On the minimal diameter of hyperbolic surfaces

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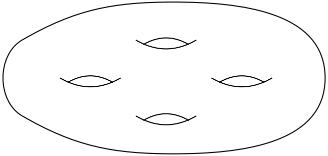
Introduction

- Goal: study the minimal possible diameter of hyperbolic surfaces with high genus g: asymptotic to $1 \times \log g$.



- Small diameter $\approx$ highly connected objects.
- Random (say, 3-regular) graphs are also very connected: **probabilistic method**, very common in combinatorics.
- Probabilistic method in hyperbolic geometry.

Program

I The diameter of 3-regular random graps

II Notions of hyperbolic geometry

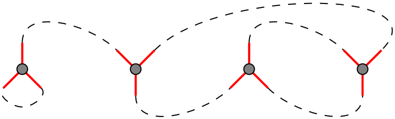
III The diameter of hyperbolic surfaces

IV Ideas of the proof

V Perspectives

The diameter of 3-regular random graphs

- G_n obtained from n vertices with 3 half-edges each, by matching the half-edges uniformly at random (connected with proba $1 - O(1/n)$).
- Diameter: maximal graph distance between two vertices.



Theorem (Bollobas–Fernandez de la Vega, 1982)

$$\frac{\mathrm{diam}(G_n)}{\log_2 n} \xrightarrow[n \to +\infty]{(P)} 1.$$

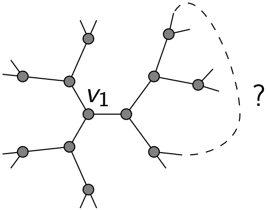
- Lower bound: a ball of radius r has size at most 3×2^r, so the diameter is $\geq \log_2 n$.

Diameter of random graphs: proof

- Upper bound: it is enough to prove that for any two fixed vertices v_1, v_2:

$$\mathbb{P}\left((1+\varepsilon)\log_2 n \leq d_{G_n}(v_1, v_2) < +\infty\right) = o\left(\frac{1}{n^2}\right).$$

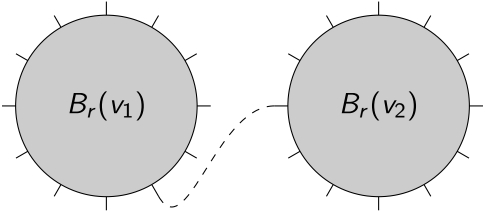
- Explore balls of radius $r = \frac{1+\varepsilon}{2}\log_2 n$ around v_1 and v_2, and try to connect them.



- If no "bad step", we would have $|\partial B_r(v_1)| = 3 \times 2^r = 3n^{\frac{1+\varepsilon}{2}}$.

Diameter of random graphs: proof

- But $\mathbb{P}(\text{bad step at time } i) \leq \frac{i+2}{3n}$, with independence over i.
- Consequence: with probability $1 - o\left(\frac{1}{n^2}\right)$:
  - $O(1)$ bad steps in the ball of radius $\frac{1-\varepsilon}{2}\log_2 n$ bad steps,
  - $o\left(n^{\frac{1+\varepsilon}{2}}\right)$ bad steps between distances $\frac{1-\varepsilon}{2}\log_2 n$ and $\frac{1+\varepsilon}{2}\log_2 n$,
- so $|\partial B_r(v_1)| \geq \delta n^{\frac{1+\varepsilon}{2}}$ w.h.p., and the same is true around v_2.



Diameter of random graphs: proof

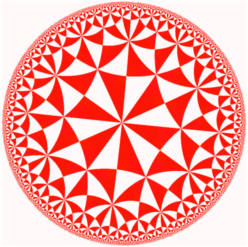
- If $B_r(v_1) \cap B_r(v_2) \neq \emptyset$, we are done.
- If not, each loose half-edge on $\partial B_r(v_1)$ has probability $\frac{|\partial B_r(v_2)|}{n} \geq \delta n^{-\frac{1-\varepsilon}{2}}$ to be connected to $B_r(v_2)$. So

$$\begin{aligned}\mathbb{P}\left(B_r(v_1) \text{ and } B_r(v_2) \text{ not directly linked}\right) &\leq \left(1 - \delta n^{-\frac{1-\varepsilon}{2}}\right)^{n^{\frac{1+\varepsilon}{2}}} \\ &\leq \exp\left(-\delta n^{\varepsilon}\right) \\ &= o\left(\frac{1}{n^2}\right),\end{aligned}$$

and $d_{G_n}(v_1, v_2) \leq 2r + 1 \leq (1+\varepsilon)\log_2 n$ with very high probability.

