The hyperbolic Brownian plane

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Planar maps

Definitions

- A *planar map* is a locally finite, connected graph embedded in the plane in such a way that :
  - no two edges cross, except at a common endpoint,
  - every compact subset of the plane intersects finitely many vertices and edges,

  considered up to orientation-preserving homeomorphism.
- The *faces* of the map are the connected components of its complementary. The *degree* of a face is the number of half-edges adjacent to this face.

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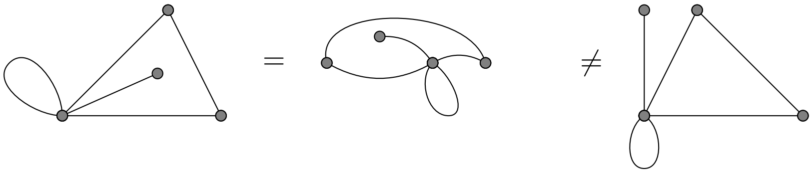
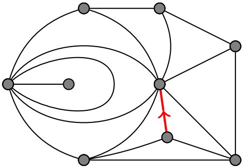


Triangulations

Definition

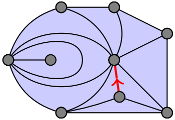
- A *triangulation of the plane* is an infinite planar map in which all the faces have degree 3. It may contain loops and multiple edges.
- A *triangulation with a hole of perimeter p* is a finite map in which all the faces have degree 3 except the external face, which has degree p.
- A *rooted* triangulation is a triangulation with a distinguished oriented edge. From now on, all the triangulations will be rooted.



Examples : a rooted triangulation with a hole of perimeter 6.

Definition

If t is a triangulation of a p-gon and T a triangulation of the plane, we write $t \subset T$ if T may be obtained by "filling" the hole of t with an infinite triangulation.



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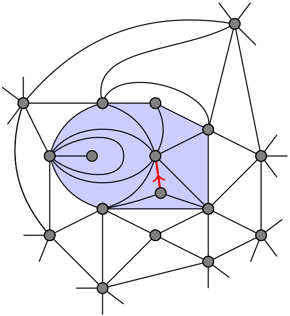


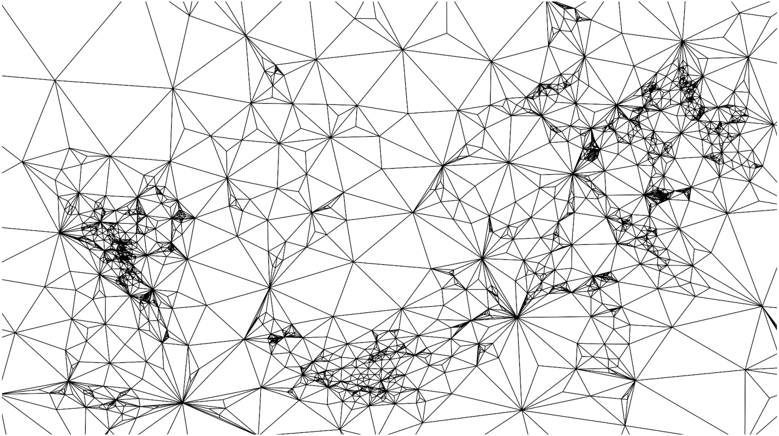
The UIPT

Theorem (≈ Angel-Schramm, 2003)

There is a random triangulation of the plane $\mathbb{T}$, called the *UIPT* (Uniform Infinite Planar Triangulation), such that for any triangulation t with a hole of perimeter p, we have

$$\mathbb{P}\big(t \subset \mathbb{T}\big) = C_p \lambda_c^{|t|},$$

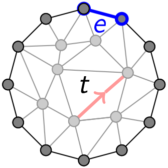
where $|t|$ is the number of vertices of t and we have $\lambda_c = \frac{1}{12\sqrt{3}}$ and $C_p = 2\sqrt{3}\frac{p(2p)!}{p!^2}3^p$.



Picture by N. Curien.

