Supercritical causal maps: geodesics and simple random walk

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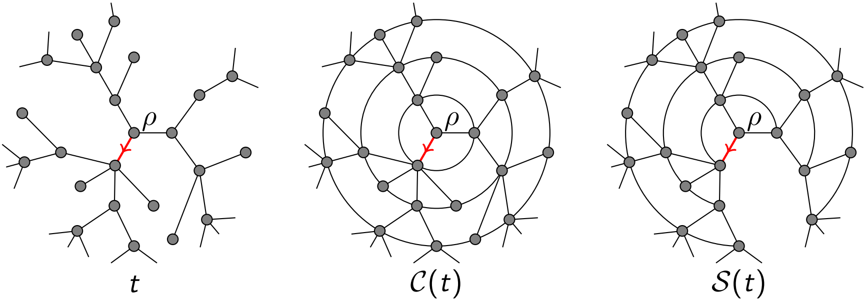
Let t be an infinite plane tree. We define the *causal map* $\mathcal{C}(t)$ and the *causal slice* $\mathcal{S}(t)$ associated to t as follows :



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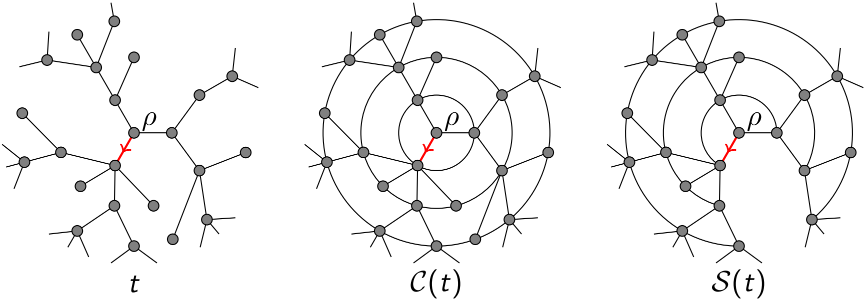

Goal : study $\mathcal{C} = \mathcal{C}(T)$, where T is a *supercritical* Galton–Watson tree conditionned to survive.

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  - some of our results can be generalized to more general maps containing a supercritical GW tree (ex : PSHIT),
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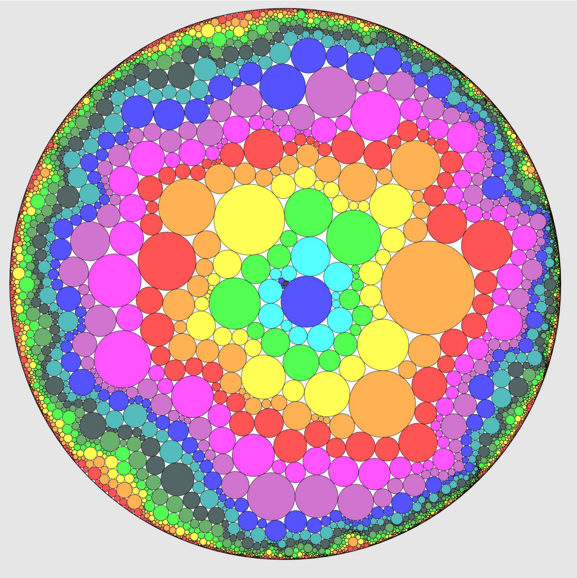
  - some of our results can be generalized to more general maps containing a supercritical GW tree (ex : PSHIT),
  - applications to the UIPT in the critical case [Curien, Ménard].
- Better understanding of the properties of supercritical GW trees : when is the tree structure necessary ?

A nice picture



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- For $\mathcal{C}(T)$, some of these properties are easy, some others are the goal of this talk.

