Markovian triangulations and robust convergence to the UIPT

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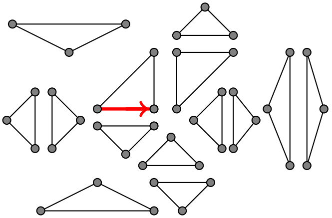
Outline

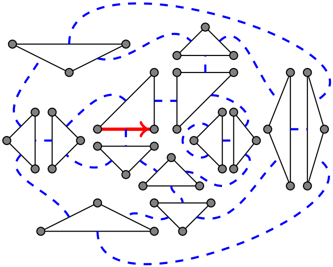
- The UIPT is a natural "uniform" model of discrete, infinite planar geometry:
  - obtained by convergence of finite models relying on enumerative combinatorics;
  - nice to study because of its *Spatial Markov property*.
- Goals:
  - classify infinite objects exhibiting a similar Markov property;
  - use this to prove the convergence of finite models to the UIPT in a "robust" way, without precise enumeration.

Triangulations of the sphere



- A *triangulation of the sphere* with $2n$ faces is a set of $2n$ triangles whose sides have been glued two by two, in a way that is homeomorphic to the sphere.
- Our triangulations are *of type I* (we may glue two sides of the same triangle), and *rooted* (oriented root edge).
- Exact, explicit enumeration **[Tutte 60s]**:

$$\#\mathcal{T}_n = 2\frac{4^n(3n)!!}{(n+1)!(n+2)!!}.$$



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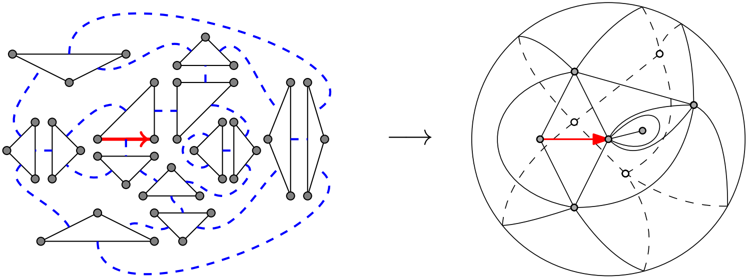

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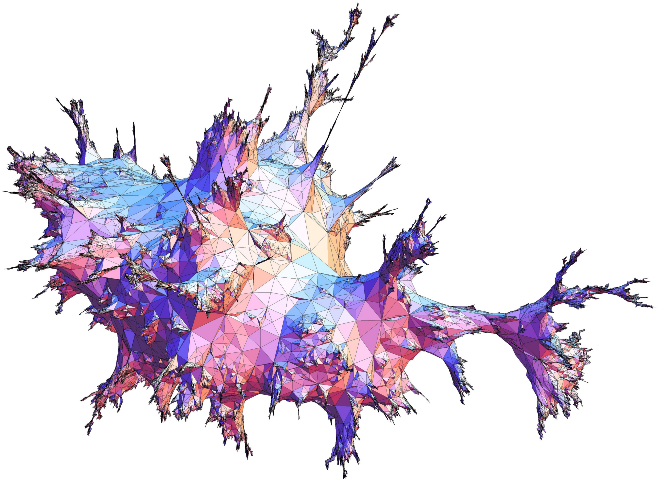
Random triangulations of the sphere

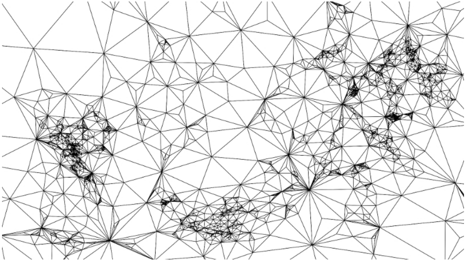
- Let T_n be a uniform triangulation of the sphere with $2n$ faces. What does T_n look like?



- *Local convergence*:

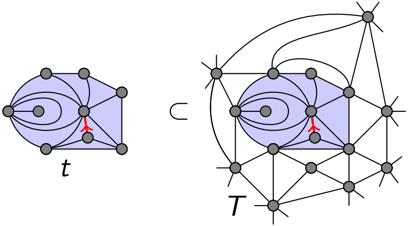
$$d_{\mathrm{loc}}(t, t') = \left(1 + \max\left\{r \geq 0 | B_r(t) = B_r(t')\right\}\right)^{-1}.$$

- Then T_n converges in distribution for the local topology to a random triangulation of the plane called the UIPT (Uniform Infinite Planar Triangulation) $\mathbb{T}$ **[Angel–Schramm 03]**.



