Recurrence of the UIHPM via duality of resistances

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UBC

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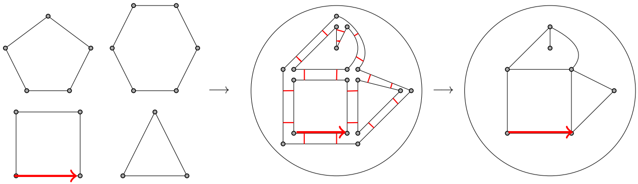
Random planar maps

- A *planar map* is a gluing of polygons homeomorphic to the sphere (finite case) or to the plane (infinite case).
- We consider *rooted maps* (distinguished oriented edge).



Local limits of uniform random maps

- Natural idea for probabilists: pick a random map uniformly at random among all maps of fixed finite size (in a certain class).
- Motivation: theoretical physics (two-dimensional quantum gravity).
- To take the limit when the size goes to $+\infty$, local convergence: two maps are close if they have large isomorphic balls around the root.
- Limiting objects are infinite planar maps such as:
  - the UIPT for triangulations **[Angel–Schramm 2003]**,
  - the UIPQ for quadrangulations **[Krikun 2005]**,
  - the UIPM for general planar maps **[Ménard–Nolin 2013]**.
- Metric properties are well understood:
  - Volume growth $\approx r^4$ **[Angel 2004]**
  - Scaling limit: Brownian plane **[Curien–Le Gall 2014]**
  - Links with Liouville Quantum Gravity **[Miller–Sheffield 2015, 2016...]**

Simple random walk on infinite random maps

- Historically, the simple random walk on these models has been much harder to understand.
- Recurrence of the SRW on a wide class of models **[Gurel-Gurevich–Nachmias 2012, Lee 2017]**.
- Speed: the distance of the SRW at time n to the root is $n^{1/4+o(1)}$ **[Gwynne–Hutchcroft 2018]**.
- Bounds on the resistance between the root and the boundary of the ball of radius r:
  - At most polynomial in $\log r$ **[Gwynne–Miller 2017]**,
  - Circle packing arguments for recurrence are expected to give lower bounds of order $\log \log r$,
  - Conjectured order of magnitude: $\log r$.

Recurrence of half-plane models

- Half-plane models: models of infinite maps with an infinite (simple) boundary.
- Recurrence of the Uniform Infinite *Half-Plane* Triangulation **[Angel–Ray 2016]**.
- UIHPM: similar model for general maps.

Theorem (B.–Lehéricy 2019)

The UIHPM M_∞ s a.s. recurrent. More precisely, there is a constant c such that a.s.:

$$R_{M_\infty}(\rho \leftrightarrow \partial B_r(M_\infty)) \geq c \log r$$

for r large enough.

Construction of the UIHPM

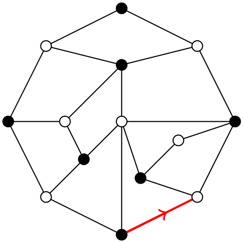
- First definition: local limit of critical Boltzmann maps with simple boundary. Let M_p be such that

$$\mathbb{P}(M_p = m) = \frac{1}{Z_p}\left(\frac{1}{12}\right)^{\#\mathrm{Edges}(m)}$$

  for all map m with a simple boundary of length p, rooted on the boundary. Then M_∞ is the local limit of M_p as $p \to +\infty$.
- Second construction: from the Uniform Infinite Half-Plane *Quadrangulation*, using the Tutte bijection.

The Tutte bijection

- Correspondence between quadrangulations and general maps:



- Start from the UIHPQ Q_∞ and apply the Tutte correspondence. There is a unique infinite 2-connected component, which has the distribution of M_∞.
- The finite 2-connected components have negligible contribution to distances and to resistances, so we can work on $\mathrm{Tutte}(Q_\infty)$.

