of systems with many interdependent components
- e.g. laser, condensed matter, cell, brain, ecosystem, climate system, transport network, internet, health service, finance, economy
- Hope for unifying principles: mathematics

Is it new? No!

- Complexity Science builds on much preceding science, e.g. statistical mechanics, nonlinear dynamics, stochastic processes, ecology/ epidemiology, game theory, evolutionary theory, many-body quantum theory…
- “The main themes in complexity theory have been studied by physicists for over a hundred years, and these scientists have evolved a toolkit of concepts and techniques to which complexity studies have added barely a handful of new items.” (P Ball, 2004)

New? continued

- “A new branch of mathematics – the theory of systems with a large number of locally interacting random components – has developed in recent years. This theory is a natural instrument for the mathematical modeling of … systems of various natures, such as complex biological, chemical, physical, and cybernetical systems and socio-economic structures.” (Dobrushin, Kryukov &Toom, 1978).
- The current wave can be considered to have started with the Santa Fe Institute in 1984.
- But the surface has only been scratched.

What are the questions?

1. Emergence
2. Robustness
3. Control and Design
4. More

1. Emergence

- Wikipedia: “Emergence is the way complex systems and patterns arise out of a multiplicity of relatively simple interactions”
- “The whole is more than the sum of its parts” (Aristotle, c330BC)
- “the whole becomes not merely more, but very different from the sum of its parts” (Anderson, 1972)

A “cathedral” mound produced by a termite colony: a classic example of emergence in nature.

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Philosophers

- JS Mill, 1843
- Weak v Strong emergence,

  e.g. Chalmers, 2006: A high-level phenomenon is weakly / strongly emergent with respect to a low-level domain when it arises from the latter but truths concerning it are

unexpected given the principles governing /

not deducible even in principle from truths in

the low-level domain.

My view: space-time phases

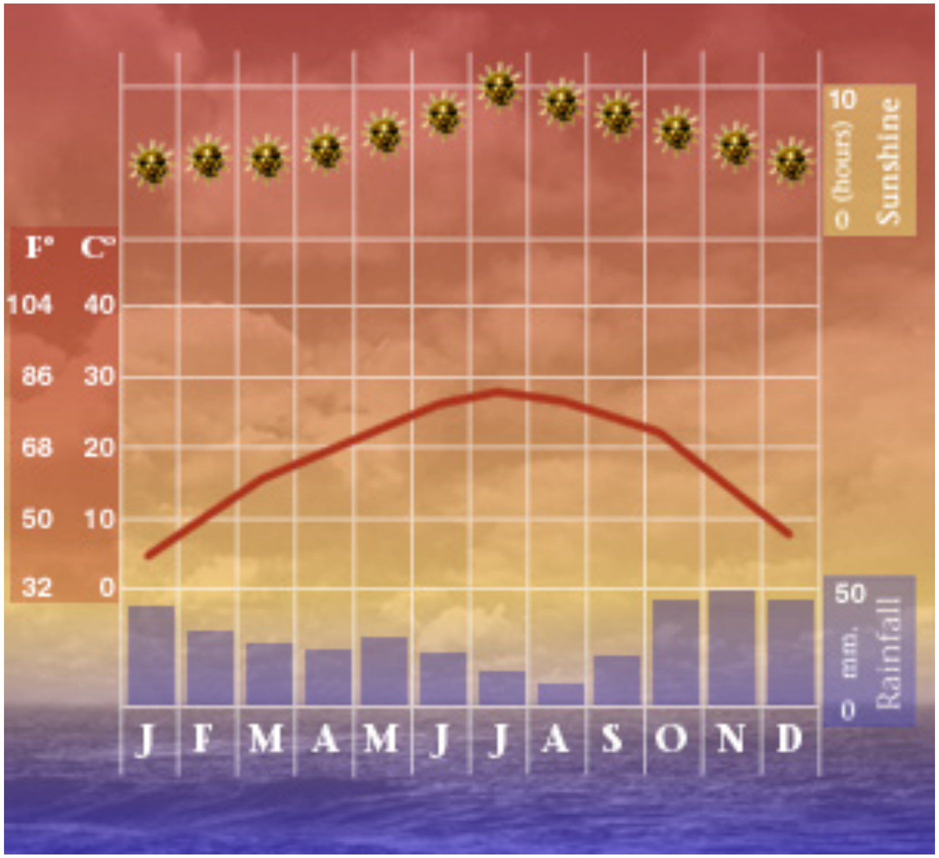
(a) What emerges from a spatially extended dynamical system is "space-time phases": probability distributions over realisations of state as function of space-time that arise from typical initial probabilities in the distant past.

