Response to external perturbations in systems with long-range interactions

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Plan

- HMF model
- Quasistationary states
- Inhomogeneous steady states
- Linear response of quasistationary states

Gravity

Einstein equations

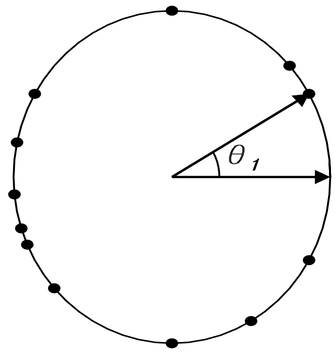
$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

Newton equations

$$\frac{d^2\mathbf{r}_i}{d\tau^2} = \nabla_i \sum_{j=1}^{N-1} \frac{Gm_j}{|\mathbf{r}_j - \mathbf{r}_i|}$$

HMF model

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i,j=1}^{N} (1 - \cos(\theta_i - \theta_j))$$



Magnetization $\mathbf{M} = \lim_{N\to\infty} \left(\frac{\sum_{i=1}^{N} \cos\theta_i}{N}, \frac{\sum_{i=1}^{N} \sin\theta_i}{N} \right) = (M_x, M_y)$

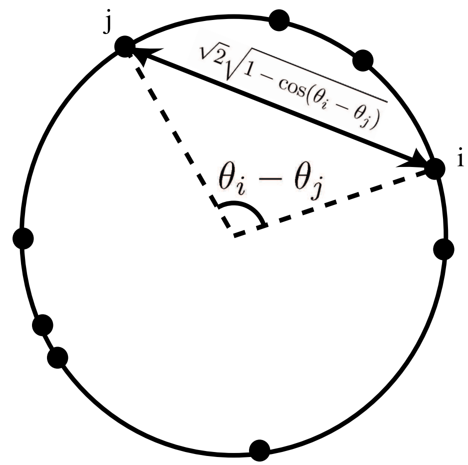
Energy $U = \lim_{N\to\infty} \frac{H}{N}$

Self-gravitating ring

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2N}\sum_{i,j} V_\varepsilon(\theta_i - \theta_j)\,,$$

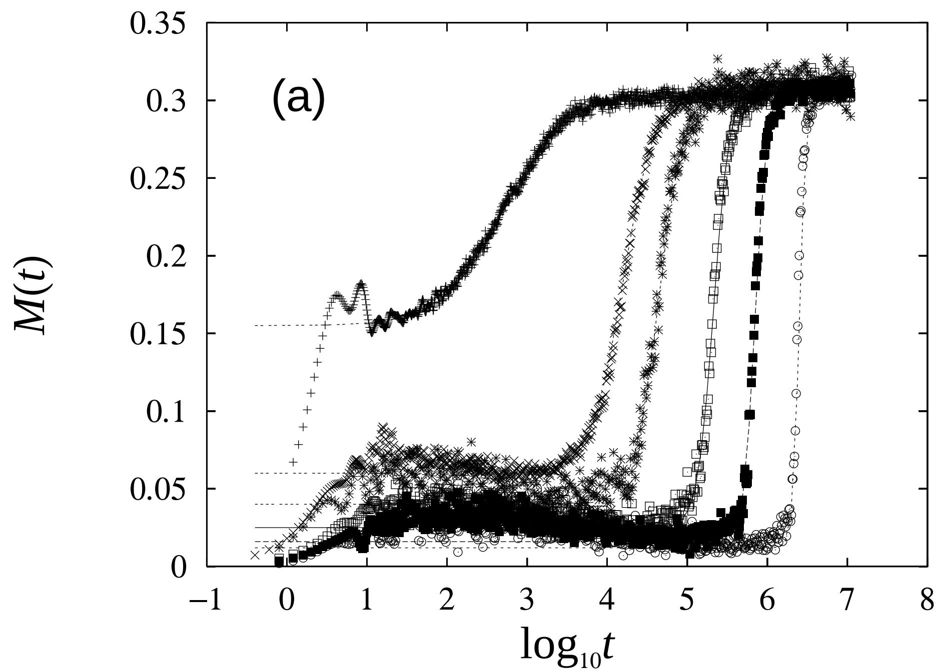
$$V_\varepsilon(\theta_i - \theta_j) = -\frac{1}{\sqrt{2}}\frac{1}{\sqrt{1-\cos(\theta_i-\theta_j)+\varepsilon}}$$

where ε is the softening parameter.



T. Tatekawa, F. Bouchet, T. Dauxois

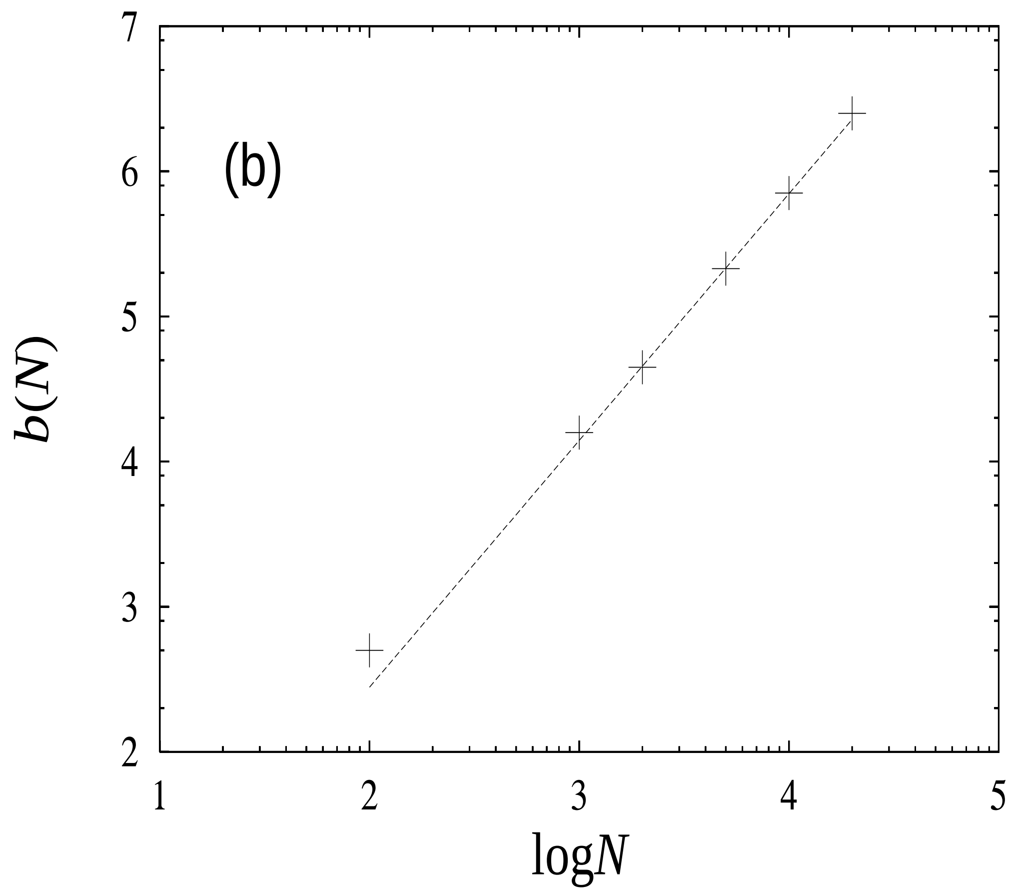
Quasi-stationary states



$U = 0.69$, from left to right $N = 10^2, 10^3, 2 \times 10^3, 5 \times 10^3, 10^4, 2 \times 10^4$.
Initially $\Delta\theta = \pi$, hence $M_0 = 0$.

