About the Boltzmann-Grad limit

Laure SAINT-RAYMOND

Joint work with Isabelle GALLAGHER
and Benjamin TEXIER

Kinetic equations as thermodynamic limits

Kinetic equations describe the dynamics of (rarefied) gases out of thermodynamic equilibrium.

- The <u>unknown</u> is the probability distribution function
$$f = f(t, x, v), \quad t \in \mathbb{R}, x \in \Omega, v \in \mathbb{R}^3$$
Such a statistical description makes sense if the number of elementary particles $N >> 1$.

- The <u>evolution equation</u> can be of different kinds depending on the nature and on the scaling of the elementary interactions

(1) Mean field versus collisional dynamics

- **weak coupling**

$$\begin{cases} \text{range}(\Phi) \sim 1 \\ \text{strength}(\Phi) \ll 1 \end{cases}$$

each particle feels the effect of the field created by all other particles.

$$F_N(x_i) = -\frac{1}{N}\sum_{j\neq i} \nabla_x \Phi(x_i - x_j)$$

$$\sim -\nabla_x \Phi * \int f \, dx$$

- **low density**

$$\begin{cases} \text{range}(\Phi) \ll 1 \\ \text{strength}(\Phi) \sim 1 \end{cases}$$

each particle moves freely until it undergoes a collision

$\rightsquigarrow$ stability?

② Collisional transport equations

- In the absence of external and interaction forces, particles have uniform rectilinear motion, and f satisfies the <u>free transport equation</u>

$$\partial_t f + v \cdot \nabla_x f = 0 .$$

- If the interactions are localized, trajectories are essentially polylines, and f satisfies an integro-differential transport equation

$$\partial_t f + v \cdot \nabla_x f = Q(f)$$

The precise form of the <u>collision operator</u> $Q(f)$ depends on the microscopic interactions (number of involved particles, conserved quantities ...)

③ Elementary collisions and scattering

- In the classical case of binary elastic collisions, the total momentum and kinetic energy are conserved

$$Q(f) = \int \left[f(v_1') f(v') - f(v) f(v_1) \right] b(v - v_1, \omega) \, d\omega \, dv_1$$

where the pre-collisional velocities v_1' and v' are parametrized by the deflection angle $\omega \in S^2$.

The cross-section b gives the statistical repartition of ω.

- For elementary waves having cubic interactions, we get the phonon equation where $Q(f)$ is cubic and involves the resonance relation.

Weak turbulence is therefore expected to be the counterpart of the Boltzmann-Grad limit.

The convergence result

Theorem (Lanford, King) Assume that the repulsive potential ϕ is compactly supported, radial, monotonic and singular at 0. Let f_0 be continuous, with exponential decay at large energies.
Consider the system of N particles, initially distributed according to f_0 and "independent", governed by

$$\frac{dx_i}{dt} = v_i \quad , \quad \frac{dv_i}{dt} = \frac{-1}{\varepsilon} \sum_{j \neq i} \nabla \phi \left(\frac{x_i - x_j}{\varepsilon} \right)$$

Then, in the Boltzmann-Grad limit $\varepsilon \to 0$ $(N\varepsilon^2 \sim 1)$, its distribution function f_ε converges to the solution f of the Boltzmann equation with initial data f_0, in the sense of observables.

(1) The notion of convergence

- By definition, "$f_\varepsilon \to f$ in the sense of observables" means that, for any test function $\varphi \in C_c(\mathbb{R}^3_v)$

$$\int f_\varepsilon \varphi(v)\, dv \to \int f \varphi(v)\, dv \quad \text{uniformly on any compact in } t, x$$

In particular, there is no incompatibility between the convergence of observables and the fact that the Boltzmann dynamics does not predict pathological trajectories involving recollisions. These bad events have vanishing probability in the Boltzmann-Grad limit.

- Lanford's result is actually more precise insofar as it gives the convergence of all marginals outside from diagonals.

② The sense of time

- The convergence result could seem paradoxical as noted by Loschmidt since the original Hamiltonian dynamics is reversible, which is not the case of the Boltzmann dynamics.
More precisely, Boltzmann's H theorem states
$$\partial_t \int f \log f \, dv + \nabla_x \cdot \int f \log f \, v \, dv \leq 0$$
- The sense of time in the thermodynamic limit is actually fixed by the (arbitrary) distinction between pre and post-collisional configurations.
At each collision, we loose part of the information.
This implies in particular that the initial data should play a special role in the proof of convergence.