Hyperbolic geometry

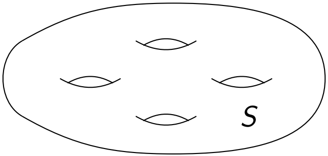
- The *hyperbolic plane* $\mathbb{H}$ can be seen as the unit disk, equipped with the metric
$$\mathrm{d}s^2 = \frac{4\mathrm{d}x^2}{1-|x|^2}.$$



- *Curvature*: $|B_\varepsilon(x)| = \pi\varepsilon^2 - \frac{\pi}{12}\varepsilon^4 K(x) + o(\varepsilon^4)$.
- Riemann uniformization theorem: $\mathbb{H}$ is the unique simply connected surface with constant curvature equal to -1.

Compact hyperbolic surfaces

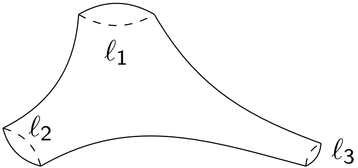
- A *compact hyperbolic surface* S is a $2d$ manifold equipped with a Riemannian metric with constant curvature -1. We consider *closed* surfaces, i.e. no boundary.
- Gauss–Bonnet formula: $\int_S K(x)\mathrm{d}x = 2\pi(2-2g)$, where g is the *genus* of the surface, i.e. the number of holes. So $g \geq 2$.



- Equivalent definitions:
  - S is locally isometric to $\mathbb{H}$,
  - S is a quotient of $\mathbb{H}$ (by a nice enough group action),
  - S is a surface equipped with a conformal structure.

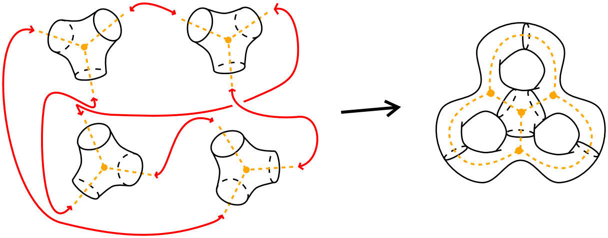
Pairs of pants

- Existence, but no uniqueness: for $g \geq 2$, hyperbolic metrics on a genus g surface form a $(6g-6)$-dimensional space M_g called the *moduli space*.
- One way to build a lot of them is to use *pants*.
- For any $\ell_1, \ell_2, \ell_3 \geq 0$, there is a unique surface isomorphic to the sphere minus 3 disjoint disks, such that:
  - the boundaries of the three disks are closed geodesics with lengths ℓ_1, ℓ_2, ℓ_3;
  - the curvature is -1 outside of the boundary.



Gluings of pants

- By gluing $2g-2$ pairs of pants such that the lengths of the boundaries match two by two, we can build many hyperbolic surfaces.
- $6g-6$ degrees of freedom: $3g-3$ for the lengths of the cycles, and $3g-3$ for the twists.
- Conversely, every hyperbolic surface of genus g can be cut by $3g-3$ closed geodesics into $2g-2$ pairs of pants.

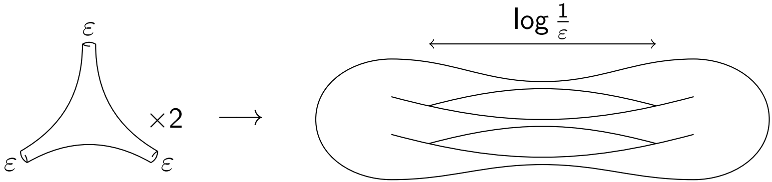


Interesting quantities for hyperbolic surfaces

- Given a hyperbolic surface S, several natural quantities to look at, and try to optimize over the moduli space:
  - diameter,
  - spectral gap (eigenvalues of the Laplacian),
  - Cheeger constant (isoperimetric inequalities),
  - systole (length of the smallest closed geodesics).
- All of these measure the "connectivity" of the surface.
- In the context of hyperbolic surfaces, non-optimal bounds (constant factors) often obtained via arithmetic constructions [Brooks, Buser, Kim, Sarnak...].
- A typical graph is very connected, so random graphs (like uniform 3-regular graphs) are close to optimal for these quantities.