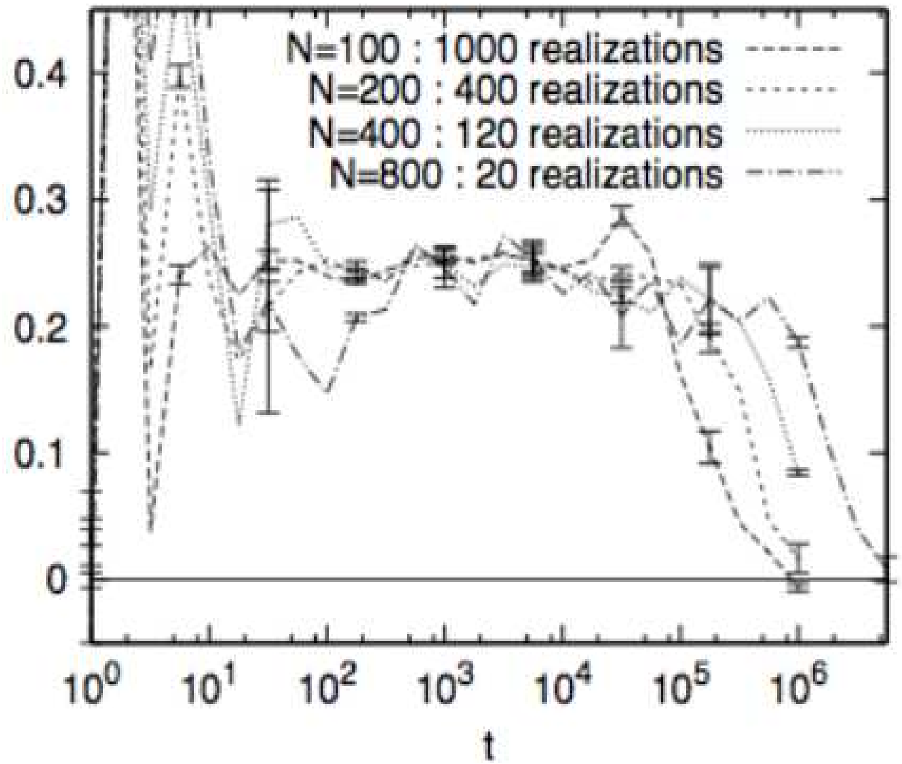
`Y. Y. Yamaguchi, J. Barre, F. Bouchet, T. Dauxois`

Time scale



Power law increase of the lifetime, exponent 1.7

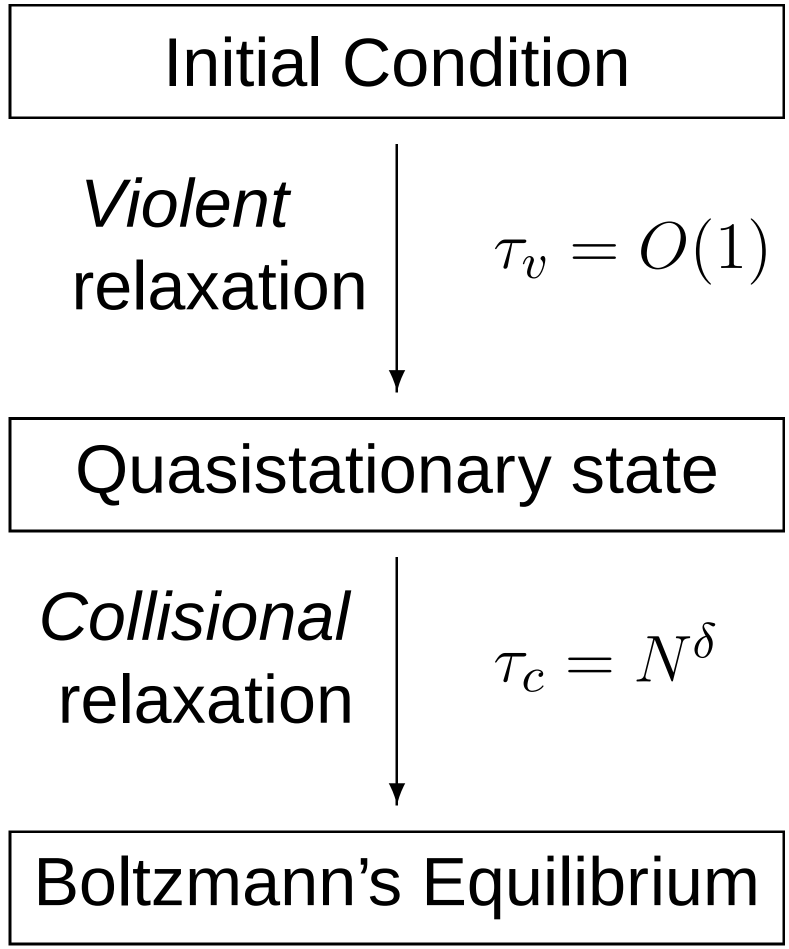
1D self-gravitating



The lifetime increases with N.

M. Joyce and T. Worrakitpoonpon

Separation of time scales



Vlasov equation

$$\frac{\partial f}{\partial t} + p\frac{\partial f}{\partial \theta} - \frac{d\Phi(\theta)[f]}{d\theta}\frac{\partial f}{\partial p} = 0$$

$$\begin{aligned}
\Phi(\theta)[f] &= 1 - M_x[f]\cos(\theta) - M_y[f]\sin(\theta) \ , \\
M_x[f] &= \int f(\theta, p, t)\ \cos\theta \mathrm{d}\theta \mathrm{d}p \quad , \\
M_y[f] &= \int f(\theta, p, t)\ \sin\theta \mathrm{d}\theta \mathrm{d}p \quad .
\end{aligned}$$

Energy

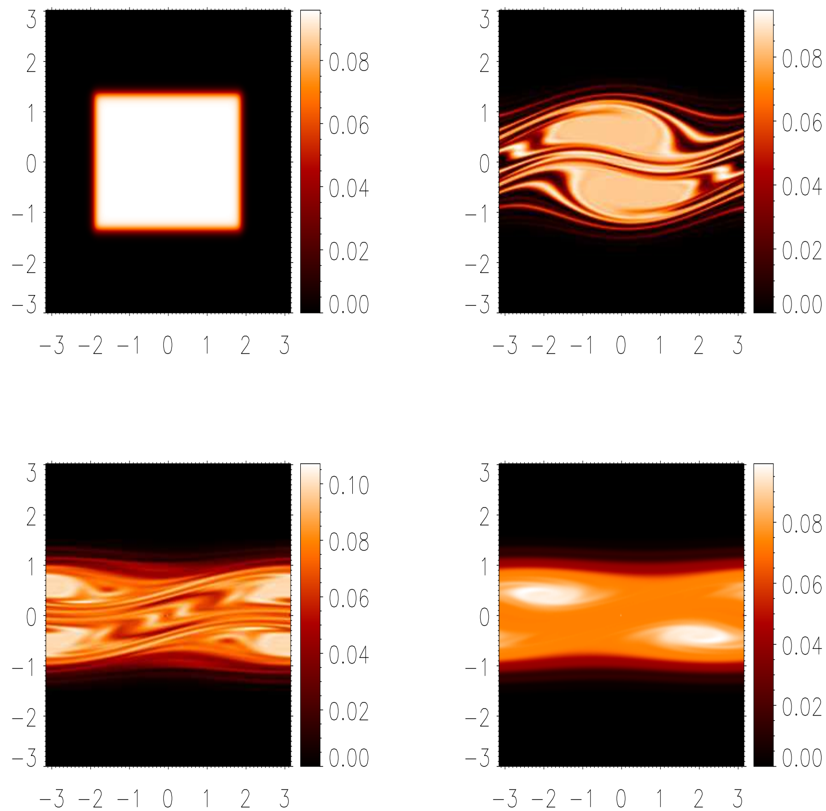
$$U[f] = \int (p^2/2) f(\theta, p, t)\mathrm{d}\theta \mathrm{d}p + 1/2 - (M_x^2 + M_y^2)/2$$