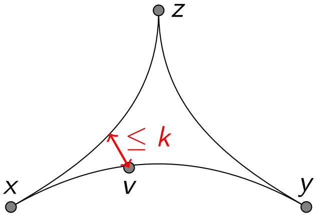
Setting

- We fix a supercritical distribution μ on $\mathbb{N}$, i.e. $\sum_{i \geq 0} i\mu(i) > 1$.
- Let T be a Galton–Watson tree with offspring distribution μ, conditionned to be infinite. Let ρ be its root.
- If G is a graph, we let d_G be the graph distance on G.
- The *height* $h(v)$ of a vertex v is its distance to the root in T, and also in $\mathcal{C}$.
- If $x \in \mathcal{C}$ has infinitely many descendants, let $\mathcal{S}[x]$ be the map formed by the descendants of x. It has the same distribution as $\mathcal{S}(T)$.

Definition

We say that a graph G is *Gromov-hyperbolic* if there is a constant $k \geq 0$ such that for every vertices x, y and z of G and every geodesics γ_{xy}, γ_{yz} and γ_{zx} from x to y, y to z and z to x, we have

$$\forall v \in \gamma_{xy}, d_G\left(v, \gamma_{yz} \cup \gamma_{zx}\right) \leq k.$$



"Usual" Gromov-hyperbolicity

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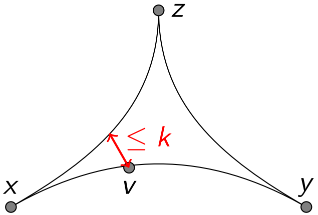


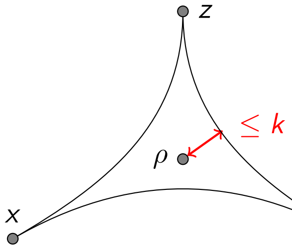
- Problem : if e.g. $\mu(1) > 0$, then $\mathcal{C}$ contains arbitrarily large portions of the square lattice, which is not hyperbolic.
- We need an "anchored" version !

Weak anchored hyperbolicity

Definition

We say that a planar map M is *weakly anchored hyperbolic* if there is a constant $k \geq 0$ such that for every vertices x, y and z of M and every geodesics γ_{xy}, γ_{yz} and γ_{zx} from x to y, y to z and z to x **such that the triangle they form surrounds** ρ, we have

$$d_M(\rho, \gamma_{xy} \cup \gamma_{yz} \cup \gamma_{zx}) \leq k.$$



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A *bi-infinite geodesic* in a graph G is a family of vertices $(\gamma(i))_{i\in\mathbb{Z}}$ such that for every $i, j \in \mathbb{Z}$,

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Theorem (B., 18)

Almost surely, the map $\mathcal{C}$ is weakly anchored hyperbolic and admits bi-infinite geodesics.

Our main tool

Let γ_ℓ (resp. γ_r) be its left (resp. right) boundaries of $\mathcal{S} = \mathcal{S}(T)$.

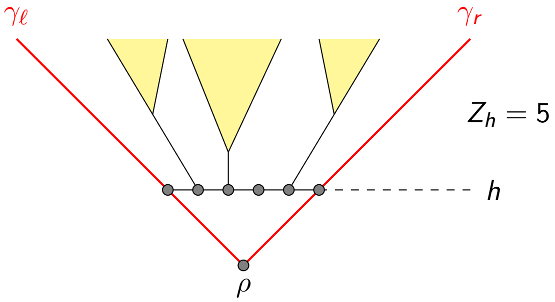
Proposition

There is a (random) $K \geq 0$ such that any geodesic in $\mathcal{S}$ from a vertex of γ_ℓ to a vertex on γ_r contains a vertex of height at most K.

Proof :

- Let γ be a geodesic in $\mathcal{S}$ from $\gamma_\ell(i)$ to $\gamma_r(j)$, and let h be the minimal height on γ.
- The path $\gamma_\ell(i) \to \rho \to \gamma_r(j)$ has length $i + j$, so $|\gamma| \leq i + j$.
- Every step of γ is either horizontal or vertical.
- Number of vertical steps $\geq (i - h) + (j - h) = i + j - 2h$.