Diameter of hyperbolic surfaces

- Diameter: maximal distance between two points of S.
- Easy: $\sup_{S \in M_g} \mathrm{diam}(S) = +\infty$.



- Lower bound for the minimal diameter: volume growth argument
  - By Gauss-Bonnet, $\mathrm{Area}(S) = 2\pi(2g-2)$.
  - In $\mathbb{H}$, the area of balls is $\mathrm{Area}(B_r(x)) = 2\pi(\cosh(r) - 1)$.
  - S is a quotient of $\mathbb{H}$, so $\mathrm{Area}(B_r(x)) \leq 2\pi(\cosh(r) - 1)$ in S.
  - So if $B_r(x)$ covers S, then $\cosh(r) - 1 \geq 2g - 2$, so

$$\inf_{S \in M_g} \mathrm{diam}(S) \geq \cosh^{-1}(2g-1) = \log g + O(1).$$

  - Best lower bound **[Bavard 1996]**: also $\log g + O(1)$.

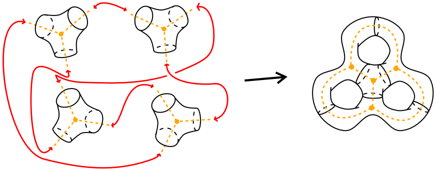
Main theorem

Theorem (B.–Curien–Petri, 2019)

We have

$$\min_{S \in M_g} \operatorname{diam}(S) = (1 + o(1)) \log g.$$

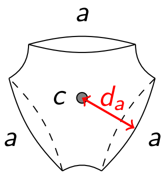
- Construction: random gluing of pants!
- Start from $2g - 2$ pants with perimeters (a, a, a), and glue the $6g - 6$ holes uniformly at random to obtain $S_{g,a}$.
- Twist 0: the "centers" of two neighbour pants have the same projections on the boundary.

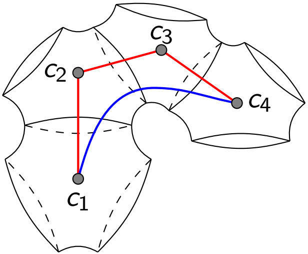


- We show $\operatorname{diam}(S_{g,a}) \sim \frac{1}{\beta_a} \log g$ w.h.p., where $\beta_a < 1$, but $\beta_a \to 1$ as $a \to \infty$.

A crude bound

- For a fixed, the diameter of a pair of pants is a constant. Enough to bound distances between the centers of the pants.
- Quick bound: $\mathrm{diam}\,(S_{a,g}) \leq 2d_a \mathrm{diam}\,(G_{2g-2}) \underset{n\to\infty}{\sim} 2d_a \log_2 n$. After computing d_a for $a \to +\infty$, we get $\approx 1.38 \log g$.

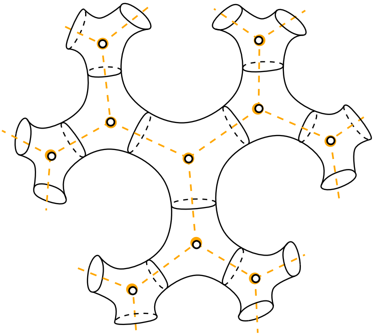




- Not optimal: sometimes, there is a much shorter path.

Adapting the explorations

- Instead of using random graphs as a black box, adapt the proof!
- Adapt the exploration to the *hyperbolic* metric, instead of the graph distance.
- Ideal situation: the neighbourhood of one center looks like an infinite tree of pants T_a. We need to understand its growth!