and momentum

$$P[f] = \int p f(\theta, p, t)\mathrm{d}\theta \mathrm{d}p$$

are conserved.

Water bag



Exact inhomogeneous steady states

Vlasov equation for the HMF model

$$\frac{\partial f}{\partial t} + p\frac{\partial f}{\partial \theta} - \frac{\partial V[f](\theta,t)}{\partial \theta}\frac{\partial f}{\partial p} = 0$$

where

$$V[f](\theta,t) = \iint d\theta' dp' f(\theta',p',t)\left(1-\cos(\theta'-\theta)\right)$$

Any function

$$f_S(\theta,p) = F(h(\theta,p)) \text{ with}$$

with

$$h(\theta,p) = \frac{p^2}{2} + V[f_S](\theta).$$

and

$$V(\theta)[f_S] = 1 - M_x[f_S]\cos(\theta) - M_y[f_S]\sin(\theta)$$

is a stationary solution of the Vlasov equation, once we require that it has the correct properties of a one-particle distribution (Bernstein-Green-Kruskal (BGK) modes).

`P. de Buyl, D. Mukamel`

Non interacting particles

Consider the dynamics of an ensemble of *uncoupled particles* moving in a fixed external field H

$$\epsilon(\theta, p) = \frac{p^2}{2} - H\cos\theta$$

For an arbitrary function $F(\epsilon(\theta, p))$ to be a steady state of the interacting model, H has to satisfy the following self-consistency condition

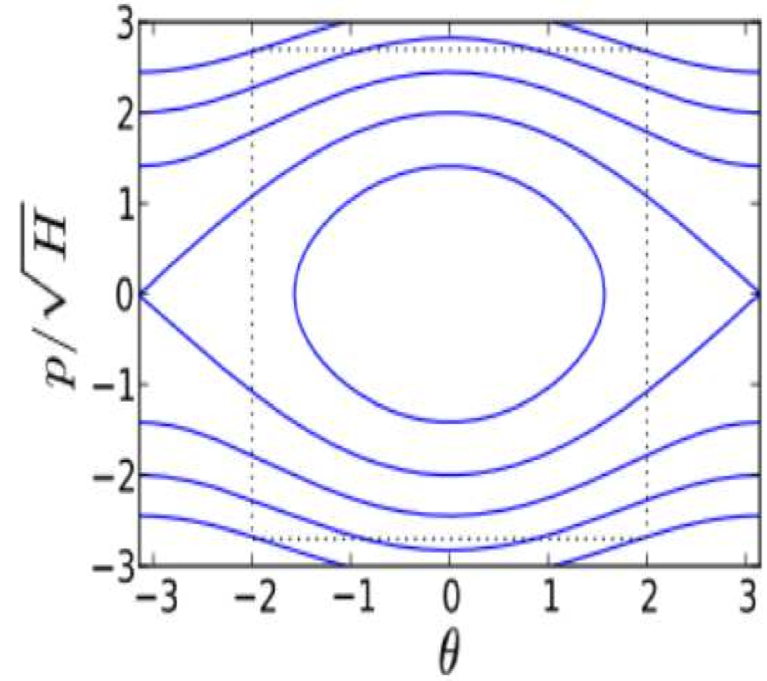
$$H = m_x = \iint d\theta dp F(\epsilon(\theta, p)) \cos\theta \quad ; \quad m_y = 0$$

To relate an initial distribution to the steady state to which it evolves, consider an initial distribution $f_0(\theta, p)$. The dynamics of the uncoupled model is such that particles in a given energy shell $[\epsilon, \epsilon + d\epsilon]$ keep moving inside that shell, eventually reaching a homogeneous distribution within it. As a result, the system attains the following steady state distribution

$$P(\theta, p) = \frac{\iint d\theta' dp' f_0(\theta', p')\, \delta\left(\epsilon(\theta', p') - \epsilon(\theta, p)\right)}{\iint d\theta' dp'\, \delta\left(\epsilon(\theta', p') - \epsilon(\theta, p)\right)}$$

Initial waterbag

$$f_0(\theta, p) = \begin{cases} (4\Delta\theta\Delta p)^{-1} & , \text{ for } |\theta| \le \Delta\theta \text{ and } |p| \le \Delta p \ , \\ 0 & , \text{ otherwise.} \end{cases}$$



Energy distribution

In order to evaluate $P(\theta, p)$, it is convenient to first consider the energy distribution $P_\epsilon(\epsilon)$.
For the waterbag initial state it is given by

$$P_\epsilon(\epsilon) = \frac{1}{4\Delta\theta\Delta p}\int d\theta \int_{-\Delta p}^{\Delta p} dp\, \delta(\frac{p^2}{2} - H\cos\theta - \epsilon)$$

Integrating over p

$$P_\epsilon(\epsilon) = \frac{1}{2\Delta\theta\Delta p}\int d\theta\, \frac{1}{\sqrt{2(\epsilon + H\cos\theta)}}\ ,$$

for $-H \leq \epsilon \leq \Delta p^2/2 - H\cos\Delta\theta$ and zero outside this range.
The integration over θ need to be done in the domain enclosed by the initial waterbag

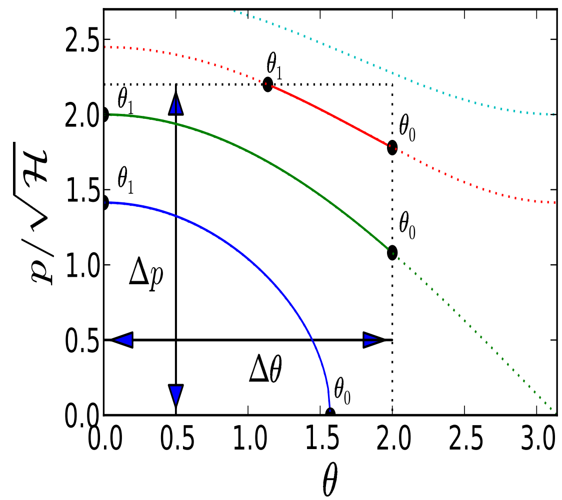
$$0 \leq \epsilon + H\cos\theta \leq \frac{\Delta p^2}{2}.$$

Thus,

$$P_\epsilon(\epsilon) = \frac{\sqrt{2}}{2\Delta\theta\Delta p}\int_{\theta_1}^{\theta_0} d\theta\, \frac{1}{\sqrt{(\epsilon + H\cos\theta)}} \qquad (1)$$

where θ_0 and θ_1 satisfy the conditions described in the next slide.

