On the Cheeger constant of hyperbolic surfaces

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Outline

- Hyperbolic surface: Riemannian surface with constant curvature -1.
- The Cheeger constant of a (compact) geometric object X measures its "expansion":

$$h(X) = \inf \left\{ \frac{|\partial A|}{|A|} \,\middle|\, A \subset X, \, |A| \leq \frac{|X|}{2} \right\}.$$

- In general, $h(X)$ is bounded by the Cheeger constant of its universal cover, i.e. by 1 for hyperbolic surfaces.
- Goal: prove that for a large hyperbolic surface X, we have $h(X) \leq \frac{2}{\pi} + o(1)$.
- First: study the same phenomenon on d-regular graphs.

Connectivity of regular graphs

- Let G be a d-regular graph with n vertices $V(G)$.
- Various ways to measure the connectivity of a d-regular graph:
  - Diameter: $\mathrm{diam}(G) = \max\{d_G(x,y) | x, y \in V(G)\}$.
  - Spectral gap: $d - \lambda_2$, where $d = \lambda_1 \geq \lambda_2 \geq \ldots$ is the spectrum of the adjacency matrix of G.
  - Cheeger constant:

$$h(G) = \inf \left\{ \frac{|\partial A|}{|A|} \,\middle|\, A \subset V(G),\ |A| \leq \frac{|V(G)|}{2} \right\},$$

where ∂A is the set of edges with one end in A and one end in $V(G) \backslash A$.

- Cheeger's inequality:

$$\frac{1}{2}(d - \lambda_2(G)) \leq h(G) \leq \sqrt{2d(d - \lambda_2(G))}.$$

Connectivity of regular graphs: diameter and spectral gap

- Diameter: at least $\log_{d-1}(n)$ because the ball of radius r has size $O\left((d-1)^r\right)$.
- For a uniform random graphs, the diameter is $(1+o(1))\log_{d-1}(n)$ w.h.p. **[Bollobas–de la Vega 82]**. Reason: balls look like trees, so they grow "as quickly as possible".
- Spectral gap: comparison with the infinite d-regular tree **[Alon–Boppana 91]**:

$$\lambda_2(G) \geq \lambda_2(\mathbb{T}_d) + o(1) = 2\sqrt{d-1} + o(1).$$

- This bound is optimal:
  - Uniform d-regular graphs satisfy $\lambda_2(G) = 2\sqrt{d-1} + o(1)$ in probability **[Friedman 03]**.
  - Arithmetic constructions of Ramanujan graphs, i.e. with $\lambda_2(G) < 2\sqrt{d-1}$ **[Lubotzky–Philipps–Sarnak 88, Margulis 88]**.

The Cheeger constant of regular graphs

- Cheeger constant:

$$h(G) = \inf \left\{ \frac{|\partial A|}{|A|} \,\middle|\, A \subset V(G),\, |A| \leq \frac{|V(G)|}{2} \right\},$$

$$c_d = \limsup_{n \to +\infty} \max\{h(G) \mid G \text{ is a } d\text{-regular graph with } n \text{ vertices}\}.$$

- We have $c_d \leq d - 2 = h(\mathbb{T}_d)$:
  - Let A be a connected, not too small subset of $V(G)$.
  - If $E(A)$ is the set of edges between vertices of A, then

$$d|A| = |\partial A| + 2|E(A)| \geq |\partial A| + 2(|A| - 1),$$

  so $\frac{|\partial A|}{|A|} \leq d - 2 + o(1)$.

- On the other hand: various families of *expanders* show $c_d > 0$. For example random graphs give $c_d \geq \frac{d}{2} - O(\sqrt{d})$ [Bollobas 88].