Spatial Markov property

Condition on $t \subset \mathbb{T}$, and let e be an edge of ∂t :



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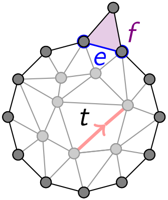

Case I

- Then $\mathbb{P}\big(\text{Case } I\big) = \dfrac{\mathbb{P}\big(t+f\subset\mathbb{T}\big)}{\mathbb{P}\big(t\subset\mathbb{T}\big)} = \dfrac{C_{p+1}\lambda_c^{|t|+1}}{C_p\lambda_c^{|t|}} = \dfrac{C_{p+1}}{C_p}\lambda_c.$

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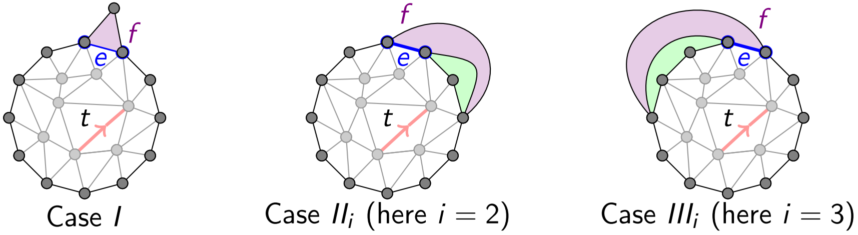

- Then $\mathbb{P}(\text{Case } I) = \frac{\mathbb{P}\left(t+f\subset\mathbb{T}\right)}{\mathbb{P}\left(t\subset\mathbb{T}\right)} = \frac{C_{p+1}\lambda_c^{|t|+1}}{C_p\lambda_c^{|t|}} = \frac{C_{p+1}}{C_p}\lambda_c.$
- $\mathbb{P}(\text{Case } II_i)$ and $\mathbb{P}(\text{Case } III_i)$ are also explicitely known, and depend only on p.

Peeling process and consequences

- Allows to discover $\mathbb{T}$, almost "face by face", in a Markovian way.
- Very flexible : the choice of e may be adapted to the information we are looking for :
  - growth in r^4 [Angel],
  - critical probabilities for percolation [Angel, Angel-Curien, Richier],
  - subdiffusivity of the random walk [Benjamini-Curien]

λ-Markovian triangulations

Definition

A random triangulation of the plane T is λ-Markovian if there are constants $(C_p)_{p\geq 1}$ such that for any triangulation t with a hole of perimeter p we have

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Proposition (Curien 2014, B. 2016)

If $\lambda > \lambda_c$ then there is no λ-Markovian triangulation. If $0 < \lambda \leq \lambda_c$ then there is a unique one (in distribution), that we write $\mathbb{T}_\lambda$. Besides we have

$$C_p(\lambda) = \frac{1}{\lambda}\left(8 + \frac{1}{h}\right)^{p-1} \sum_{q=0}^{p-1} \binom{2q}{q} h^q,$$

where $h \in (0, \frac{1}{4}]$ is such that $\lambda = \frac{h}{(1+8h)^{3/2}}$.

Hyperbolic behaviour

- Exponential volume growth [Curien]
- Anchored expansion : if A is a finite, connected set of vertices containing the root, then $|\partial A| \geq c|A|$ [Curien].
- The simple random walk has positive speed [Curien, Angel-Nachmias-Ray].

Scaling limit of $\mathbb{T}$

- A planar map can be seen as a (discrete) metric space, equipped with its graph distance and the counting measure on its vertices.
- The set of all (classes of) locally compact measured metric spaces can be equipped with the local Gromov-Hausdorff-Prokhorov distance.

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Theorem (Curien-Le Gall 14, B. 16)

Let $\mu_{\mathbb{T}}$ be the counting measure on the set of vertices of $\mathbb{T}$. We have the following convergence in distribution for the local Gromov-Hausdorff-Prokhorov distance :

$$\left(\frac{1}{n}\mathbb{T}, \frac{1}{n^4}\mu_{\mathbb{T}}\right) \xrightarrow[a\to+\infty]{(d)} \mathcal{P}$$

where $\mathcal{P}$ is a random (pointed) measured metric space homeomorphic to the plane called the *Brownian plane*.

Scaling limit of $\mathbb{T}_\lambda$?

- For $\lambda < \lambda_c$ fixed $\frac{1}{n}\mathbb{T}_\lambda$ cannot converge because $\mathbb{T}_\lambda$ "grows too quickly".

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Theorem (B. 16)

Let $(\lambda_n)_{n\geq 0}$ be a sequence of numbers in $(0, \lambda_c]$ such that

$$\lambda_n = \lambda_c\left(1 - \frac{2}{3n^4}\right) + o\left(\frac{1}{n^4}\right).$$

Then

$$\left(\frac{1}{n}\mathbb{T}_{\lambda_n}, \frac{1}{n^4}\mu_{\mathbb{T}_{\lambda_n}}\right) \xrightarrow[n\to+\infty]{(d)} \mathcal{P}^h$$

where $\mathcal{P}^h$ is a random (pointed) measured metric space homeomorphic to the plane that we call the *hyperbolic Brownian plane*.