Integration limits



$$\theta_0 = \begin{cases} \arccos(-\epsilon/H) & , \text{ for } -H < \epsilon < -H\cos\Delta\theta \\ \Delta\theta & , \text{ for } \epsilon \geq -H\cos\Delta\theta\ , \end{cases}$$

$$\theta_1 = \begin{cases} 0 & , \text{ for } -H < \epsilon \leq \Delta p^2/2 - H \\ \arccos(\frac{\Delta p^2/2-\epsilon}{H}) & , \text{ for } \Delta p^2/2 - H \leq \epsilon < \\ & \quad \Delta p^2/2 - H\cos\Delta\theta \\ \Delta\theta & , \text{ for } \epsilon \geq \Delta p^2/2 - H\cos\Delta\theta\ . \end{cases}$$

Steady state distribution

In the steady state, the distribution is such that all the microstates corresponding to a given energy are equally probable. The boundaries on (θ, p) imposed by the initial waterbag are no longer valid. Thus, the steady state distribution $P(\theta, p)$ may be expressed as

$$P(\theta, p) = \frac{1}{4\Delta\theta\Delta p} \frac{P_\epsilon(\epsilon(\theta, p))}{Q_\epsilon(\epsilon(\theta, p))} \equiv \bar{P}_\epsilon(\epsilon(\theta, p)) ,$$

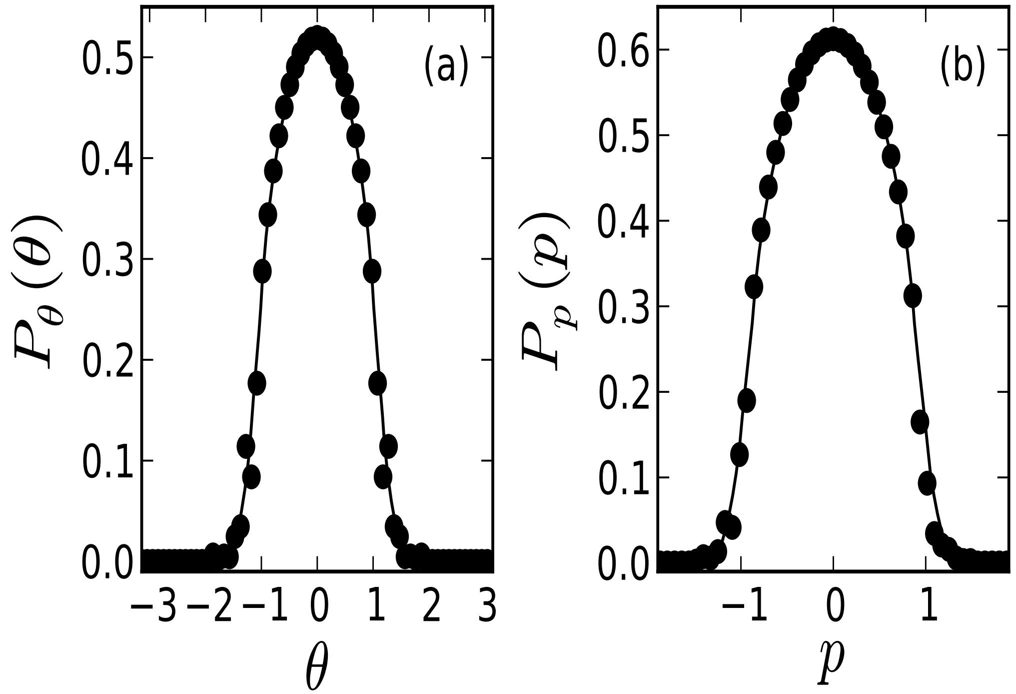
where $Q(\epsilon)$ is given by $P(\epsilon)$ with the bounds given by the waterbag removed. Integrating over p, it is straightforward to express, without any approximation, the marginal in θ as

$$P_\theta(\theta, H) = \sqrt{2} \int_{-H \cos\theta}^{\infty} d\epsilon \, \frac{1}{\sqrt{(\epsilon + H \cos\theta)}} \, \bar{P}_\epsilon(\epsilon) .$$