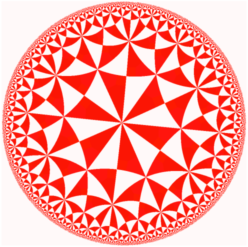
The Cheeger constant of regular graphs: upper bound

- Actually $c_d \leq \frac{d}{2}$ [Bollobas 88]:
  - Choose $A \subset V(G)$ randomly, by taking each vertex with probability $\frac{1}{2}$ in an independent way.
  - Then $|A| \approx \frac{n}{2}$ with high probability...
  - ...and $\mathbb{E}[|\partial A|] = \frac{1}{2} \times \frac{dn}{2}$.
  - So there is A with $|A| \approx \frac{n}{2}$ and $|\partial A| \leq \frac{dn}{4}$, so $h(G) \leq \frac{d}{2}$.
- Improvement [Alon 97]:
  - First make $1 \ll k \ll n$ connected groups of size $\frac{n}{k}$.
  - Keep each group in A with probability $\frac{1}{2}$.
  - Then at least $\approx n$ of the edges are inside a group, and $\frac{d-2}{2} n$ of them are on the boundary between two groups.
  - We obtain $c_d \leq \frac{d-2}{2}$.
- We have $c_d \sim \frac{d}{2}$ as $d \to +\infty$, but c_3 is still unknown.

The hyperbolic plane

- The *hyperbolic plane* $\mathbb{H}$ can be seen as the unit disk, equipped with the metric

$$\mathrm{d}s^2 = \frac{4\,\mathrm{d}x^2}{1-|x|^2}.$$

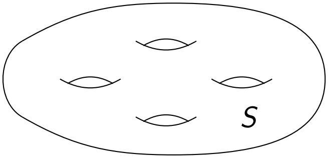


- *Curvature*: $|B(x,r)| = \pi\varepsilon^2 - \frac{\pi}{12}\varepsilon^4 K(x) + o(\varepsilon^4)$.
- Riemann uniformization theorem: $\mathbb{H}$ is the unique simply connected surface with constant curvature equal to -1.
- Perimeter and volumes of balls:

$$|B_{\mathbb{H}}(x,r)| = 2\pi\left(\cosh(r)-1\right), \quad |\partial B_{\mathbb{H}}(x,r)| = 2\pi\sinh(r).$$

Compact hyperbolic surfaces

- A *compact hyperbolic surface* S is a $2d$ manifold equipped with a Riemannian metric with constant curvature -1. We consider *closed* surfaces, i.e. no boundary.
- Gauss–Bonnet formula: $\int_S K(x)\mathrm{d}x = 2\pi(2-2g)$, where g is the *genus* of the surface, i.e. the number of holes. So $g \geq 2$.



- Equivalent definitions:
  - S is locally isometric to $\mathbb{H}$,
  - S is a quotient of $\mathbb{H}$ (by a nice enough group action),
  - S is a surface equipped with a conformal structure.
- Hyperbolic surfaces with genus g form a $(6g-6)$-dimensional space $\mathcal{M}_g$.

Expander properties for hyperbolic surfaces

- Diameter of a hyperbolic surface S: at least $(1+o(1))\log g$ (because balls are not larger in S than in $\mathbb{H}$).
- There is a random model of hyperbolic surfaces (random gluing of pants) where the diameter is $(1+o(1))\log g$ **[B.–Curien–Petri 19]**.
- Spectral gap: $\lambda_1(S)$ is the smallest nonzero eigenvalue of the Laplacian on S.
- We have $\lambda_1(S) \leq \lambda_1(\mathbb{H}) + o(1) = \frac{1}{4} + o(1)$ as $g \to +\infty$ **[Huber 74]**.
- There is a random model of hyperbolic surfaces (random cover of a fixed, small surface) where the spectral gap is $\frac{1}{4} + o(1)$ **[Hide–Maggee 21]**.
- Selberg conjecture: $\lambda_1 > \frac{1}{4}$ for *arithmetic* surfaces (Selberg proved $\frac{3}{16}$).

The Cheeger constant of hyperbolic surfaces

- $h(S) = \inf \left\{ \frac{|\partial A|}{|A|} \middle| A \subset S, |A| \leq \frac{|S|}{2} \right\}$, where $|A|$ is the area and $|\partial A|$ the boundary length (maybe $+\infty$).
- Cheeger–Buser inequality: $\frac{h(S)^2}{4} \leq \lambda_1(S) \leq 2h(S) + 10h(S)^2$.
- Various families of hyperbolic surfaces with Cheeger constant bounded from below:
  - Random models: random surfaces built from 3-regular graphs **[Brooks–Makover 04]**, Weil–Petersson random surfaces **[Mirzakhani 13]**...
  - Arithmetic surfaces: $h(S) \geq 0,168...$ **[Brooks 99 + Kim–Sarnak 03]**.