Our main tool (proof)



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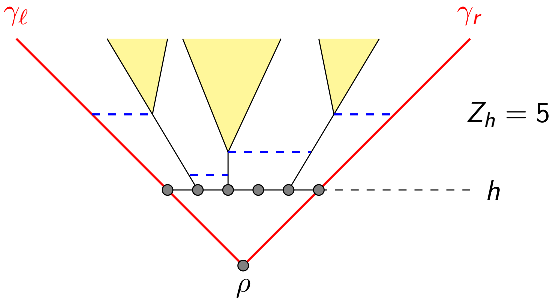

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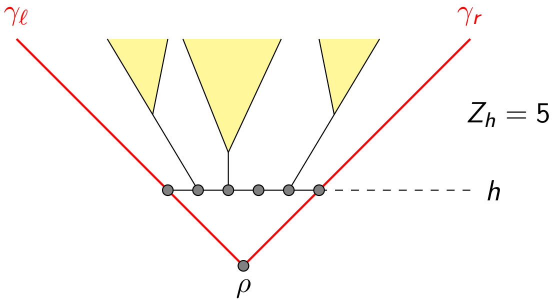
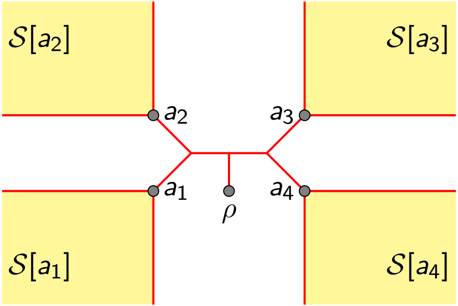


- Let Z_h be the number of vertices at height h with infinitely many descendants.
- γ stays at height $\geq h$, so it must cross Z_h slices, which requires at least $Z_h - 1$ horizontal steps.
- We obtain $i + j \geq |\gamma| \geq i + j - 2h + Z_h - 1$, so $Z_h \leq 2h + 1$, which is only true for finitely many h by exponential growth.

Proof of weak anchored hyperbolicity



- Let $(a_i)_{1 \leq i \leq 4}$ be four vertices with infinitely many descendants, neither of which is an ancestor of another.

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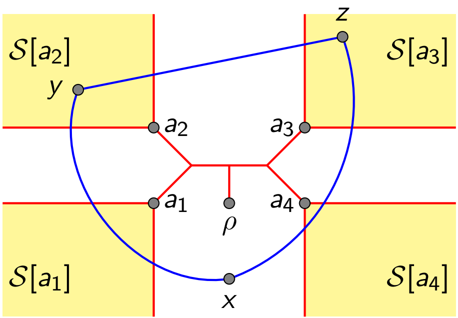

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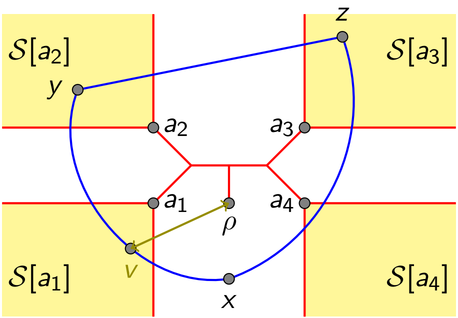
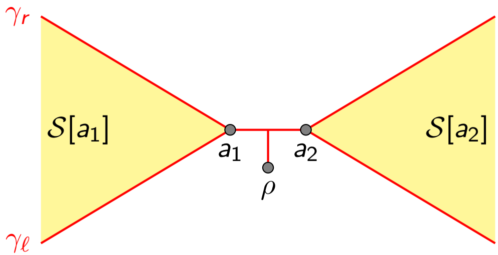


- Let $(a_i)_{1 \leq i \leq 4}$ be four vertices with infinitely many descendants, neither of which is an ancestor of another.
- If x, y, z form a geodesic triangle, assume none of them is in $\mathcal{S}[a_1]$.
- Then the geodesic from x to y must cross $\mathcal{S}[a_1]$, so it contains a vertex v with $d(\rho, v) \leq d(\rho, a_1) + K_1$.