and then impose the *consistency* relation

$$H = \int_{-\pi}^{+\pi} d\theta P_\theta(\theta, H) \cos\theta .$$

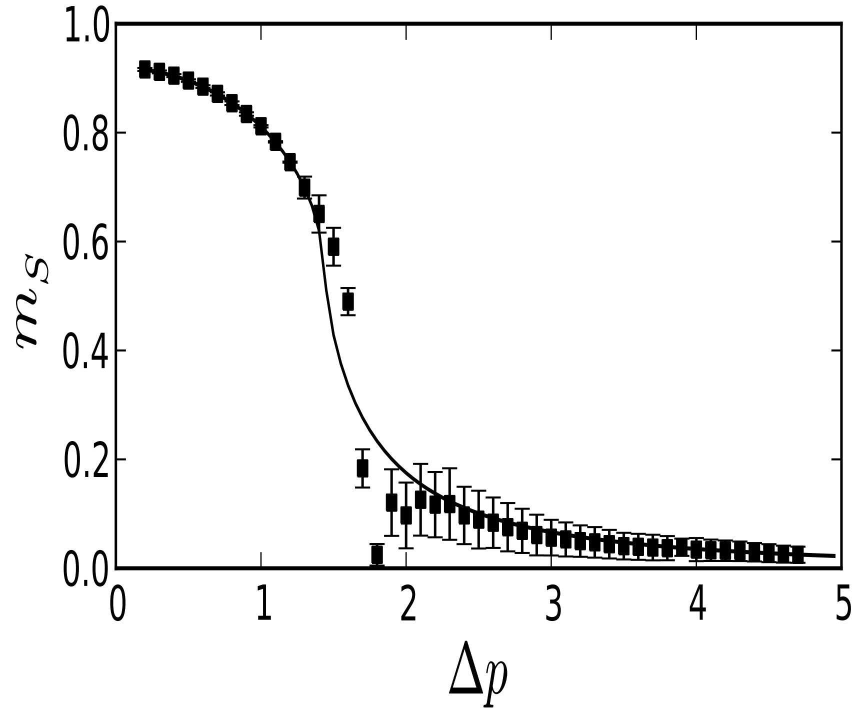
Marginals



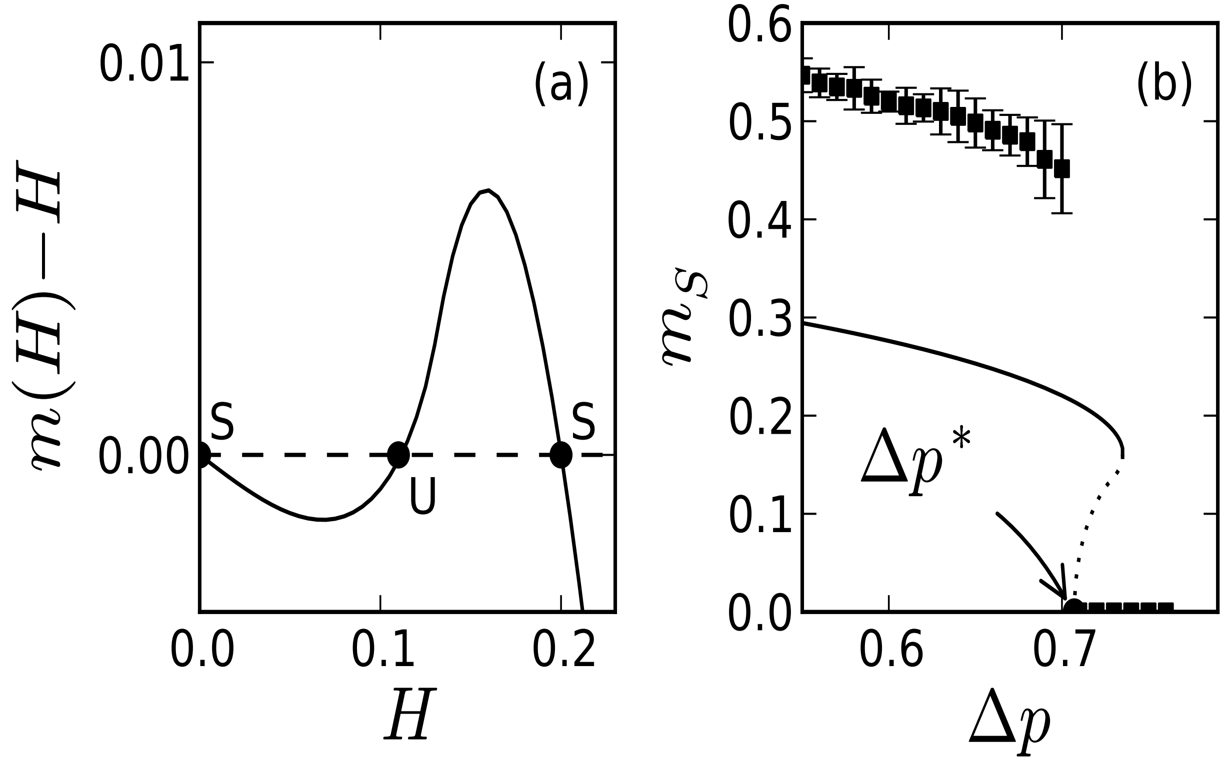
Marginals in θ and in p of the steady state distribution for $\Delta\theta = 1$ and $\Delta p = 1$.

Results

Unless $\Delta\theta = \pi$, the r.h.s of the consistency relations is proportional to $\sqrt{H}$. This implies that there is always a tail in magnetization at large values of Δp. Here, $\Delta\theta = 1$.



$\Delta\theta = \pi$



Generic homogeneous distribution

$$f_0(\theta, p) = \frac{\phi_0(p)}{2\pi}$$

$$P_{QSS}(\epsilon) = \frac{1}{2\pi} \frac{\int d\theta' \phi_0(\sqrt{2(H\cos\theta' + \epsilon)})(H\cos\theta' + \epsilon)^{-1/2}}{\int d\theta' (H\cos\theta' + \epsilon)^{-1/2}}$$

Formally expanding around $H = 0$

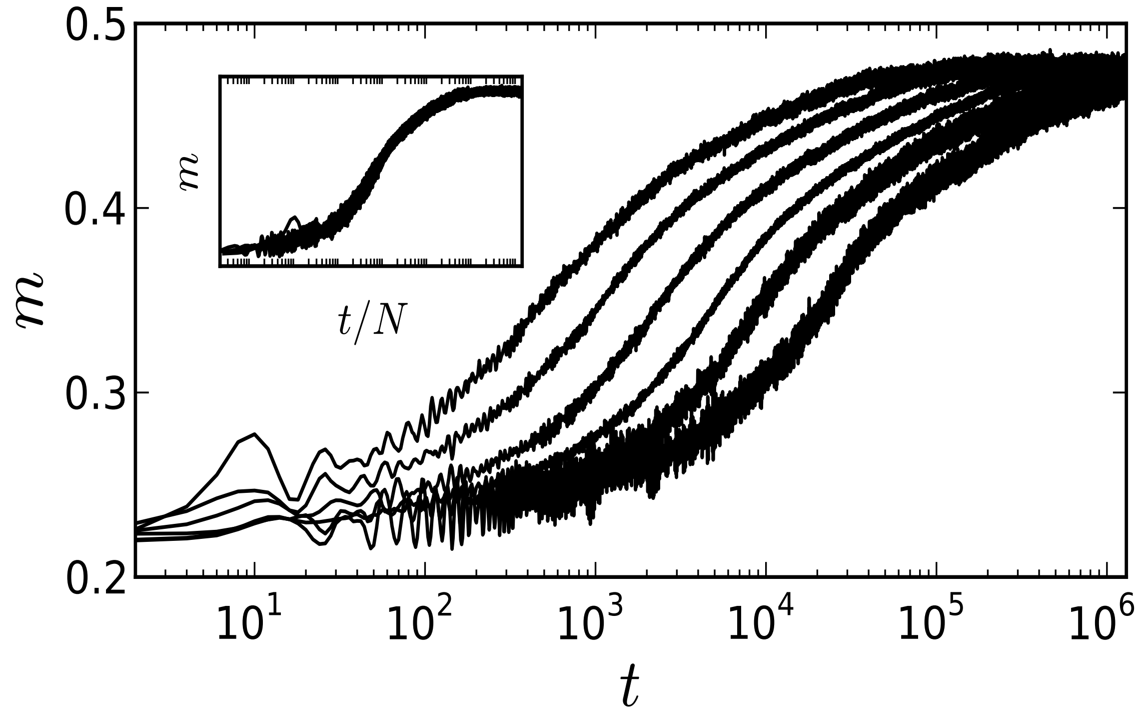
$$P_{QSS}(\theta, p) = \frac{\phi_0(p)}{2\pi} - \frac{\phi_0^{'}(p)\cos\theta}{2\pi p} H + \mathrm{O}(H^2)$$

$$H = \int d\theta dp P_{QSS}(\theta, p)\cos\theta = \int d\theta dp \frac{\phi_0(p)}{2\pi}\cos\theta - \int d\theta dp \frac{\phi_0^{'}\cos^2\theta}{2\pi p} H + O(H^2)$$

which gives

$$1 + \frac{1}{2}\int dp \frac{\phi_0^{'}(p)}{p} = 0$$

Relaxation time scales



Perturbed hamiltonian

$$H(t) = H_0 + H_{\rm ext} = H_0 - K(t) \sum_{i=1}^{N} b(q_i)$$

Vlasov equation as a Liouville equation for f

$$\frac{\partial f}{\partial t} - \mathcal{L}(q,p,t)[f]f = 0$$

$$\mathcal{L}(q,p,t)[f] \equiv -p\frac{\partial}{\partial q} + \frac{\partial \Phi(q,t)[f]}{\partial q}\frac{\partial}{\partial p} - K(t)\frac{\partial b}{\partial q}\frac{\partial}{\partial p}$$

Stationary state of the unperturbed Hamiltonian H_0

$$\mathcal{L}_0(q,p)[f_0]f_0 = 0$$

$$\mathcal{L}_0(q,p)[f_0] = -p\frac{\partial}{\partial q} + \frac{\partial \bar{\Phi}(q)[f_0]}{\partial q}\frac{\partial}{\partial p}$$