(b) Amount of emergence is the "distance" of a space-time phase from the set of products for independent units.

(c) Strong emergence means non-unique space-time phase (but not due to decomposability).

(a) Examples of phases

- "Climate" is a probability distribution for temperature, precipitation etc over space-time, compatible with the laws of weather.



Thessaloniki

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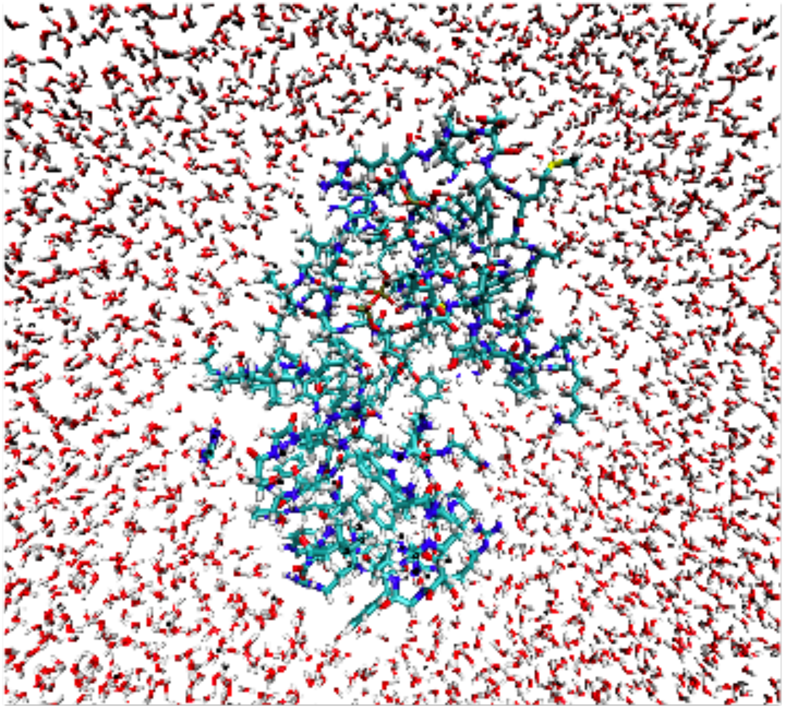
Equilibrium statistical mechanics

- The allowed probability distributions are the "Gibbs phases" for βH where H represents the sum of contributions h to the energy and β is coolness (1/temperature).
- i.e. probability density

$$\frac{1}{Z} e^{-\Sigma \beta h}$$

wrt reference measure, where Z is a normalisation constant, or better those whose conditionals for all finite subsystems and external states satisfy this (Dobrushin,Lanford,Ruelle).



Dihydrofolate reductase in water (Dmitry Nerukh)

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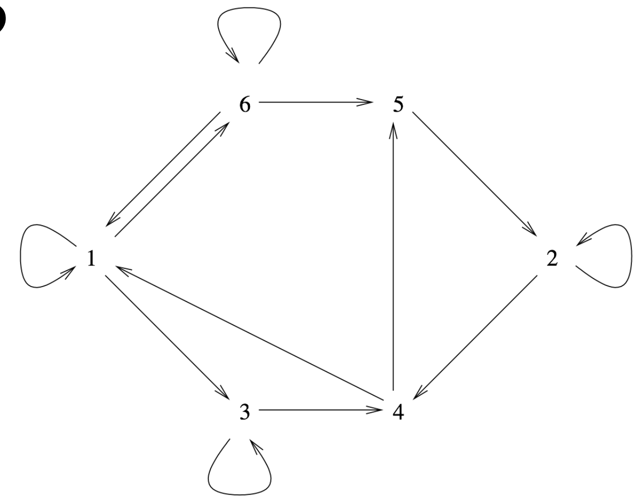
Stochastic dynamics