The argument of Angel and Schramm

- Let t be a small triangulation with a hole:



- Then $\mathbb{P}(t \subset T_n) = \frac{\#\text{ways to fill the hole}}{\#\mathcal{T}_n}$ depends on the perimeter and volume of t. It is explicit by the enumeration of Tutte, and converges as $n \to +\infty$.

The argument of Angel and Schramm

- Before that:
  - Tightness (control vertex degree): uses that $\frac{\#\mathcal{T}_{n+1}}{\#\mathcal{T}_n}$ is bounded.
  - One-endedness, i.e. in the limit there is no finite set separating two infinite regions: uses that

$$\sum_{\substack{k+\ell=n \\ k,\ell \gg 1}} \#\mathcal{T}_k \times \#\mathcal{T}_\ell \ll \#\mathcal{T}_n.$$

- Can be mimiced for many models, as long as exact enumeration is known.
- What if add "perturbations" that make the model too hard to count? For example, if we start with n triangles and $o(n)$ quadrangles?

The spatial Markov property of the UIPT $\mathbb{T}$

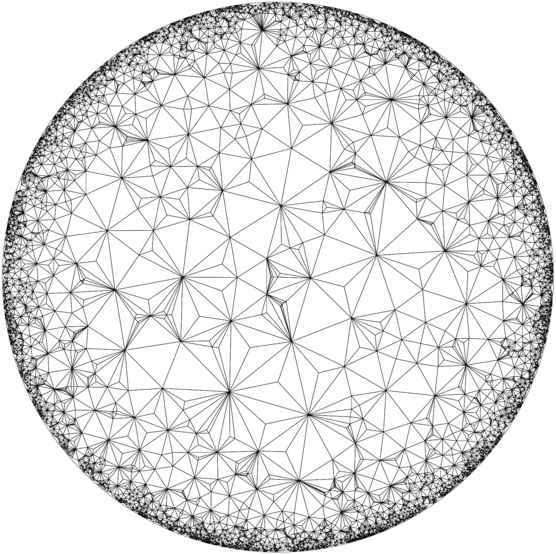
- By the Angel–Schramm argument $\mathbb{P}(t \subset \mathbb{T}) = \lim_{n\to+\infty} \mathbb{P}(t \subset T_n)$ only depends on the perimeter and volume of t.
- Consequence: when we explore the UIPT "face by face" using *peeling explorations*, the perimeter and volume of the explored region follow a Markov chain with values in $\mathbb{N}^2$.
- Useful tool to study the fractal-like properties of $\mathbb{T}$ (e.g. volume growth in r^4 **[Angel 04]**).

- The PSHT (Planar Stochastic Hyperbolic Triangulations) are the local limits of uniform triangulations of genus g with $2n$ faces when g is proportional to n [B.–Louf 19].
- They form a one-parameter family $(\mathbb{T}_\lambda)_{0 \leq \lambda \leq \lambda_c}$, where $\lambda_c = \frac{1}{12\sqrt{3}}$. Characterized by a stronger version of the Markov property [Curien 14]:

$$\mathbb{P}\left(t \subset \mathbb{T}_\lambda\right) = C_{|\partial t|}(\lambda) \times \lambda^{|t|}.$$

- What do they look like?
  - $\lambda = \lambda_c$: the UIPT;
  - $0 < \lambda < \lambda_c$: hyperbolic (mean degree > 6, exponential volume growth...);
  - $\lambda = 0$: dual of a complete binary tree (degenerate object with infinite vertex degrees).

A PSHT $\mathbb{T}_\lambda$ with $0 < \lambda < \lambda_c$



Convergence of high genus triangulations

- The proof does not rely on asymptotic enumeration!
- Main ingredients:
  - Tightness: "robust" adaptation of the tightness argument of Angel–Schramm;
  - Planarity and one-endedness of subsequential limits: use the Goulden–Jackson recursion (coming from algebraic combinatorics);
  - Any limit must have the spatial Markov property;

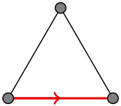
Theorem (B.–Louf 19)

Let T be a Markovian planar, one-ended, random infinite triangulation. Then T is of the form $\mathbb{T}_\Lambda$, where Λ is a random variable on $[0, \lambda_c]$ (PSHT with a random parameter).

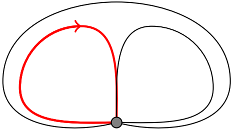
  - Compare estimates on the PSHT and surgery arguments to prove that Λ is deterministic.
- Almost "enumeration-free" proof: if we want to replicate this sketch on planar models, the weak point is one-endedness.