The Cheeger constant of hyperbolic surfaces

- Hyperbolic plane: $h(\mathbb{H}) = 1$, attained for balls.
- If $h(S) > 1 + \varepsilon$, then for all $r \geq 0$ such that $|B_S(x,r)| < \frac{|S|}{2}$:

$$\frac{d}{dr}\,|B_S(x,r)| = |\partial B_S(x,r)| \geq (1+\varepsilon)\,|B_S(x,r)|\,,$$

so $|B_S(x,r)| \geq c\, e^{(1+\varepsilon)r}$, absurd for r large enough, so

$$\limsup_{g\to+\infty}\ \sup_{S\in\mathcal{M}_g}\ h(S) \leq 1.$$

Theorem (B.–Curien–Petri 22)

We have

$$\limsup_{g\to+\infty}\ \sup_{S\in\mathcal{M}_g}\ h(S) \leq \frac{2}{\pi} \approx 0,637.$$

Strategy of proof

- Like on graphs: cut the surface S into k regions ($1 \ll k \ll g$), color each region in black or white with probability $\frac{1}{2}$ and let A be the union of black regions.
- Strategy already used on specific expander models, with a clever choice of the regions:
  - Arithmetic surfaces : $h(S) \lesssim 0,44$ **[Brooks–Zuk 02]**,
  - A natural model built from a random 3-regular graph: $h(S) \leq \frac{2}{3} + o(1)$ **[Shen–Wu 22]**.
- If S is partitionned into regions C_i with $\max(|C_i|) \ll |S|$, then $|A| \approx \frac{|S|}{2}$.
- We have

$$\mathbb{E}\left[|\partial A|\right] = \frac{1}{2}|\partial C| := \frac{1}{2}\left|\bigcup_i \partial C_i\right|,$$

so $h(S) \leq \frac{|\partial C|}{|S|}$.

Poisson–Voronoi tesselation

- Let (x_i) be a Poisson point process with intensity λ on S, i.e. for all $R \subset S$, the number of points x_i in R has law $\mathrm{Poisson}(\lambda|R|)$, with independence between disjoint regions.
- Voronoi tesselation:

$$C_i = \{z \in S \,|\, \forall j,\ d_S(x_i, z) \leq d_S(x_j, z)\}\,.$$

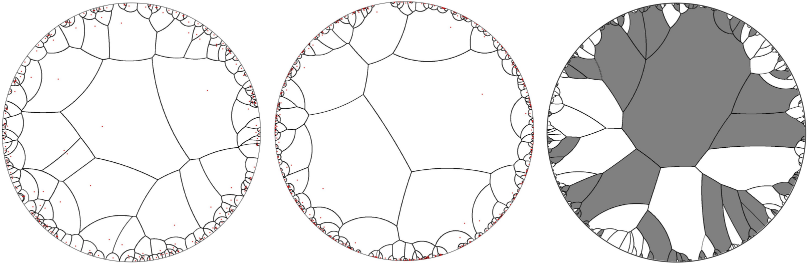
- Finally, take λ small. We need to prove that for λ small enough:

$$\limsup_{g \to +\infty} \sup_{S \in \mathcal{M}_g} \mathbb{E}\left[\left|\bigcup_i \partial C_i\right|\right] \leq \left(\frac{2}{\pi} + \delta\right)|S|.$$

Thickening the boundary

- Let $\partial^\varepsilon C$ be the ε-neighbourhood of ∂C. Then
$$|\partial C| = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} |\partial^\varepsilon C|,$$
so
$$\mathbb{E}\left[|\partial C|\right] \leq \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_S \mathbb{P}\left(x \in \partial^\varepsilon C\right) \mathrm{d}x,$$
so we want $\mathbb{P}\left(x \in \partial^\varepsilon C\right) \leq \frac{2}{\pi} + \delta$.
- Easy version of the argument: assume that S has a large injectivity radius around x, i.e. $B_S(r,x) = B_{\mathbb{H}}(r,x)$ for some $r \gg 1$. We first get the bound 1 instead of $\frac{2}{\pi}$.

- Local picture around x: Poisson–Voronoi tesselation of the hyperbolic plane (for decreasing values of λ).