- For Markov chains the phases are the Gibbs phases (over time) for - log *p(i,j)*: probability of sequence $i_0, i_1, \ldots, i_n$ =

$$\prod_t p(i_t, i_{t+1}) = e^{-\sum_t -\log p(i_t, i_{t+1})}$$

- Probabilistic cellular automata (PCA): update state σ_s^t at spatial site *s* and time *t* by independent probabilities conditional on current state σ^t
- Demonstration: Toom's NEC majority voter PCA with error rate $p = 0.15$, by Marina Diakonova.
- The phases of a PCA are the space-time Gibbs phases for – log $p(\sigma_s^{t+1}|\sigma^t)$.

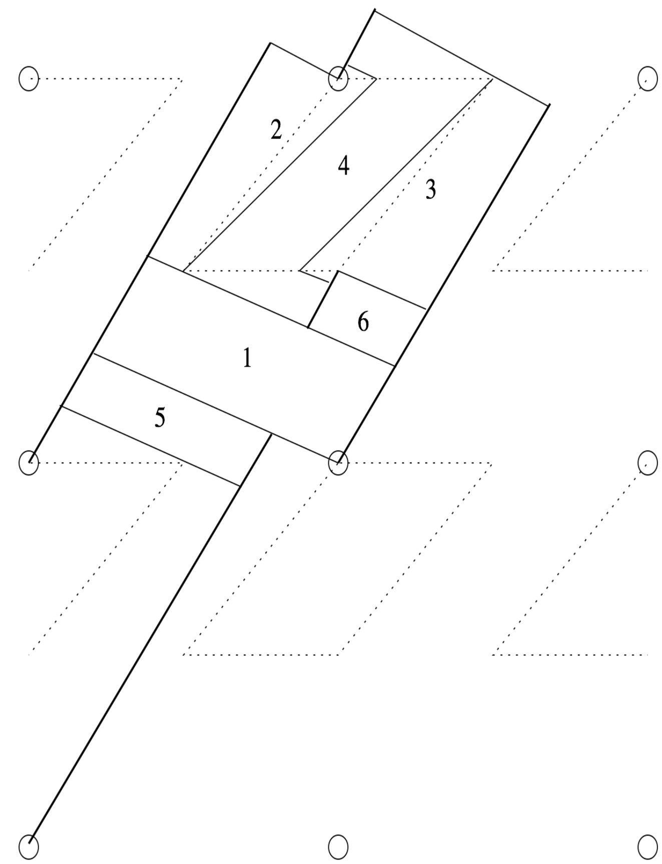


Deterministic dynamics

- Sensitive dependence on initial conditions makes individual trajectories unpredictable but often leads to a unique probability distribution on an attractor for random initial conditions in its basin.
- e.g. trajectories on a topologically mixing uniformly hyperbolic attractor for a map f can be coded by symbol strings σ, and random initial conditions in distant past in the basin give trajectories distributed according to unique Gibbs phase for

  $$\beta H = \sum_t \log |\det Df_{E^-}(x^t(\sigma))|$$

  on time [Sinai, 1967/72]
- Analogous results for continuous-time.