The BBGKY hierarchy

- The starting point of the derivation is the Liouville equation for the N-particles distibution
$$\partial_t F_N + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} F_N - \sum_{i \neq j} \nabla_x \Phi(x_i - x_j) \cdot \nabla_{v_i} F_N = 0$$
that we integrate to get the equations for the marginals $F_N^{(s)}$.
- For hard spheres, the collision operator appears as a boundary term. By analogy, for compactly supported potentials we introduce truncated marginals so that the collision operator has the same form. We then obtain the coupled system
$$\partial_t F_N^{(s)} + \sum_{i=1}^{s} v_i \cdot \nabla_{x_i} F_N^{(s)} - \sum_{i,j=1}^{s} \nabla_x \Phi(x_i - x_j) \cdot \nabla_{v_i} F_N^{(s)} = \sum_{k=1}^{N-s} C_N^{s,s+k} F_N^{(s+k)}$$
- Mild solutions to this hierarchy are defined by Duhamel's formula
$$F_N^{(s)}(t) = H_s(t) F_{N,0}^{(s)} + \int_0^t H_s(t-\tau) \sum_{k=1}^{N-s} C_N^{s,s+k} F_N^{(s+k)}(\tau)\, d\tau$$

① Uniform loss continuity estimates

To deal with the general form of the hierarchy, we need summability estimates which come from cluster expansions. For the sake of simplicity, we skip this technical step and consider the case of hard spheres

$$C_N^{s,s+1} F_N^{(s+1)} = \sum_{i=1}^{s} (N-s)\varepsilon^2 \int_{\Sigma^{i,s+1}} \nu^{i,s+1} \cdot (v_{s+1}-v_i)\, F_N^{(s+1)}\, d\sigma_{s+1}\, dv_{s+1}$$

- The transport operator satisfies some maximum principle as well as the conservation of the s-particles Hamiltonian, so that the group H_s is an isometry in suitable functional spaces.
- The collision operators are not bounded in these functional spaces
  - because of the sum with respect to i (loss on high correlations)
  - because of the factor $\nu^{i,s+1} \cdot (v_{s+1}-v_i)$ (loss on high energies)

② A Cauchy-Kowalewski argument

- Define more precisely the Banach space $X_{\varepsilon,\beta,\mu}$

$$\| F_N \|_{\varepsilon,\beta,\mu} = \sup_s \exp(\mu s) \sup_{X_s,V_s} |F_N^{(s)}| \exp(\beta E_\varepsilon(X_s,V_s))$$

with $E_\varepsilon(X_s,V_s) = \frac{1}{2}\sum_{i=1}^s |v_i|^2 + \sum_{1\leq i<j\leq s} \Phi_\varepsilon(x_i-x_j)$

(grand canonical formalism : μ chemical potential, β^{-1} temperature)

- We then expect the pointwise loss

$$|C_N^{s,s+1} F_N^{(s+1)}| \leq C(\beta,\mu)\left(s+\sum_{i=1}^s |v_i|^2\right) e^{-\mu s-\beta E_\varepsilon(Z_s)} \|F_N\|_{\varepsilon,\mu,\beta}$$

to be compensated by the time integration.

- A fixed point argument gives the local existence of a unique solution to the BBGKY hierarchy, together with the uniform estimate

$$\sup_{t\in[0,T]} \| F_N \|_{\varepsilon,\beta_0-ct,\mu_0-ct} \leq C_0 \text{ as long as } \beta_0 - cT > \frac{\beta_0}{2}$$

Strategy of the convergence proof

What will be proved is the convergence of the BBGKY hierarchy to the Boltzmann hierarchy, which admits as particular solutions - tensorized marginals $f^{\otimes s}$ with f solution to the Boltzmann equation. (Note that all solutions are actually convex combinations of such tensorized solutions)

Given the special role of the initial data, the starting point here is the iterated Duhamel formula.

$$F_N^{(s)}(t) = \sum_{k=0}^{N-s} \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} H_s(t-t_1) C_N^{s,s+1} \dots C_N^{s+k-1,s+k} H_{s+k}(t_k) F_{N,0}^{(s+n)}$$

Provided that $F_{N,0}$ converges in some "strong" sense to F_0, we have only to prove the convergence of transport and collision operators in the Boltzmann-Grad limit.

① Reductions via dominated convergence.