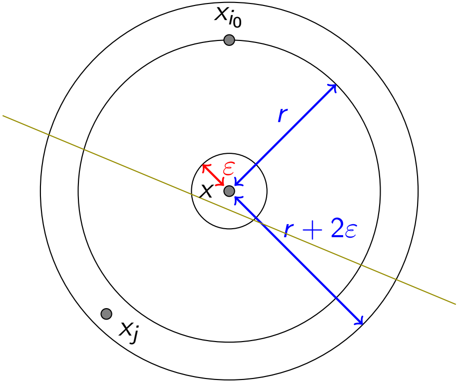
- Large injectivity radius: we can do all the computations in $\mathbb{H}$ instead of S.
- For a typical cell C_i [Isokawa 00]:

$$\mathbb{E}\left[|C_i|\right] = \frac{1}{\lambda}, \quad \mathbb{E}\left[|\partial C_i|\right] = \frac{8}{\sqrt{\pi\lambda}} \int_0^\infty \mathrm{e}^{-u} \sqrt{u + \frac{u^2}{4\pi\lambda}}\, \mathrm{d}u \sim \frac{4}{\pi\lambda},$$

but not robust enough for the general case.

The closest point to x

- Condition on the closest point of the Poisson process to x, say x_{i_0}, with $d_{\mathbb{H}}(x, x_{i_0}) = r$.



- If $x \in \partial^\varepsilon C$, then there is $j \neq i_0$ such that the bisector between x_{i_0} and x_j intersects $B_{\mathbb{H}}(x, \varepsilon)$, so $r \leq d_{\mathbb{H}}(x, x_j) \leq r + 2\varepsilon$, so

$$\mathbb{P}\left(x \in \partial^\varepsilon C \,|\, x_{i_0}\right) \lesssim \lambda \times 2\varepsilon \times |\partial B_{\mathbb{H}}(x, r)| .$$

Computation

- On the other hand, law of $d_{\mathbb{H}}(x, x_{i_0})$:

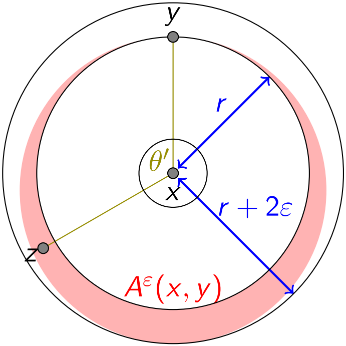
$$\mathbb{P}\left(d_{\mathbb{H}}(x, x_{i_0}) \geq r\right) = \exp\left(-\lambda \left|B_{\mathbb{H}}(x, r)\right|\right).$$

- For λ small, we have r large so $|\partial B_{\mathbb{H}}(r)| \approx |B_{\mathbb{H}}(r)|$, so

$$\begin{aligned}
&\mathbb{P}\left(x \in \partial^{\varepsilon} C\right) \\
&\leq 2\varepsilon\lambda^2 \int_0^{+\infty} \left(\frac{d}{dr}\left|B_{\mathbb{H}}(r)\right|\right) \times |\partial B_{\mathbb{H}}(r)| \times e^{-\lambda|B_{\mathbb{H}}(r)|}\,\mathrm{d}r \\
&\approx 2\varepsilon\lambda^2 \int_0^{+\infty} \left(\frac{d}{dr}\left|B_{\mathbb{H}}(r)\right|\right) \times |B_{\mathbb{H}}(r)| \times e^{-\lambda|B_{\mathbb{H}}(r)|}\,\mathrm{d}r \\
&= 2\varepsilon\lambda^2 \left[\left(-\frac{|B_{\mathbb{H}}(r)|}{\lambda} + \frac{1}{\lambda^2}\right) e^{-\lambda|B_{\mathbb{H}}(r)|}\right]_0^{+\infty} \\
&= 2\varepsilon.
\end{aligned}$$

- So $\mathbb{E}\left[|\partial C|\right] \lesssim |S|$ and $h(S) \lesssim 1$.

- Let $A^\varepsilon(x,y)$ be the set of points $z \in \mathbb{H}$ such that the bisector between y and z intersects $B_{\mathbb{H}}(x,\varepsilon)$.