Markov partition for Cerbelli-Giona map

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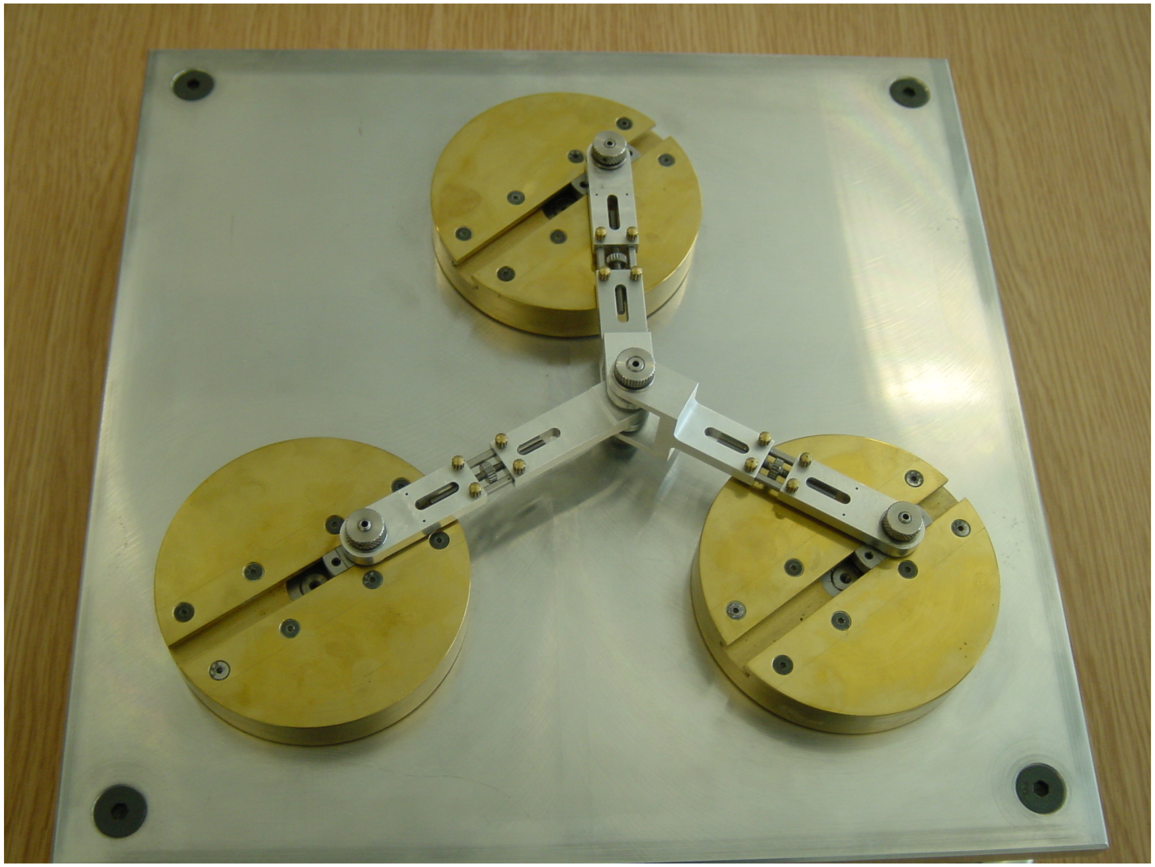
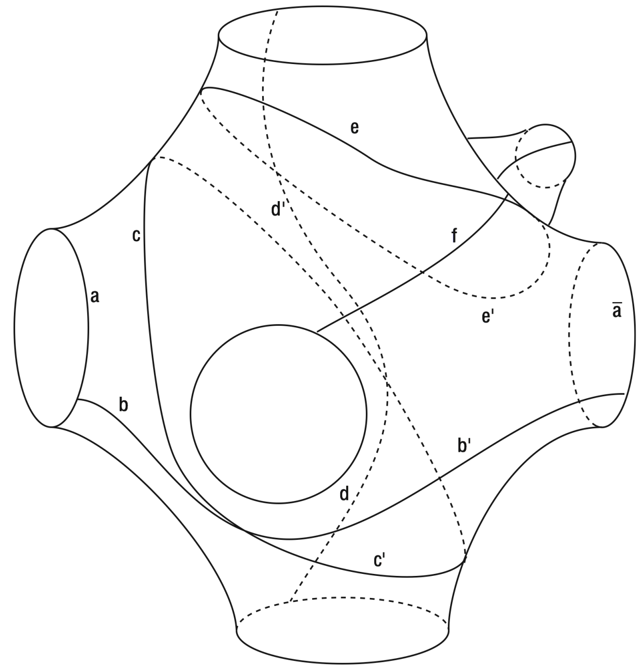
A physical uniformly hyperbolic system



Show video



Minimal geodesics on configuration space from which to make a 40 element Markov partition

Spatially extended deterministic dynamics

- Trajectories of uniformly hyperbolic spatially extended discrete-time system *f* (coupled map lattice) can be coded by space-time symbol tables $\sigma = (\sigma_s^t)$.
- Random initial conditions in distant past lead to distribution of trajectories given by Gibbs phases of

$$\beta H = \sum_{s,t} \mathrm{tr}(\log Df_{E^-}(x_s^{\ t}(\sigma)))_{ss}$$

(M,1995; Bricmont & Kupiainen, 1996).