Infinite triangulations

- Infinite triangulation: family of countably many triangles glued along their edges and vertices. We do not assume one-endedness, nor finite vertex degrees.
- Examples of "degenerate" planar infinite triangulations:



$\mathbb{T}_0$ (PSHT of parameter 0)



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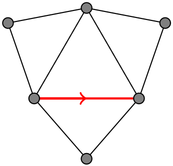

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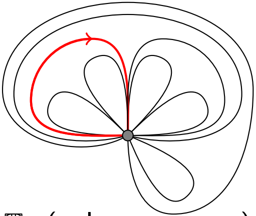

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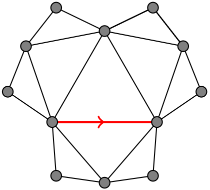

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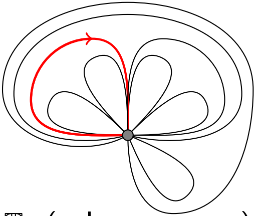

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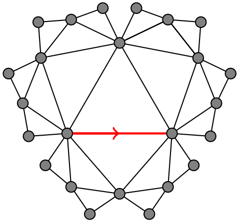

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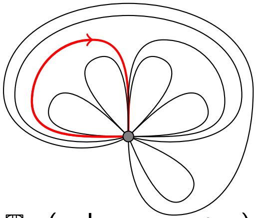

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Spatial Markov property (general version)

Definition

A random, infinite, planar triangulation T is *Markovian* if for any finite planar triangulation t with one or several holes, the probability $\mathbb{P}(t \subset T)$ only depends on the perimeters of the holes of t and its total number of faces.

Theorem (B.21+)

Let T be an infinite, planar, Markovian random triangulation. Then T is of the form $\mathbb{T}_\Lambda$, where Λ is a random variable with values in $[0, \lambda_c] \cup \{\star\}$.

- Consequences:
  - No nice notion of "uniform planar multi-ended triangulation".
  - The UIPT is the only Markovian planar triangulation where the expected inverse degree is $1/6$.

Robust convergence to the UIPT

- Sketch of a "combinatorics-free" proof of convergence of triangulations of the sphere to the UIPT:
  - "Dual local topology": use dual distance instead of graph distances.

$$d^*_{\mathrm{loc}}(t,t') = \left(1 + \max\left\{r \geq 0 \middle| B^*_r(t) = B^*_r(t')\right\}\right)^{-1}$$

  - Makes tightness immediate, but limits may have infinite vertex degrees;
  - The finite model is Markovian, so any subsequential limit is Markovian;
  - The expected inverse of the root degree in a triangulation of the sphere is $1/6$, and this passes to the limit;
  - So the UIPT is the only possible subsequential limit;
  - In particular the limit has finite vertex degrees, so convergence for the dual local topology implies convergence for the usual local topology.
- Robust argument: still works if we add a perturbation "small compared to the size n".

Examples of applications

- For example, this sketch allows to prove the convergence to the UIPT of:
  - "Triangulations with defects", i.e. maps with prescribed face degrees where triangles represent a proportion $1 - o(1)$ of the edges;
  - High temperature Ising triangulations, i.e. triangulations of size n equipped with an Ising model on the faces with inverse temperature $\beta_n \to 0$;
  - Uniform triangulations of size n and genus $g_n = o(n)$.
- "Meta-theorem": If we perturb the uniform measure by factors $e^{o(n)}$, we still have convergence to the UIPT.

Large deviations for pattern occurences in uniform triangulations

- For t_0 a triangulation with a hole and T a triangulation of the sphere, let $\mathrm{occ}_{t_0}(T)$ be the number of occurences of t_0 in T.
- Fix t_0, and let $T_n^{(\beta)}$ be a triangulation of the sphere of size n, picked with probability proportional to $e^{\beta\,\mathrm{occ}_{t_0}(T)}$.
- The previous sketch shows that if $\beta_n \to 0$, then $T_n^{(\beta_n)}$ converges locally to the UIPT.

Corollary

Let T_n be a uniform triangulation of the sphere with $2n$ triangles. Then for every $\varepsilon > 0$, the probability that

$$\left| \frac{\mathrm{occ}_{t_0}(T_n)}{6n} - \mathbb{P}\left(t_0 \subset UIPT\right) \right| > \varepsilon$$

decreases exponentially in n.