Existence of bi-infinite geodesics



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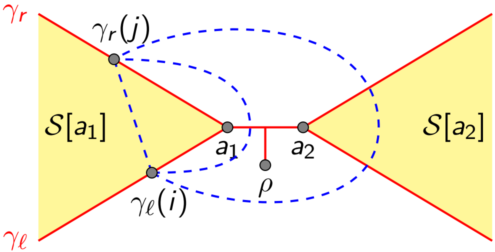

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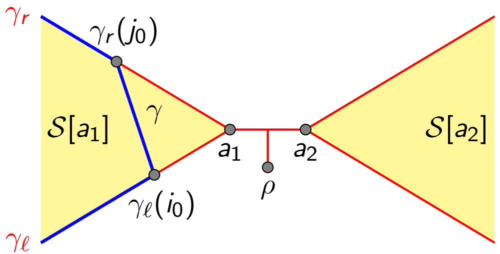

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- Take (i_0, j_0) such that $a_{i_0,j_0} = \max\{a_{i,j} | i, j \geq 0\}$, and concatenate a geodesic from $\gamma_\ell(i_0)$ to $\gamma_r(j_0)$ with γ_ℓ and γ_r.

Robustness

Theorem (B., 18)

Any planar map containing (an injective embedding of) a supercritical Galton–Watson tree conditionned to survive is weakly anchored hyperbolic and admits bi-infinite geodesics.

- The conclusion from the Proposition is almost the same as in the particular case of $\mathcal{C}$.

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- To prove the Proposition, two obstacles :
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- The proof of the Proposition relies on the fact that a supercritical Galton–Watson process survives even if a reasonable number of individuals are killed at each generation.
- This general setting includes the PSHIT, which are hyperbolic variants of the UIPT [B., 18].

Positive speed for the simple random walk

Let (X_n) be the simple random walk on $\mathcal{C}$ started from ρ.

Theorem (B., 18)

Assume $\mu(0) = 0$. Then there is a constant $v > 0$ such that

$$\frac{d_{\mathcal{C}}(\rho, X_n)}{n} \xrightarrow[n \to +\infty]{a.s.} v.$$

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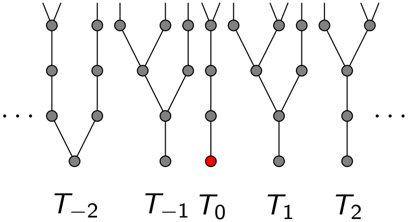
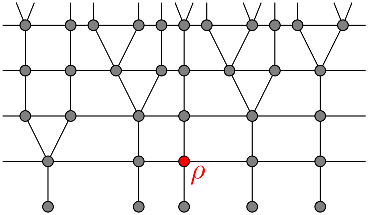
- Proved in 1995 for Galton–Watson trees, by finding a stationary environment [Lyons, Pemantle, Peres].
- All the similar proofs make heavy use of the tree structure.
- Two main tools in our proof :
  - an exploration method of $\mathcal{C}$ guarantees that we do not discover "very bad" points,
  - a regeneration argument gives the positive speed.

To gain stationarity, we work in the following half-plane model $\mathcal{H}$, where $(T_i)_{i\in\mathbb{Z}}$ are i.i.d. Galton–Watson trees :

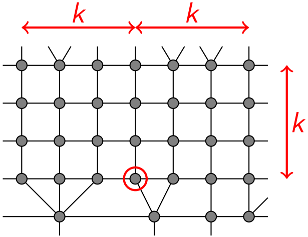




- We will prove positive speed away from the boundary on $\mathcal{H}$. To pass from $\mathcal{H}$ to $\mathcal{C}$, show that the SRW on $\mathcal{H}$ stays in the same tree eventually.