(b) Distance between multivariate probabilities

- Most metrics on spaces of probabilities do not behave well for large number of variables.
- e.g. product of N independent Bernoulli *B(p,1-p)* variables on $\{0,1\}^N$ in total variation metric:

  $D_{TV}(\rho,\sigma) = \sup(\rho(A) - \sigma(A))$ over measurable subsets A,

  moves with speed ~ $\sqrt{\dfrac{N}{2\pi p(1-p)}}$ wrt *p*.

  Can't save it by dividing by $\sqrt{N}$ because diameter in TV = 1.

  "Total variation convergence essentially never occurs for particle systems" (Liggett, 1985).

Dobrushin metric

- If X is a product of (complete separable, bounded diameter) metric spaces (X_s,d_s) over s in a countable set S, define Dobrushin's functions $f : X \to R$, continuous wrt product topology and "summably component-wise Lipschitz": $\|f\| = \Sigma_s \Lambda_s(f) < \infty$, where $\Lambda_s(f)$ is the Lipschitz constant for f wrt x_s.
- For (Borel wrt product topology) probabilities ρ,σ on X, define

$$D(\rho,\sigma) = \sup \frac{\rho(f) - \sigma(f)}{\|f\|}$$

over non-constant Dobrushin f, where $\rho(f)$ is the mean of f wrt ρ.

\+ Gives speed of product of N Bernoulli $B(p,1-p)$ variables = 1.

\+ Streamlines Dobrushin's proof of unique phase for "weakly dependent" PCA (e.g. Toom NEC for p in (1/3,2/3).

\+ Proves uniformly smooth change of phase wrt parameters for PCA with spectral gap (I-P invertible on space of neutral measures).

\- Not easy to compute (yet)

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Amount of emergence

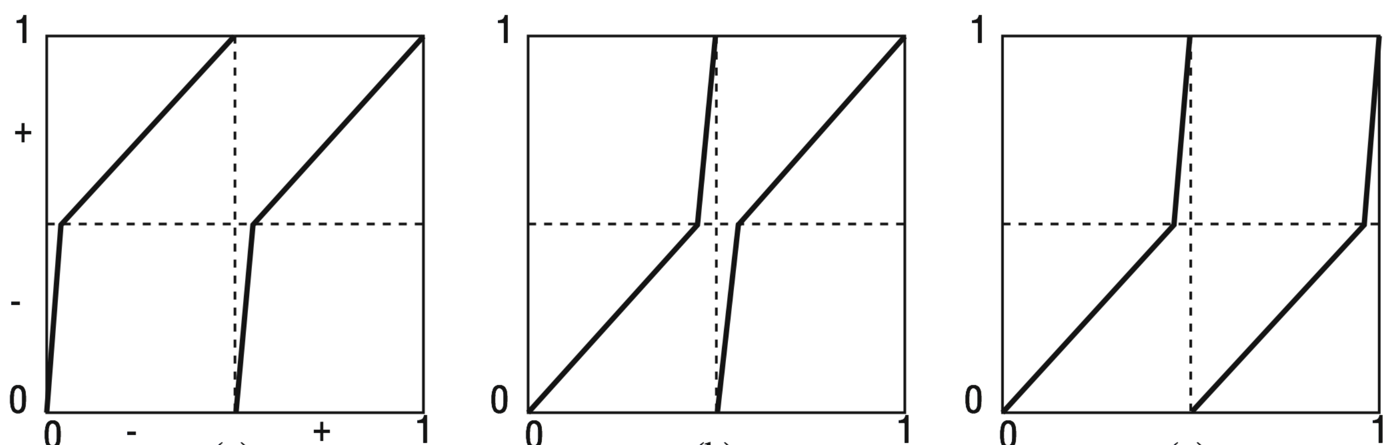
- is the distance of a space-time phase from the set of product probabilities for independent units.
- measures how far the behaviour is from mean-field approximations
- does not capture what some people want to mean by “emergence”, e.g. law of averages, selection of Maxwellian velocity distribution
- but does capture a likely consequence of having interdependent components
- More interesting would be to determine correlation structure of the phase