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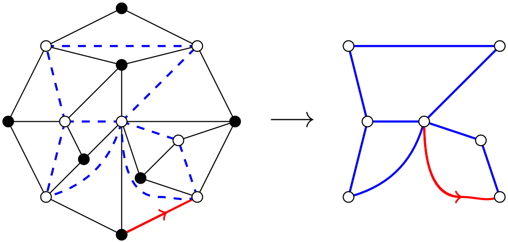

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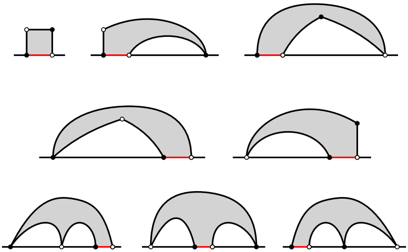
Sketch of the proof

- Central idea: use the self-duality of the model. We find a self-dual block in which the resistance is bounded from below with positive probability.
- Using peeling estimates in the UIHPQ, we build a logarithmic number of these self-dual blocks between the root and infinity.
- Similarities with RSW theory in percolation. Also used in the context of SRW on $\mathbb{Z}^2$ with weights given by the exponential of a Gaussian Free Field [Biskup–Ding–Goswami 2016].

Spatial Markov property of the UIHPQ Q_∞

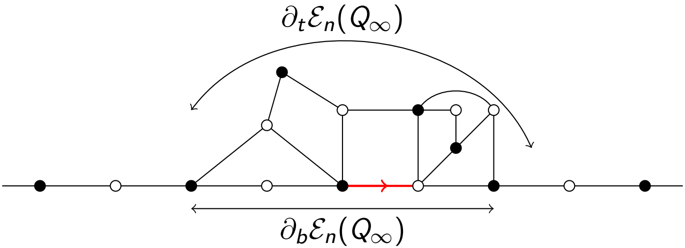
- Spatial Markov property: the quadrilateral incident to the root edge has one of the following shapes:



- The probability of each case is known and, conditionally on the grey quadrilateral:
  - Finite regions are filled with *critical Boltzmann quadrangulations*.
  - The infinite region has the law of Q_∞.

Peeling explorations of the UIHPQ Q_∞

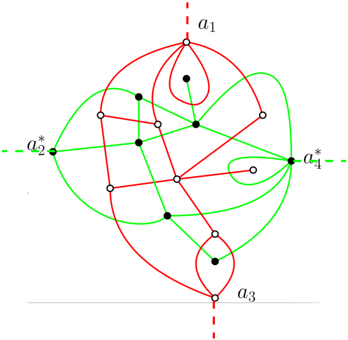
- Consequence: we can explore Q_∞ almost face by face. Explored region after n steps:



- The boundary length variation $X_n = |\partial_t \mathcal{E}_n(Q_\infty)| - |\partial_b \mathcal{E}_n(Q_\infty)|$ is a random walk with explicitly known transitions.
- Scaling limit of (X_n): stable Lévy process with index $3/2$ and only negative jumps.
- Flexibility: at each step, we can choose the next boundary edge to explore (peeling algorithm).

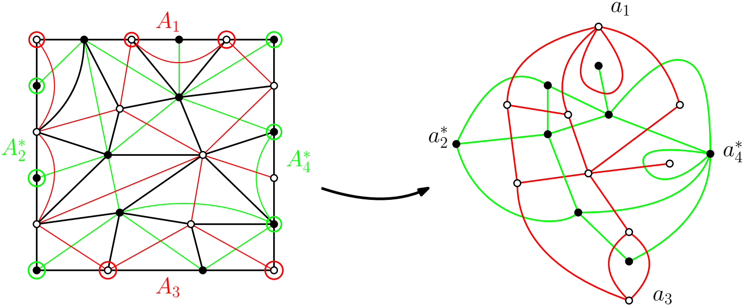
Duality of resistances

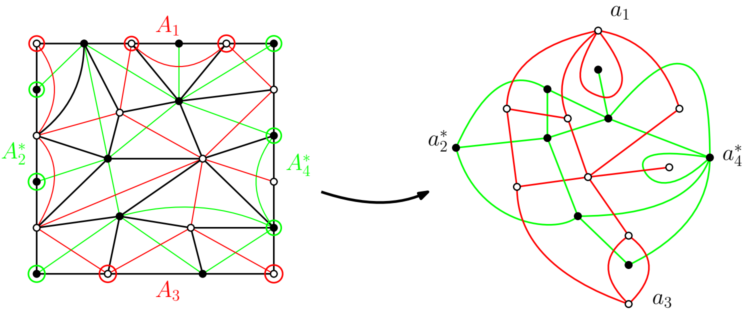
- Let M be a planar map with two vertices a_1, a_3 on the unbounded face. Let (M^*, a_2^*, a_4^*) be the dual map.