Bad vertices

- The drift at a vertex with i children is $\frac{i-1}{i+3} \geq 0$, so we need (X_n) to spend a lot of time at vertices with ≥ 2 children.
- We define a k-bad vertex as follows :



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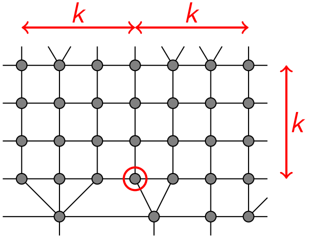

- Exemple : $\mathbb{P}\left(\rho \text{ is } k\text{-bad}\right) = \mu(1)^{k(2k+1)} \approx e^{-k^2}$.

Lemma

There is a constant c such that almost surely, for n large enough, none of the vertices $X_0, X_1, \ldots, X_n$ is $c\sqrt{\log n}$-bad.

Proof of the "bad vertex lemma"

- We "explore" $\mathcal{H}$ along the walk X_n. At time n, we discover X_n and all its ancestors.

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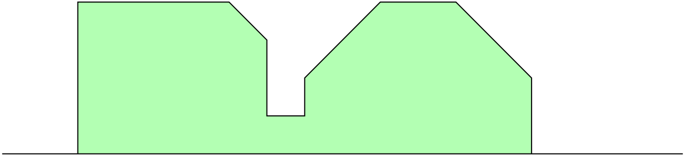

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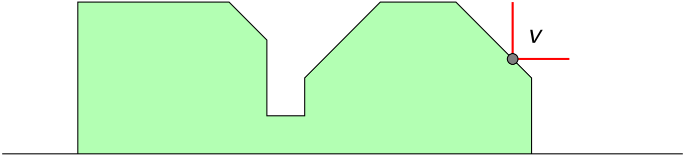

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- Let $k = c\sqrt{\log n}$. When we discover a new vertex v, if its k neighbours on the left (or on the right) are undiscovered, then $\mathbb{P}\left(v \text{ is } k\text{-bad}\right) \leq \mu(1)^{k^2}$.

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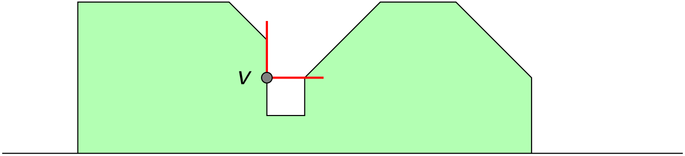

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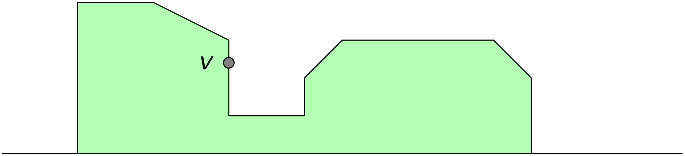

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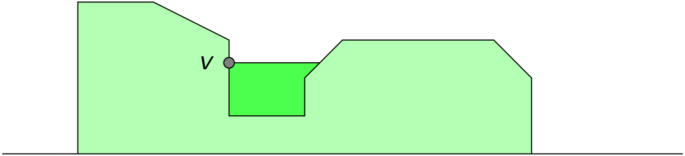

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Quasi-positive speed

- A vertex is *good* if it has ≥ 2 children.
- For every $0 \leq i \leq n$, the vertex X_i is at distance at most $c\sqrt{\log n}$ from a good vertex.
- Hence, the probability to reach a good vertex from X_i in $c\sqrt{\log n}$ steps is at least

$$4^{-c\sqrt{\log n}} = n^{-o(1)}.$$

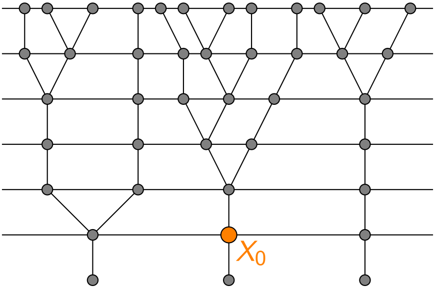
- Hence, the number of good vertices visited between time 0 and n is $n^{1-o(1)}$, so the drift accumulated is $n^{1-o(1)}$.
- We obtain $h(X_n) = n^{1-o(1)}$ almost surely.
- More careful computations : $h(X_n)$ is "almost increasing", with explicit, subpolynomial tail bounds.