A. Patelli, S. Gupta, C. Nardini

Linearized Vlasov equation

$$f(q,p,t) = f_0(q,p) + \Delta f(q,p,t)$$

$$\Delta f(q,p,0) = 0$$

$$\frac{\partial \Delta f}{\partial t} - \mathcal{L}_0(q,p)[f_0]\Delta f = \mathcal{L}_{\rm ext}(q,p,t)[\Delta f] f_0(q,p)$$

$$\mathcal{L}_{\rm ext}(q,p,t)[\Delta f] = \frac{\partial v_{\rm eff}(q,t)[\Delta f]}{\partial q}\frac{\partial}{\partial p}$$

$$v_{\rm eff}(q,t)[\Delta f] = \Phi(q,t)[\Delta f] - K(t)b(q)$$

Evolution of an observable

Formal solution of the linearized equation

$$\Delta f(q,p,t) = \int_0^t d\tau \; e^{(t-\tau)\mathcal{L}_0(q,p)[f_0]} \mathcal{L}_{\mathrm{ext}}(q,p,\tau)[\Delta f] f_0(q,p)$$

$$\langle \Delta a(q) \rangle (t) \equiv \langle a(q) \rangle (t) - \langle a(q) \rangle_{f_0} = \int dqdp \; a(q) \Delta f(q,p,t)$$

$$\langle \Delta a(q) \rangle (t) = -\int_0^t d\tau \int dqdp \left\langle \frac{\partial a(t-\tau)}{\partial p} \frac{\partial v_{\mathrm{eff}}(q,\tau)[\Delta f]}{\partial q} \right\rangle_{f_0}$$

with

$$\langle a(q) \rangle_{f_0} \equiv \iint dqdp \; a(q) f_0(q,p), \quad a(t-\tau) = e^{-(t-\tau)\mathcal{L}_0(q,p)[f_0]} a(q)$$

Solution in Laplace-Fourier

$$\widehat{\Delta f}(k,p,\omega) = \frac{1}{2\pi}\int_0^\infty dt \int dq \exp(-ikq + i\omega t)\Delta f(q,p,t)$$

For homogeneous stationary states $f_0 = P(p)$, since we are interested in observables that depend only on q

$$\int dp\, \widehat{\Delta f}(k,p,\omega) = \frac{\widehat{K}(\omega)\widetilde{b}(k)}{2\pi\widetilde{v}(k)}\left[\frac{\epsilon(k,\omega)-1}{\epsilon(k,\omega)}\right]$$

where $\widetilde{v}(k)$ is the Fourier transform of the two-body potential

$$\epsilon(k,\omega) = 1 - 2\pi k\widetilde{v}(k)\int \frac{dp}{kp-\omega}\frac{\partial P(p)}{\partial p}$$

is the so-called plasma response dielectric function and $K(\omega)$ is the Laplace transform of $K(t)$.

Application to HMF

$$v(q) = 1 - \cos q, \quad \widetilde{v}(k) = \left[\delta_{k,0} - \frac{\delta_{k,-1} + \delta_{k,1}}{2}\right]$$

$$b(q) = \cos q, \quad \widetilde{b}(k) = \frac{\delta_{k,-1} + \delta_{k,1}}{2}$$

$$\widehat{K}(\omega) = -\frac{h}{i\omega}$$

$$\int dp\, \widehat{\Delta f}(\pm 1, p, \omega) = \frac{ih}{2\pi\omega}\left[\frac{1 - \epsilon(\pm 1, \omega)}{\epsilon(\pm 1, \omega)}\right]$$

$$\int dp\, \widetilde{\Delta f}(\pm 1, p, t) = \frac{ih}{4\pi^2}\int_L d\omega\, \frac{1}{\omega}\left[\frac{1}{\epsilon(\pm 1, \omega)} - 1\right]e^{-i\omega t}$$

$$\langle m_x \rangle(t) = \frac{ih}{2\pi}\int_L d\omega\, \frac{1}{\omega}\left[\frac{1}{\epsilon(\pm 1, \omega)} - 1\right]e^{-i\omega t}$$

while $\langle m_y \rangle(t) = 0$ for all times.

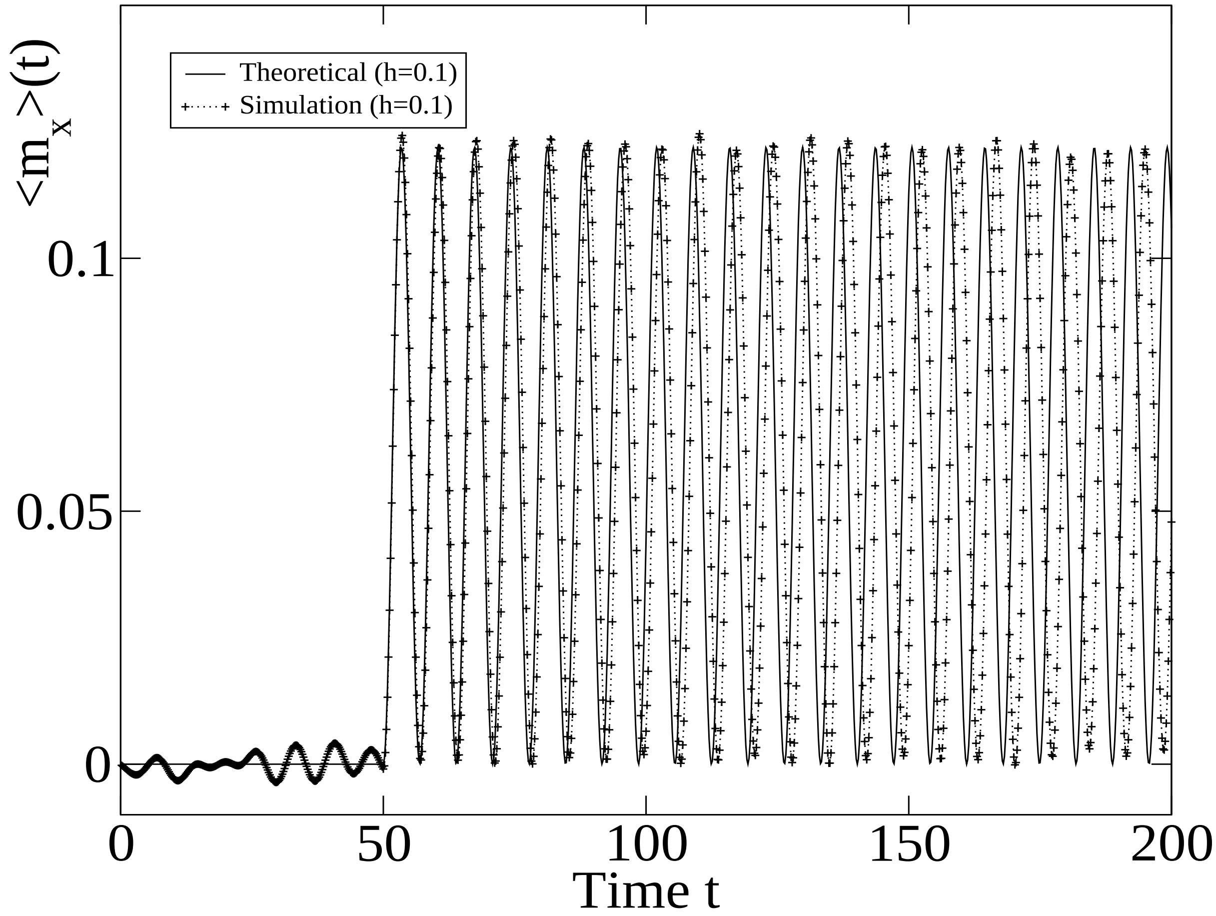
Homogeneous waterbag

$$P(p) = \frac{1}{2\pi}\frac{1}{2p_0}\Big[\Theta(p+p_0) - \Theta(p-p_0)\Big]; \quad p \in [-p_0, p_0]$$

