Definition of the functions

In[1]:=
```
f[z_] := E^z - 1 - z
```

In[2]:=
```
PhiA[x_, v_, s_, z_] := - 1 - x / (x + 2 v + s)  + v^2 z^4 E^z / ((x + 2 v + s) ^2 f[z] ^2) -
   x v z^2 E^z / ((x + 2 v + s) ^2 f[z]) ; PhiB[x_, v_, s_, z_] :=
  - 1 + x / (x + 2 v + s)  - v^2 z^4 E^z / ((x + 2 v + s) ^2 f[z] ^2) ;
PhiC[x_, v_, s_, z_] :=  1 - x / (x + 2 v + s)  - 2 v z^2 E^z / ((x + 2 v + s) f[z]) +
    v^2 z^4 E^z / ((x + 2 v + s) ^2 f[z] ^2)  + x v z^2 E^z / ((x + 2 v + s) ^2  f[z]);
```

Lemma 7: Asymptotics of the derivatives near the end

We compute the derivative of PhiA with respect to x, where z is an implicit parameter characterized by z (E^z - 1)/f[z] == 2 + s/v

In[4]:=
```
ImplicitD[PhiA[x, v, s, z], z (E^z - 1) / f[z] == 2 + s / v, z, x] // FullSimplify
```

Out[4]=

$$-\frac{1}{(s+2v+x)^3\,(1-e^z+z)^2}\Big(s^2\,(1-e^z+z)^2+s\,(-1+e^z-z)\,\big(-((4v+x)\,(1+z))+e^z\,\big(x+v\,(4+z^2)\big)\big)+v\,\Big(x\,(1-e^z+z)\,\big(2\,(1+z)+e^z\,(-2+z^2)\big)+2v\,\big(2\,(1+z)^2+e^{2z}\,(2+z^2)+e^z\,(-4+z\,(-4+z\,(-1+(-1+z)\,z)))\big)\Big)\Big)$$

We compute the three derivatives of PhiA with respect to x, v and s respectively, parametrize them using β *and then compute their series expansion at* $\beta = \left(\frac{1}{e}\right)^-$

In[5]:=
```
Series[{ImplicitD[PhiA[x, v, s, z], z (E^z - 1) / f[z] == 2 + s / v, z, x],
        ImplicitD[PhiA[x, v, s, z], z (E^z - 1) / f[z] == 2 + s / v, z, v],
        ImplicitD[PhiA[x, v, s, z], z (E^z - 1) / f[z] == 2 + s / v, z, s]} /.
      v → beta E^(-z) f[z] /. x → z^2 / E - z beta (1 - E^(-z)) /.
    s → beta (z + z E^(-z) + 2 E^(-z) - 2) /. z → E beta + Log[beta],
  {beta, 1 / E, -1}, Direction → "FromBelow"] // FullSimplify
```

Out[5]=

$$\left\{-\frac{1}{e\left(\text{beta}-\frac{1}{e}\right)^2}-\frac{3}{4\left(\text{beta}-\frac{1}{e}\right)}+O\left[\text{beta}-\frac{1}{e}\right]^0,\ O\left[\text{beta}-\frac{1}{e}\right]^0,\ -\frac{1}{2\left(\text{beta}-\frac{1}{e}\right)}+O\left[\text{beta}-\frac{1}{e}\right]^0\right\}$$

Recall that $\beta - e^{-1} = e^{-1/2}\,\epsilon^{1/2} + \frac{5}{12}\,\epsilon + O(\epsilon^{3/2})$*. The second derivative (the one with respect to v) is O(1). We replace beta as a function of epsilon in the first and third term.*

In[6]:=
```
Series[{- 1/(e (beta - 1/e)^2) - 3/(4 (beta - 1/e)) + O[eps]^0 , - 1/(2 (beta - 1/e)) + O[eps]^0} /.
    beta → 1 / E + E^(-1 / 2) eps^(1 / 2) + 5 / 12 eps + C eps^(3 / 2),
  {eps, 0, -1}, Direction → "FromBelow"] // FullSimplify
```

Out[6]= $\left\{-\frac{1}{\text{eps}}+\frac{1}{\sqrt{O[\text{eps}]}},\ \frac{1}{\sqrt{O[\text{eps}]}}\right\}$

This gives the first three derivatives of Lemma 7. Let us move on to PhiB

In[7]:=
```
Series[{ImplicitD[PhiB[x, v, s, z], z (E^z - 1) / f[z] == 2 + s / v, z, x],
        ImplicitD[PhiB[x, v, s, z], z (E^z - 1) / f[z] == 2 + s / v, z, v],
        ImplicitD[PhiB[x, v, s, z], z (E^z - 1) / f[z] == 2 + s / v, z, s]} /.
      v → beta E^(-z) f[z] /. x → z^2 / E - z beta (1 - E^(-z)) /.
    s → beta (z + z E^(-z) + 2 E^(-z) - 2) /. z → E beta + Log[beta],
  {beta, 1 / E, -2}, Direction → "FromBelow"] // FullSimplify
```

Out[7]= $\left\{\frac{3}{4\,e\,\left(\text{beta}-\frac{1}{e}\right)^2}+\frac{1}{O\left[\text{beta}-\frac{1}{e}\right]},\ \frac{1}{O\left[\text{beta}-\frac{1}{e}\right]},\ \frac{1}{O\left[\text{beta}-\frac{1}{e}\right]}\right\}$

The second and third terms are $O(1/(\beta-1/e)) = O(\epsilon^{-1/2})$*. For the first term, replacing* β *as a function of* ϵ*, we obtain*

In[8]:=
```
Series[3/(4 e (beta - 1/e)^2) /.
    beta → 1 / E + E^(-1 / 2) eps^(1 / 2) + 5 / 12 eps + C eps^(3 / 2),
  {eps, 0, -1}, Direction → "FromBelow"] // FullSimplify
```

Out[8]= $\frac{3}{4\,\text{eps}}+\frac{1}{\sqrt{O[\text{eps}]}}$

We finally move to the derivation of PhiC.