Regeneration times

- We say that $n > 0$ is a *regeneration time* if

$$\forall k < n,\ h(X_k) < h(X_n) \text{ and } \forall k \geq n,\ h(X_k) \geq h(X_n).$$

- We list these times as $0 < \tau_1 < \tau_2 < \dots$.



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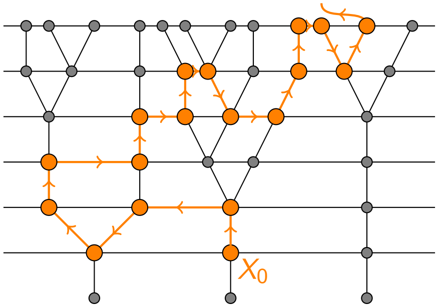

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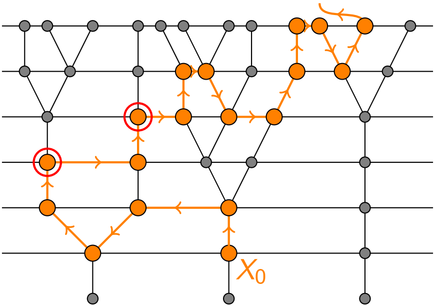
- We say that $n > 0$ is a *regeneration time* if

$$\forall k < n,\ h(X_k) < h(X_n) \text{ and } \forall k \geq n,\ h(X_k) \geq h(X_n).$$

- We list these times as $0 < \tau_1 < \tau_2 < \ldots$.



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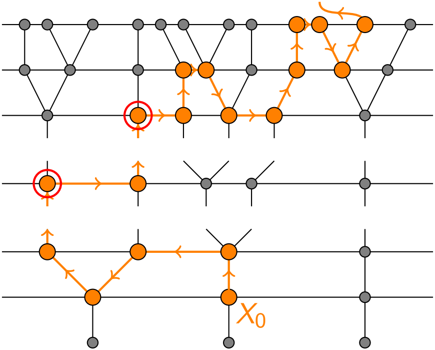

Regeneration times

- We know that $h(X_n) \to +\infty$ a.s., so every new height has a positive probability to be a regeneration time, so all the τ_i are finite.
- The blocs between τ_i and τ_{i+1} are i.i.d.! In particular, the variables $(\tau_{i+1} - \tau_i)$ and $\left(h(X_{\tau_{i+1}}) - h(X_{\tau_i})\right)$ are i.i.d..
- By the law of large numbers, we obtain

$$\frac{X_n}{n} \to \frac{\mathbb{E}\left[h(X_{\tau_2}) - h(X_{\tau_1})\right]}{\mathbb{E}\left[\tau_2 - \tau_1\right]},$$

  so it is enough to prove $\mathbb{E}\left[\tau_2 - \tau_1\right] < +\infty$.
- By the "quasi-positive speed" estimates, "fresh" times occur often and, when they do, we know quickly if they are regeneration times. We obtain that $\tau_2 - \tau_1$ has subpolynomial tail.

Robustness and other questions

- Highly non-robust proof : relies heavily on the fact that the local drift is nonnegative at *every* vertex.
- With different ideas, results about the Poisson boundary :
  - even if $\mu(0) > 0$, description of the Poisson boundary,
  - if T is filled with i.i.d. slices, the map we obtain is non-Liouville.
- Heat kernel decay ? Our proof of positive speed gives $\mathbb{P}(X_n = \rho) = o(n^{-\beta})$ for every β, we expect $\exp(-n^{1/3})$.
- λ-biased random walk ? No regime where the positive drift is "too strong" ?

THANK YOU !