Sketch of proof in the one-ended case

- Let T be a one-ended, planar, infinite Markovian triangulation. For $v \geq p \geq 1$, let
$$a_v^p = \mathbb{P}(t \subset T)$$
for t a triangulation with perimeter p and v vertices in total.
- Peeling equations:
$$a_v^p = a_{v+1}^{p+1} + 2 \sum_{i=0}^{p-1} \sum_{j \geq 0} a_{v+j}^{p-i} \# \mathcal{T}_{i+1,j}.$$
- In particular, the law of T is determined by the numbers a_v^1.
- For a mixture of PSHT $\mathbb{T}_\Lambda$, we have $a_v^1 = \mathbb{E}\left[\Lambda^{v-1}\right]$.
- So we need to show that $\left(a_v^1\right)_{v \geq 1}$ is the sequence of moments of some variable Λ.

The Hausdorff moment problem

- Let Δ be the discrete derivative operator:

$$(\Delta u)_n = u_n - u_{n+1}.$$

Theorem (Hausdorff)

Let (u_n) be a sequence of real numbers. Then (u_n) is the sequence of moments of some $[0,1]$-valued random variable if and only if

$$\forall k \geq 0, \forall n \geq 0, \ \Delta^k u_n \geq 0.$$

- By induction on k, we prove $\left(\Delta^k a^p\right)_v \geq 0$ for all $k \geq 0$ and $v \geq p \geq 1$.
- So there is a variable $\Lambda \in [0,1]$ such that $a_v^1 = \mathbb{E}[\Lambda^{v-1}]$.
- The convergence of the sum in the peeling equation implies $\Lambda \in \left[0, \frac{1}{12\sqrt{3}}\right]$.

The multi-ended case

- Let T be a multi-ended Markovian triangulation. For a triangulation t with volume v and k holes of perimeters $p_1, \ldots, p_k$, we write

$$a_v^{p_1,p_2,\ldots,p_k} = \mathbb{P}\left(t \subset T \text{ and } T \backslash t \text{ has } k \text{ infinite components}\right).$$

- New peeling equation:

$$a_v^{p_1,\ldots,p_k} = a_v^{p_1+1,p_2,\ldots,p_k} + 2\sum_{i=0}^{p_1-1}\sum_{j\geq 0} a_{v+i+j}^{p_1-i,p_2,\ldots,p_k} \times \#\mathcal{T}_{i+1,j} + \sum_{i=0}^{p_1-1} a_v^{i+1,p_1-i,p_2,\ldots,p_k}.$$

- The first term in the RHS has "larger" perimeters, so the law of T is determined by terms of the form $a_v^{1,\ldots,1} = a_v^{k\otimes 1}$.

The multi-ended case

- The law of T is determined by terms of the form $a_v^{1,\dots,1} = a_v^{k\otimes 1}$.
- Hausdorff moment problem : there is (Λ, Γ) such that, for all $k, v \geq 1$:

$$a_v^{k\otimes 1} = \mathbb{E}\left[\Lambda^{v-1}\Gamma^{k-1}\right].$$

- We want either $\Gamma = 0$ (PSHT $\mathbb{T}_\lambda$) or $(\Lambda, \Gamma) = (0, 1)$ (degenerate $\mathbb{T}_\star$).
- From here, we can solve completely the peeling equations:

$$a_v^{p_1,\dots,p_k} = \mathbb{E}\left[\Lambda^{v-1}\Gamma^{k-1}\prod_{i=1}^{k} C_{p_i}(\Lambda, \Gamma)\right],$$

where $C_p(\lambda, \gamma)$ is given by the induction

$$C_p = C_{p+1} + 2\sum_{i=0}^{p-1}\lambda^i Z_{i+1}(\lambda) C_{p-i} + \gamma \sum_{i=0}^{p-1} C_{i+1} C_{p-i}.$$

The multi-ended case

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- In particular, the generating function $\sum_{p\geq 1} C_p(\lambda,\gamma)x^p$ is explicit.
- Behaviour of the generating function near its first singularity $\rightarrow$ if $\gamma > 0$ and $(\lambda,\gamma) \neq (0,1)$, then $C_p(\lambda,\gamma) < 0$ for some p.
- Recall $a_v^{p,(k-1)\otimes 1} = \mathbb{E}\left[\Lambda^{v-1}\Gamma^{k-1}C_p(\Lambda,\Gamma)\right]$.
- When k, v get large, this is close to $\lambda_{\max}^{v-1}\gamma_{\max}^{k-1}C_p(\lambda_{\max},\gamma_{\max})$, which is negative for some p if $\gamma_{\max} > 0$.
- So almost surely $\Lambda = 0$ or $\Gamma = 0$.
- Similar argument to exclude $\Lambda = 0$ and $0 < \Gamma < 1$.

More questions

- Different kinds of face degrees ?
- What if we remove the planarity assumption? Conjectures:
  - There are no Markovian nonplanar triangulations with finite vertex degrees...
  - ...but there should be nonplanar, degenerate objects.

THANK YOU !