- Then $R_{M^*}(a_2^* \leftrightarrow a_4^*) = (R_M(a_1 \leftrightarrow a_3))^{-1}$.

- Consider Q_p critical Boltzmann quadrangulation of the $2p$-gon, i.e. $\mathbb{P}(Q_p = q) = \frac{1}{Z_p}\left(\frac{1}{12}\right)^{\#\mathrm{Faces(q)}}$.
- Split the boundary into 4 parts A_1, A_2^*, A_3, A_4^*.
- Let M_p (resp. M_p^*) be the map obtained by the Tutte bijection on white (resp. black) vertices.



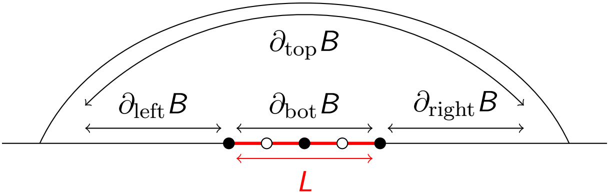


- Then $R_{M_p}(A_1 \leftrightarrow A_3) = \left(R_{M_p^*}(A_2^* \leftrightarrow A_4^*)\right)^{-1}$.
- Symmetry black/white: if $|A_1|, |A_3| \leq |A_2^*|, |A_4^*|$, then $R_{M_p}(A_1 \leftrightarrow A_3)$ stochastically dominates $R_{M_p^*}(A_2^* \leftrightarrow A_4^*)$.
- Consequence: for all p,

$$\mathbb{P}\left(R_{M_p}(A_1 \leftrightarrow A_3) \geq 1\right) \geq \frac{1}{2}.$$

Construction of a self-dual block at a given scale

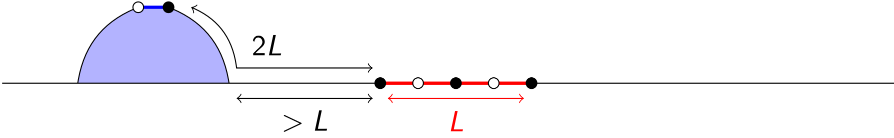
- Fix a red segment of length $L \geq 1$ on ∂Q_∞. We want to build a self-dual Boltzmann block B which:
  - separates the red segment from infinity,
  - has top and bottom boundaries smaller than the left and right boundaries.



- Explore Q_∞ with the following peeling algorithm: always peel the edge at distance $2L$ on the left of the red segment.
- Stop the exploration at the time τ where the boundary length process makes a jump $\leq -L$.

Construction of a self-dual block at a given scale

- Stop the exploration at the time τ where the boundary length process makes a jump $\leq L$.
- In particular, if a point at distance $< L$ to the red segment is hit, the exploration stops.



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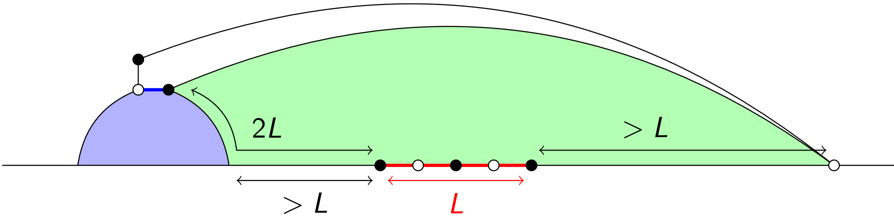
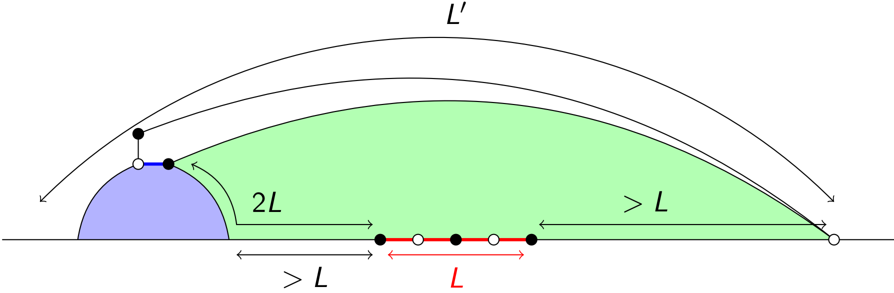