In[9]:=
```
Series[{ImplicitD[PhiC[x, v, s, z], z (E^z - 1) / f[z] == 2 + s / v, z, x],
        ImplicitD[PhiC[x, v, s, z], z (E^z - 1) / f[z] == 2 + s / v, z, v],
        ImplicitD[PhiC[x, v, s, z], z (E^z - 1) / f[z] == 2 + s / v, z, s]} /.
      v → beta E^(-z) f[z] /. x → z^2 / E - z beta (1 - E^(-z)) /.
    s → beta (z + z E^(-z) + 2 E^(-z) - 2) /. z → E beta + Log[beta],
  {beta, 1 / E, 0}, Direction → "FromBelow"] // FullSimplify
```

Out[9]= $\left\{\frac{3}{4\left(\text{beta}-\frac{1}{e}\right)}+\frac{29\,e}{48}+O\left[\text{beta}-\frac{1}{e}\right]^1,\ \frac{1}{\text{beta}-\frac{1}{e}}+\frac{5\,e}{4}+O\left[\text{beta}-\frac{1}{e}\right]^1,\ -\frac{3}{2\,e\left(\text{beta}-\frac{1}{e}\right)^2}-\frac{7}{4\left(\text{beta}-\frac{1}{e}\right)}+\frac{37\,e}{160}+O\left[\text{beta}-\frac{1}{e}\right]^1\right\}$

The first term is $O(\epsilon^{-1/2})$*. For the last two terms, replacing* β *as a function of* ϵ*, we obtain*

In[10]:=
```
{Series[1/(beta - 1/E) + 5 E/4 /.
    beta → 1 / E + E^(-1/2) eps^(1/2) + 5/12 eps + C eps^(3/2), {eps, 0, 0},
   Direction → "FromBelow"], Series[-3/(2 E (beta - 1/E)^2) - 7/(4 (beta - 1/E)) /.
    beta → 1 / E + E^(-1/2) eps^(1/2) + 5/12 eps + C eps^(3/2),
   {eps, 0, -1}, Direction → "FromBelow"]} // FullSimplify
```

Out[10]=

$$\left\{\frac{\sqrt{e}}{\sqrt{eps}}+\frac{5\,e}{6}+\sqrt{O[eps]},\ -\frac{3}{2\,eps}+\frac{1}{\sqrt{O[eps]}}\right\}$$

Note that we do not need the term of constant order in the derivative with respect to v

Lemma 8 : Last estimate

We write bx = x/v and bs = s/v

In[11]:=
```
PhiA[bx v, v, bs v, z] // FullSimplify
```

Out[11]=

$$-1+\frac{-bx\,(2+bs+bx)+\frac{e^z z^4}{(1-e^z+z)^2}+\frac{bx\,e^z z^2}{1-e^z+z}}{(2+bs+bx)^2}$$

The expression does not depend on v and we thus define PsiA as follows

In[12]:=
```
PsiA[bx_, bs_, z_ ] := PhiA[bx v, v, bs v, z] // FullSimplify
```

We compute its derivative with respect to bs, where again z is an implicit parameter

In[13]:=
```
ImplicitD[PsiA[bx, bs, z], z (E^z - 1) / f[z] == 2 + bs, z, bs] // FullSimplify
```

Out[13]=

$$\frac{1}{(2+bs+bx)^3}\left(-\frac{2\,z^4\,(1+z)}{(1-e^z+z)^2}+\frac{2\,z^2\left(2+bs+2\,bx+(2+bs+2\,bx)\,z-z^2\right)}{-1+e^z-z}+bx\left(2+bs+bx-2\,(2+bs+bx)\,z+2\,z^2\right)+\frac{(2+bs+bx)\,e^{-z}\,z\,(bx+z)\,(-2+e^z\,(2+(-2+z)\,z))}{2+z^2-2\,\mathrm{Cosh}[z]}\right)$$

Since z depends on bs and goes to 0 as bs goes to 0, we replace bs as function of z in the previous expression...

In[14]:=
```
% /. bs → z (E^z - 1) / f[z] - 2 // FullSimplify
```

Out[14]=

$$-\left(\left(e^{-z}\,(1-e^{z}+z)^{2}\left(e^{z}\,z^{4}\,(2+e^{z}\,(-2+z)+z)+bx^{2}\,(1-e^{z}+z)\left(1+e^{z}\left(-2+e^{z}\,(1-2\,z)+z\left(2+z+z^{2}\right)\right)\right)+bx\,z\left(1+e^{z}\left(-3-e^{2\,z}+5\,z^{2}+z^{3}\,(4+z)+e^{z}\left(3+z^{2}\,(-5+2\,z)\right)\right)\right)\right)\right)\Big/\left((bx+z+bx\,z-e^{z}\,(bx+z))^{3}\left(2+z^{2}-2\,\mathrm{Cosh}[z]\right)\right)\right)$$

... and compute the limit when z -> 0 and bx -> 0

In[15]:=
```
Limit[%, {z → 0, bx → 0}]
```

Out[15]=

0