- Good approximation in polar coordinates (for r large):

$$\left\{ (r';\theta') \,|\, r \le r' \le r + 2\varepsilon \left| \sin \frac{\theta'}{2} \right| \right\}.$$

- So $|A^\varepsilon(x,y)| \approx 2\varepsilon \times \frac{2}{\pi} \times |\partial B_{\mathbb{H}}(x,r)|$.

$$\mathbb{P}\left(x \in \partial^\varepsilon C \,|\, x_{i_0}\right) \approx \lambda \,|A^\varepsilon(x,x_{i_0})| \approx 2\varepsilon\lambda \times \frac{2}{\pi} \,|\partial B_{\mathbb{H}}(x,r)|\,.$$

Dirichlet domains

- Reminder: S is a quotient of $\mathbb{H}$ by the action of a discrete isometry group G, i.e. there is a surjective isometry $p : \mathbb{H} \to S$ such that $p(x') = p(y')$ iff $\exists g \in G, y' = g \cdot x'$.
- Fundamental domain: $D \subset \mathbb{H}$ such that p is a bijection from D to S.
- Let $x \in S$, and assume $p(0) = x$. The *Dirichlet domain* of S around x is

$$D = D(S, x) = \{x' \in \mathbb{H} \,|\, \forall g \in G, d_{\mathbb{H}}(0, x) \leq d_{\mathbb{H}}(0, g \cdot x)\}.$$

- In other words, it is the Voronoi cell around 0 of the point set $\{g \cdot 0 \,|\, g \in G\}$.

- Roughly speaking, the argument still works, replacing $|B_{\mathbb{H}}(x,r)|$ by $|B_D(x,r)| = |B_{\mathbb{H}}(x,r) \cap D|$.
- Now, when we evaluate the contribution of points at distance r from x, we need to integrate over the set

$$I_r = \{\theta \in [0, 2\pi] \,|\, [r,\theta] \in D\},$$

and not just over $[0, 2\pi]$:

$$\begin{aligned}\mathbb{P}\left(x \in \partial^{\varepsilon} C \,|\, x_{i_0} = [r,\theta]\right) &\approx \lambda \,|A^{\varepsilon}(x, x_{i_0}) \cap D| \\ &\approx \int_{I_r} 2\varepsilon \sinh(r) \left|\sin \frac{\theta' - \theta}{2}\right| \mathrm{d}\theta'\end{aligned}$$

and similarly, express the law of x_{i_0} in polar coordinates:

$$\exp\left(-\lambda \,|B_D(x,r)|\right) \mathbb{1}_{\theta \in I_r} \sinh(r) \,\mathrm{d}r \,\mathrm{d}\theta.$$

The general case

- In the contribution of $\{d(x, x_{i_0}) = r\}$, the following integral appears:

$$\int_{I_r \times I_r} \left| \sin \frac{\theta - \theta'}{2} \right| \, \mathrm{d}\theta \, \mathrm{d}\theta'.$$

- Useful lemma: for all finite measure μ on $[0, 2\pi]$,

$$\int_0^{2\pi} \int_0^{2\pi} \left| \sin \frac{\theta_1 - \theta_2}{2} \right| \mu(\mathrm{d}\theta_1) \, \mu(\mathrm{d}\theta_2) \leq \frac{2}{\pi} \mu([0, 2\pi])^2,$$

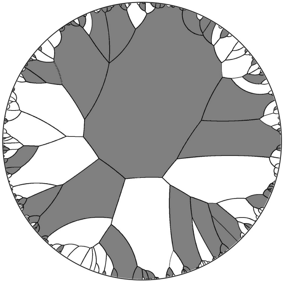
with equality for the uniform measure [Toth 56].

- We find again

$$\mathbb{P}\left(x \in \partial^{\varepsilon} C\right) \leq 2\varepsilon\lambda^2 \times \frac{2}{\pi} \int_0^{+\infty} \left(\frac{d}{dr} |B_D(r)| \right)^2 \times e^{-\lambda |B_D(r)|} \, \mathrm{d}r$$

and finish the computation as before.

Further questions



- The bound $\frac{2}{\pi}$ should not be sharp, as the interfaces are not straight geodesics. Can we get to $\frac{1}{2}$ or below?
- On $\mathbb{H}$, Poisson–Voronoi has a nontrivial limit when $\lambda \to 0$: points go to infinity but interfaces stay there ("pointless Voronoi diagram"). Study this object?
  - Already known: $p_c \sim \frac{\pi}{3}\lambda$ as $\lambda \to 0$ **[Hansen–Müller 20]**.

THANK YOU !