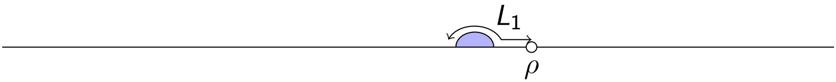
- Assume the jump at time τ is $< -4L$ and is caused by swallowing a region B on the right of the peeled edge.
- Then the hole in green is filled by a Boltzmann quadrangulation with top and bottom boundaries $\leq L$ and left, right boundaries $\geq L$.

Construction of a self-dual block at a given scale



- $\mathbb{P}$(the first jump $< -L$ is $< -4L$) $\to \delta > 0$ as $L \to +\infty$.
- So with probability $\geq \frac{\delta}{2}$, the green block has the right dimensions.
- Scaling limit results on the boundary length process: $\mathbb{E}\left[\frac{L'}{L}\right]$ is bounded.
- Iterate this construction in the unexplored region, with L' playing the role of the new L.

- Repeat this construction, starting from $L_0 = 1$:



- At each scale, conditionally on the previous ones, probability to have a "nice" green block $\geq \frac{\delta}{2}$, so the probability to have a resistance at least 1 is $\geq \frac{\delta}{4}$.
- Hence $R\left(\rho \leftrightarrow n\text{-th blue region}\right) \geq \frac{\delta}{4} n$
- On the other hand $\mathbb{E}[L_n] \leq e^{cn}$, so the segment of length L_n is at exponential distance from ρ.

Construction of a logarithmic number of blocks

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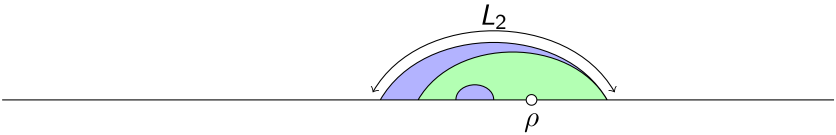

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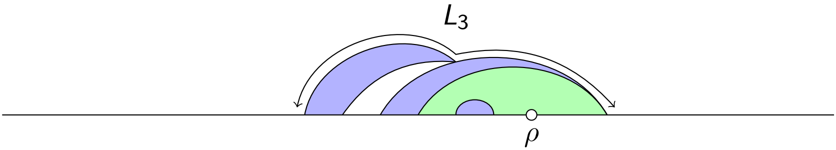

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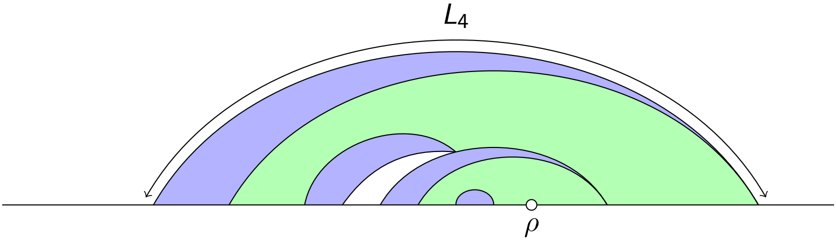

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Perspectives

- Self-duality is crucial, so no hope to generalize the approach to other classes of maps (e.g. triangulations).
- Full plane topology?
- Matching lower bound? Natural first step: prove that the resistance in a large self-dual block is typically of order 1.
- Self-dual random maps equipped with statistical physics models?

THANK YOU !