$$\epsilon(\pm 1, \omega) = 1 - \frac{1}{2(p_0^2 - \omega^2)}$$

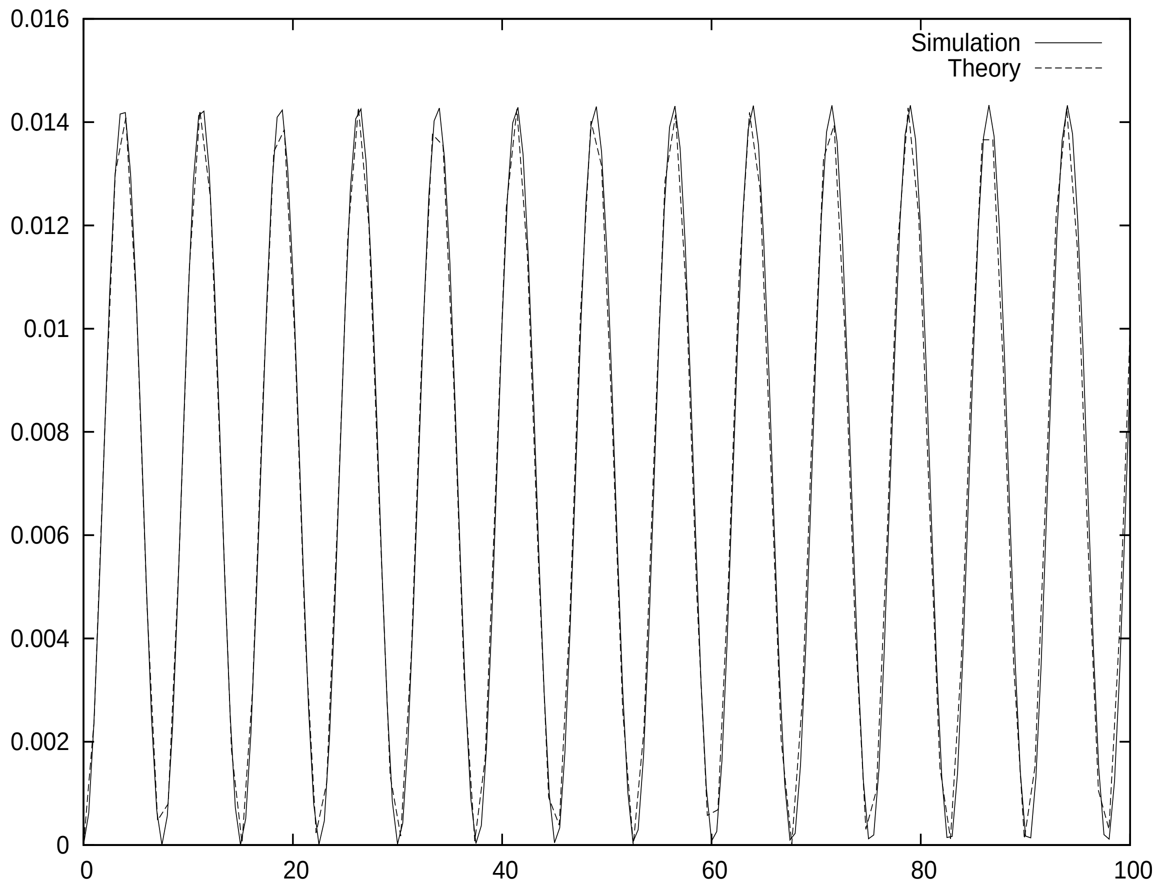
$$\langle m_x \rangle (t) = \frac{2h}{2p_0^2 - 1} \sin^2\left(\frac{t}{2}\sqrt{p_0^2 - \frac{1}{2}}\right)$$

Permanent oscillations N-body



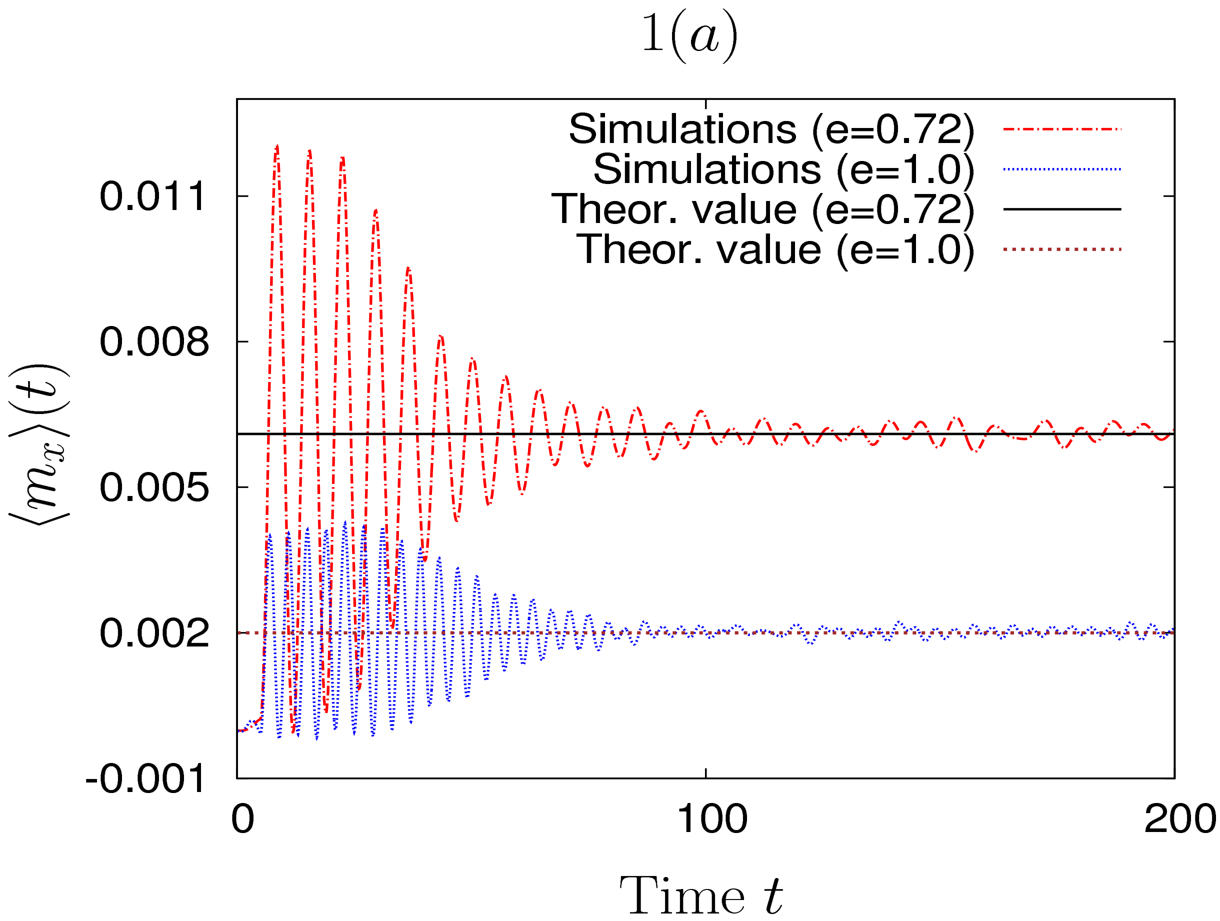
N-body simulation, $N = 10^5$, $p_0 = 1.1$, $U = .7$