(c) Strong emergence

- More than one possible phase ("phase transition")
- Example: 2D Ising model (Peierls)
- Example: Toom's majority voter PCA with error rate 0.05
- Say amount of strong emergence is the diameter of the set of phases. An alternative is the persistent mutual information between well separated parts of space-time (Ball, Diakonova, M)
- Non-unique phase can arise for topological reasons, e.g. more than one attractor, or 2-piece attractor; more generally, because system is "decomposable". Don't count as strong emergence.
- A system with a space-time symbolic description is "indecomposable" if any allowable configurations on two sufficiently separated space-time patches can be joined into an allowable configuration ("specification property").
- Non-trivial strong emergence requires infinite system, but is reflected in long-range correlations for finite versions.

Proved examples of strong emergence

- Ferromagnetic phases of 2D Ising model
- Ferromagnetic phases of Toom's NEC voter PCA
- Period-2 phases of Toom's NEC voter (error rate 0.95)
- Examples with (at least) 2^n extremal phases [demo], and also non-monotonic examples, e.g. 3 phases [demo]
- Endemic infection v disease-free phases of contact processes (Stavskaya…)
- Coupled map lattices based on these (Sakaguchi, Gielis&MacKay, Bardet&Keller)



(a) (b) (c)

2. Robustness

(i) How does a phase respond to a shock?

- Exponential decaying response to shocks in case of PCA with spectral gap, but more generally?

(ii) How does the phase or set of phases (closed, convex) vary with parameters?

- For PCA with a spectral gap, under small changes the unique phase stays unique and varies smoothly (cf. Ruelle for SRB measure of a uniformly hyperbolic dynamical system)
- For systems whose phases are Gibbsian, the set of phases varies upper hemi-continuously.
- But not always lower, e.g. 2D Ising as magnetic field crosses 0.

Bifurcations

- In equilibrium statistical mechanics, co-existence of N (extremal) phases is of codimension N-1 (Gibbs phase rule).
- But for space-time phases, non-unique phase can be robust, e.g. Toom PCA.
- Does the set of phases generically vary smoothly? Perhaps there is a spectral projection that contains all the dynamics of domains?
- Proved examples of bifurcation: kinetic 2D Ising, Keller's globally coupled maps
- Universality classes? Renormalisation (=aggregation + rescaling)?

3. Control and Design

- What changes to a phase can be achieved with local control? A zealot can have huge effect in opinion-copying models [Mobilia, 2003]
- Boundary control can have a large effect when phase is non-unique.
- What changes to the set of phases can be obtained with infinitesimal (but high gain) control? (cf. “control of chaos”)
- How to design a complex system so that its phases optimise some objective function or partial order?

4. More questions

- More realistic systems, e.g. general network instead of a lattice, interaction of mobile units via proximity in space (swarms)
- Special classes, e.g. multi-agent games, number-conserving systems, many-body quantum systems, quantum gravity?
- Systems that never settle down (evolution?)?
- Aggregation procedures
- Reduction to macroscopic models
- Fitting to data

Conclusion

- Complexity Science offers a lot of serious and worthwhile challenges to Mathematics.
- Complexity Science needs serious input from Mathematics.

Advertisements

- Warwick EPSRC Mathematics Research Centre Symposium Year on "The Mathematics of Complexity Science and Systems Biology", Sept 09 - Sep 10. Remaining workshop:

  14-16 September 2010: Ecology, epidemiology and evolution
- UK Complex Systems Dynamics LMS network (CoSyDy).
- Warwick EPSRC Doctoral Training Centre in Complexity Science.
- ERASMUS MUNDUS 2-year MSc in Complexity Science, joint with Ecole Polytechnique, Chalmers University and University of Goteborg.