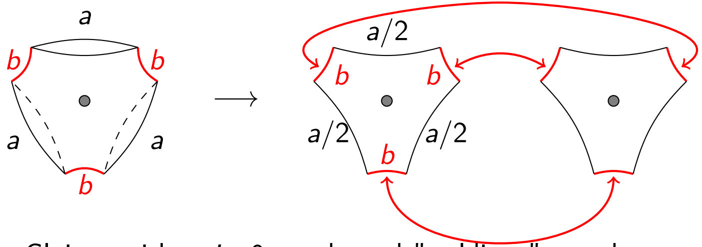
Growth of the infinite tree of pants

- B_r: ball of radius r around a center of the pants tree T_a. Let $|B_r|$ be the number of pants whose center is in B_r.

Lemma

We have $|B_r(T_a)| \underset{r \to +\infty}{\sim} C_a e^{\beta_a r}$, where $\beta_a \to 1$ as $a \to +\infty$.

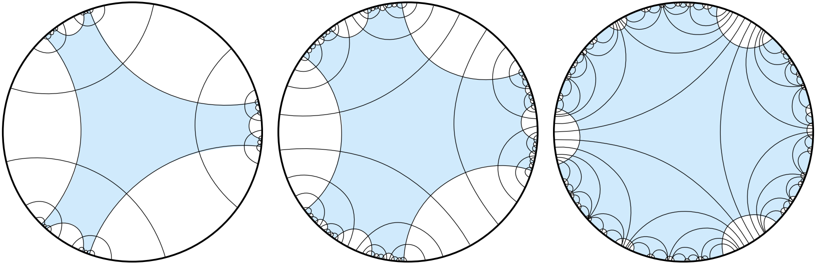
- Sketch of proof: pants can be decomposed in two right-angled hexagons.



- Gluings with *twist* 0, so the red "weldings" match on neighbour pants.

Growth of the infinite pair of pants

- Hence, the tree of pants is the gluing of two copies of an infinite tree of right-angled hyperbolic hexagons:



- Above: infinite tree of hexagons for increasing values of a.
- The growth of hexagon trees corresponds to orbital counting for a subgroup of $PSL_2(\mathbb{R})$ generated by reflexions. This is well understood by geometers [Patterson–Sullivan, McMullen...]

Bounding the diameter

- As for graphs, we want to show, for any centers c_1, c_2:

$$\mathbb{P}\left(d_{\mathrm{hyp}}(c_1, c_2) \geq \left(\frac{1+\varepsilon}{\beta_a}\right)\log g\right) = o\left(\frac{1}{g^2}\right).$$

- We explore the balls of radius $r = \frac{1+\varepsilon}{2\beta_a}\log g$ around c_1 and c_2 for the *hyperbolic metric* on the infinite tree of pants.
- As for graphs, we can bound the number of "bad" steps: the volume and boundary of $B_r(c_1)$ are at least a constant times what they would be in the tree of pants.
- So $|\partial B_r(c_1)| \geq \delta \exp\left(\beta_a \frac{1+\varepsilon}{2\beta_a}\log g\right) = \delta g^{\frac{1+\varepsilon}{2}}$, and the same is true for c_2.
- As for graphs, this implies that with very high probability, there is an edge between $B_r(c_1)$ and $B_r(c_2)$, so $d_{\mathrm{hyp}}(c_1, c_2) \leq 2r + O(1) = \frac{1+\varepsilon}{\beta_a}\log g + O(1)$.

Perspectives

- Error term? For random graphs $O(\log\log n)$. It should also be true here (for $a = \log\log n$).
- Other natural models of random surfaces:
  - Brooks–Makover surfaces (built by uniformizing random triangulations with unconstrained genus): diameter $\sim 2\log g$ **[BCP 2019+]**,
  - Weil–Petersson random surfaces ($\approx$ "Lebesgue measure" on the space of hyperbolic surfaces): diameter $\leq 40\log g$ **[Mirzakhani 2013]**.
- Study other quantities? Maximal spectral gap for hyperbolic surfaces? Cheeger constant?
- Hyperbolic manifolds in higher dimensions?

THANK YOU !