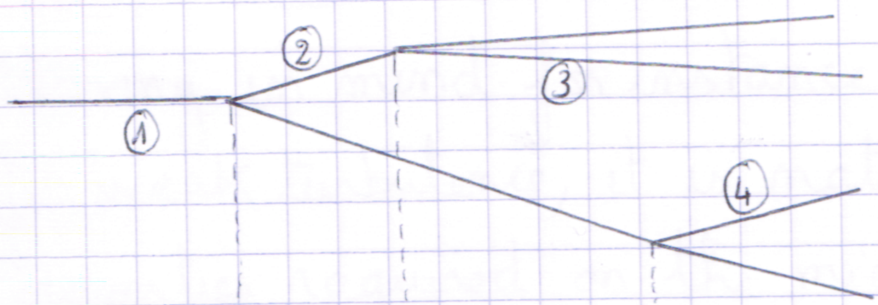
- From cluster estimates, we deduce that we can neglect the contributions of multiple collisions

- Using the uniform a priori estimates
$$|F_N^{(s)}(t, X_s, V_s)| \leq C_0 \exp(-\mu(t)s) \exp(-\beta(t) E_\varepsilon(X_s, V_s))$$
we can further restrict to
  - a finite number n of collisions
  - involving particles with bounded energy $E_\varepsilon(X_{s+n}, V_{s+n}) \leq R^2$
  - at times separated at least by δ

- We are then reduced to the study of elementary functionals, the sum of which are good approximations of the observables (with an error depending on $\varepsilon, n, R, \delta$)

② Reformulation in terms of pseudo-trajectories

- Each one of the elementary functionals can be associated to some pseudo-dynamics
  - characteristics are followed backwards in time between t_i and t_{i+1}
  - an additional particle is "adjoined" at t_{i+1}



⚠ the total number of particles is not conserved!

- Transport operators H_{s+i} coincide with free transport S_{s+i} if and only if there is no recollision

Pathological trajectories (involving recollisions will be eliminated by truncations of the domains of integration.

③ Role of the scattering.

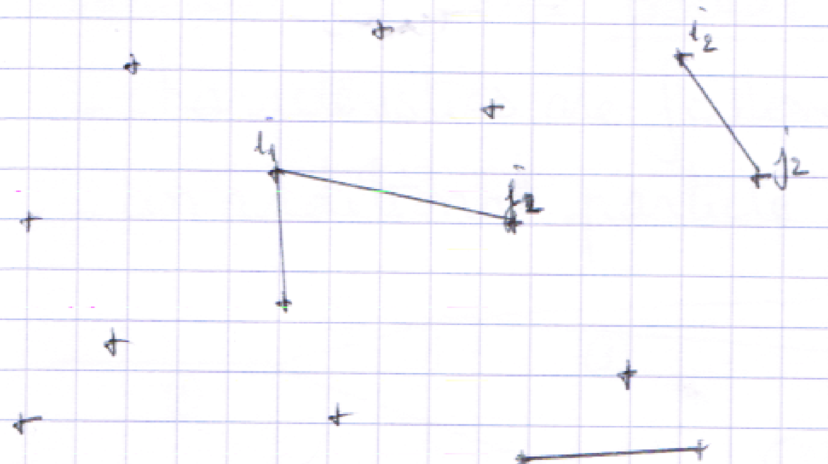
The key argument of the proof is therefore based on some geometrical lemmas, describing the set of velocities leading to some recollision. Because we focus here on transport in the whole space, the geometric sets to be excluded are very simple (cones and cylinders).

- The first point is to check that these sets are stable under small translations.

- The second question is to understand how they are modified by the inverse scattering (the integration being on precollisional velocities)

Limitations and perspectives

① About the time of validity

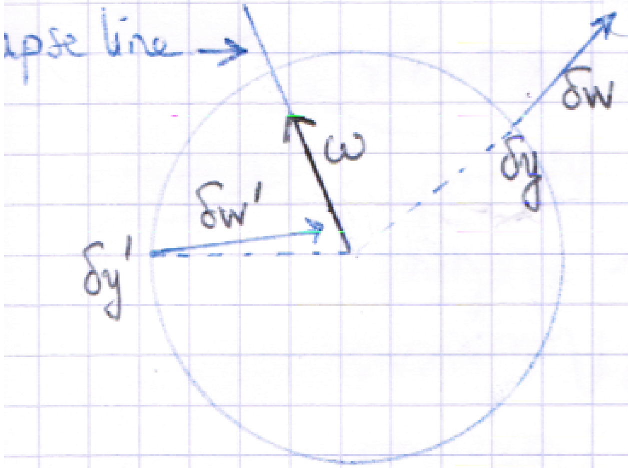


Propagation of chaos can be valid for at most $N/2$ collisions (phase transition to a giant component)

Representation of collisions without dynamics

- How correlations are taken into account by norms $X_{\varepsilon,\beta,\mu}$?
- Starting with a much larger number of particles, can we use the geometry of the flow (dispersion) to prove some decay of correlations?
- Is there a good notion of "fluctuation around equilibrium"?

② About the scattering



In the Boltzmann equation, quadratic collisions are parametrized by the deflection angle.

Having in mind for instance the derivation of kinetic models for weak turbulence, it is natural to investigate the properties required on the microscopic interactions.

- What about cubic or quartic (instead of quadratic) interactions?
- How to deal with long range interactions?