Hull process of $\mathcal{P}$

For $r \geq 0$ we write $\overline{B_r}(\mathcal{P})$ for the *hull* of radius r of $\mathcal{P}$, that is, the reunion of its ball of radius r and all the bounded connected components of its complementary.

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Theorem (Curien-Le Gall 14)

- There is a natural notion of "perimeter" of $\overline{B_r}(\mathcal{P})$, that we write $P_r(\mathcal{P})$, and $\left(P_r(\mathcal{P})\right)_{r \geq 0}$ is a time-reversed stable branching process (in particular it is càdlàg with only negative jumps).

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- If $V_r(\mathcal{P})$ is the volume of $\overline{B_r}(\mathcal{P})$, then

$$\left(V_r(\mathcal{P})\right)_{r\geq 0} = \left(\sum_{t_i \leq r} \xi_i |\Delta P_r(\mathcal{P})|^2\right)_{r\geq 0},$$

where (t_i) is a measurable enumeration of the jumps of $\left(P_r(\mathcal{P})\right)_{r\geq 0}$, and the ξ_i are i.i.d. with density $\frac{e^{-1/2x}}{\sqrt{2\pi x^5}}\mathbb{1}_{x>0}$.

Theorem

For all $r \geq 0$, the random variable $\overline{B_r}(\mathcal{P}^h)$ has density

$$e^{-2V_{2r}(\mathcal{P})} e^{P_{2r}(\mathcal{P})} \int_0^1 e^{-3P_{2r}(\mathcal{P})x^2} \mathrm{d}x$$

with respect to $\overline{B_r}(\mathcal{P})$.

Sketch of proof

- We use the convergence of $\mathbb{T}$ to $\mathcal{P}$ and the absolute continuity relations between $\mathbb{T}$ and $\mathbb{T}_\lambda$:

$$\frac{\mathbb{P}\big(\overline{B_r}(\mathbb{T}_\lambda) = t\big)}{\mathbb{P}\big(\overline{B_r}(\mathbb{T}) = t\big)} = \frac{C_p(\lambda)}{C_p(\lambda_c)} \Big(\frac{\lambda}{\lambda_c}\Big)^{|t|}$$

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- Two main tools :
  - precise asymptotics for the $C_p(\lambda)$,
  - a reinforcement of the convergence of $\mathbb{T}$ to $\mathcal{P}$.

Asymptotics for the absolute continuity relations

Proposition

Fix $r > 0$. Let (λ_n), (p_n) and (v_n) be such that :

- $\lambda_n = \lambda_c\left(1 - \frac{2}{3n^4}\right) + o\left(\frac{1}{n^4}\right)$,
- $\frac{v_n}{n^4} \longrightarrow 3v$,
- $\frac{p_n}{n^2} \longrightarrow \frac{3}{2}p$.

Let t_n be a possible value of $\overline{B_{rn}}(\mathbb{T})$ such that t_n has v_n vertices and a hole of perimeter p_n. Then

$$\frac{\mathbb{P}\left(\overline{B_{rn}}(\mathbb{T}_{\lambda_n}) = t_n\right)}{\mathbb{P}\left(\overline{B_{rn}}(\mathbb{T}) = t_n\right)} \longrightarrow e^{-2v} e^{p} \int_0^1 e^{-3px^2} dx.$$

Reinforced convergence to the Brownian plane

Theorem

The three following convergences hold **jointly** in distribution as $n \to +\infty$:

$$\begin{cases} \frac{1}{n}\mathbb{T} & \longrightarrow \quad \mathcal{P} \\ \left(\frac{1}{n^4}|\overline{B_{rn}}(\mathbb{T})|\right)_{r\geq 0} & \longrightarrow \quad \left(3V_r(\mathcal{P})\right)_{r\geq 0} \\ \left(\frac{1}{n^2}|\partial\overline{B_{rn}}(\mathbb{T})|\right)_{r\geq 0} & \longrightarrow \quad \left(\frac{3}{2}P_r(\mathcal{P})\right)_{r\geq 0}. \end{cases}$$

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- Joint convergence of the last two marginals. [Curien-Le Gall]
- To conclude : show that $\left(P_r(\mathcal{P})\right)_{r\geq 0}$ is determined by $\left(V_r(\mathcal{P})\right)_{r\geq 0}$.

THANK YOU !