Permanent oscillations Vlasov



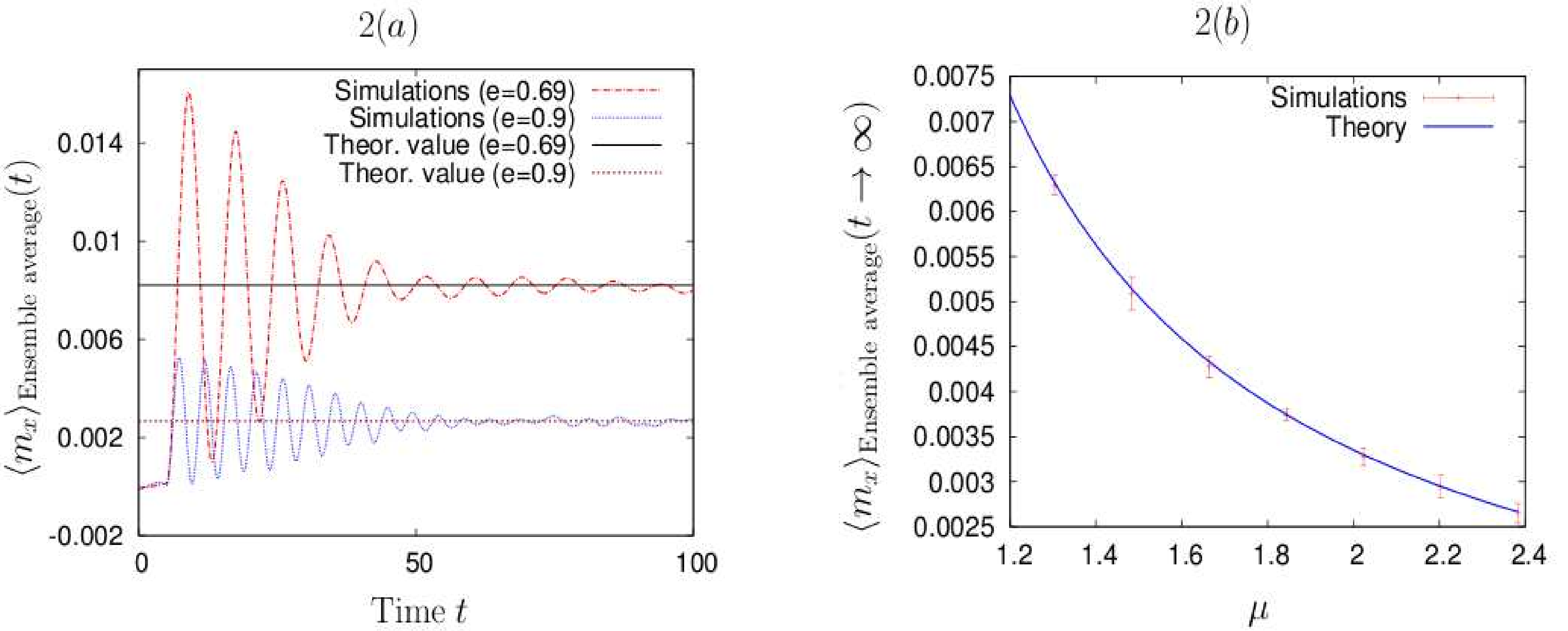
Vlasov simulation ($N = \infty$), $U = 0.7$.

Average over initial conditions



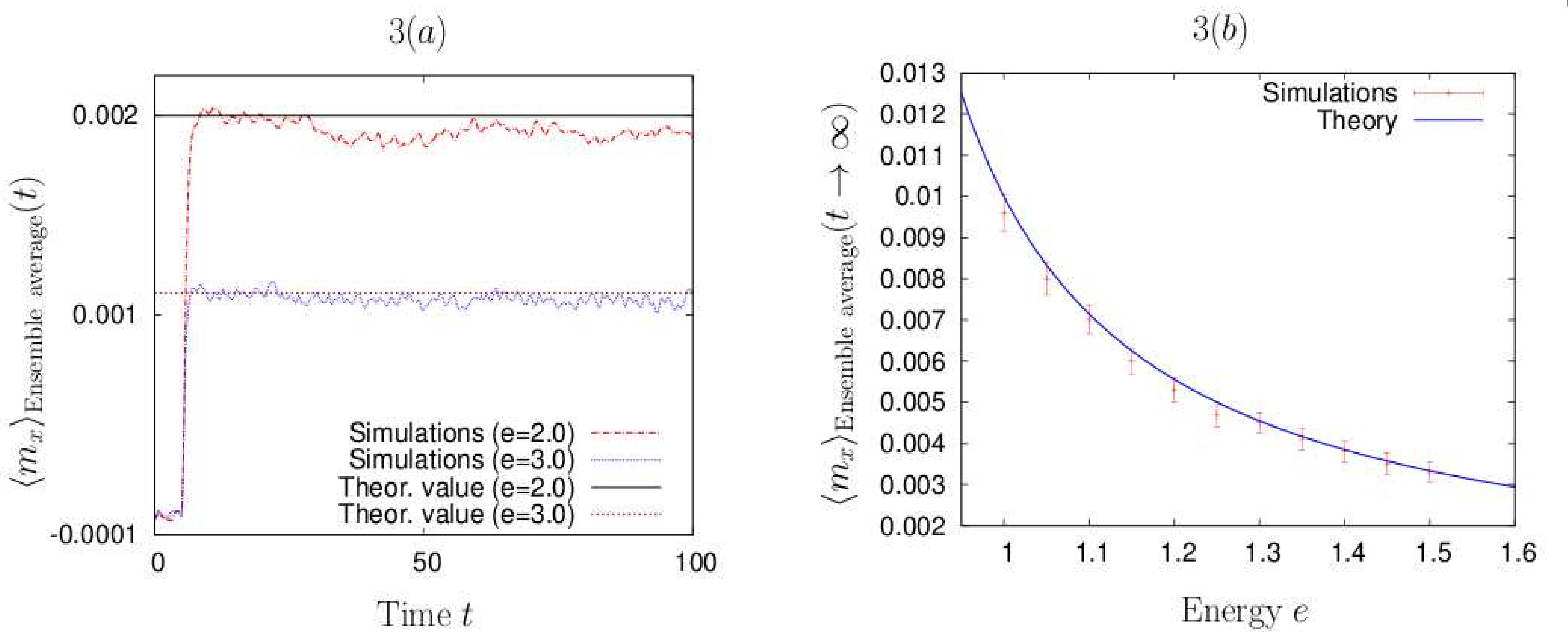
$N = 10^5$

Fermi-Dirac



$$P(p) = \frac{A}{2\pi} \frac{1}{1 + \exp(\beta(p^2 - \mu))}$$

Gaussian



$$P(p) = \sqrt{\frac{\beta}{2\pi}} \exp(-\beta p^2/2)$$

Relaxation to equilibrium