Forcing clique immersions via chromatic number

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1 Introduction

2 Sketch of proof
Immersion – a variant of minor

Definition by mapping: map a graph $H$ into another $G$:

**Minor: $H \leq_m G$**
- Vertices of $H \rightarrow$ disjoint self-connected sets of vertices of $G$.
- Edges of $H \rightarrow$ distinct edges of $G$.

**Immersion: $H \leq_i G$**
- Vertices of $H \rightarrow$ distinct vertices of $G$.
- Edges of $H \rightarrow$ edge-disjoint paths of $G$. 
Alternative definition of immersion

- To **split off** a path, we replace the path by an edge between its endpoints.
- **G** contains **H** as an **immersion** if **H** can be obtained from a subgraph of **G** by a series of splitting off paths.

Figure: Example: **G** contains **H** as an immersion.
Hadwiger’s conjecture and its immersion variant

Hadwiger’s conjecture’43

\[ \chi(G) \geq t \implies G \geq_m K_t. \]

Conjecture 1 (Lescure–Meyniel’89, Abu-Khzam–Langston’03)

\[ \chi(G) \geq t \implies G \geq_i K_t. \]

- \( n \leq 6 \): proved by Lescure–Meyniel’89,
- \( n = 7 \): by Devos, Kawarabayashi, Mohar and Okamura ’10.
Conjecture 1

Relaxation of Conjecture 1

Find some good function \( \varphi \) such that

\[
\chi(G) \geq \varphi(t) \implies G \geq_i K_t.
\]

We can attack the relaxation via minimum degree.

Observation

If the following holds true for any \( G \),

\[
\delta(G) \geq \varphi(t) \implies G \geq_i K_t,
\]

then the following also holds true for any \( G \),

\[
\chi(G) \geq \varphi(t) + 1 \implies G \geq_i K_t.
\]
## Main result

<table>
<thead>
<tr>
<th>Theorem (Devos, Dvořak, Fox, McDonald, Mohar and Scheide’14)</th>
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<tbody>
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<td>[ \delta(G) \geq 200t \implies G \geq_i K_t. ]</td>
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<th>Theorem (Dvořák–Yepremyan’15+)</th>
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<td>[ \delta(G) \geq 11t + 7 \implies G \geq_i K_t. ]</td>
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<th>Theorem (Le–Wollan’16+)</th>
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<td>[ \delta(G) \geq 7t + 7 \implies G \geq_i K_t. ]</td>
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<td>[ \chi(G) \geq 3.54t + 4 \implies G \geq_i K_t. ]</td>
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1 Introduction

2 Sketch of proof
Lemma 1

Given $G$ of $n$ vertices and $A \subseteq V(G)$ of $t$ vertices.

- $\overline{d}_G(A)$: average degree (in $G$) of vertices in $A$.
- $\delta_G(A)$: min degree (in $G$) of vertices in $A$.

Lemma 1

If $\overline{d}_G(A) + \delta_G(A) \geq n + 2 + t$, then $G \geq_i K_t$.

- Idea of proof: repeatedly splitting off paths of length 2 and 4 with endpoints in $A$, noting that the degree of endpoints are unchanged after a splitting off.
Proof of $\delta(G) \geq 7t + 7 \implies G \geq_i K_t$.

• Let $G^*$ be a smallest (number of vertices) counterexample.

• Then $G^*$ contains an immersion of some eulerian graph $G$ such that
  • max-deg of $G$ is at least $7t$,
  • $\sum_{x:d_G(x)<7t} (7t - d_G(x)) \leq 7t$.

• Let $x$ be a max-deg vertex in $G$, let $A = \{x\}$ and $B = N_G(x)$.

• Repeatedly add vertices (who have many edges to $B$) into $A$.

• We can always find such vertices, otherwise there is some counterexample smaller than $G$.

• Stop when $|A| = t$. 
Proof of $\delta(G) \geq 7t + 7 \implies G \geq_i K_t$.

- Let $H = G[A \cup B]$. There are two cases:

  **CASE 1:**
  - All vertices in $A$ have high enough degree (in $H$).
  - Then $\overline{d}_H(A) + \delta_H(A)$ is high enough.
  - Then apply Lemma 1.

  **CASE 2:**
  - Most of vertices in $A$ have high degree, but some have low degree.
  - Then $\overline{d}_H(A)$ is high, but $\delta_H(A)$ is low.
  - Then use a specific strategy (similar to Lemma 1) to split off paths of length 2 and 4 to obtain $K_t$. $\square$
Lemma 2

- Let $\alpha = \frac{2|E|}{n^2}$ (density of $G$), we have:

\\
\begin{align*}
\text{Lemma 2} \\
\text{If } 0.5 \leq \alpha \leq 0.75, \text{ then } G \geq_i K_t, \text{ where } t = (2\alpha - 1)n.
\end{align*}
\\
- For instance, if $\alpha = 0.7$, then $G \geq_i K_{0.4n}$
Proof of $\chi(G) \geq 3.54t + 4 \iff G \geq_i K_t.$

- Let $G^*$ be a smallest (number of vertices) critical counterexample.
- Let $x$ be a min-deg vertex, and $N = N_{G^*}(x)$.
- Let $\chi(G^*) = \ell$, and $G = G^* \backslash \{x\}$, then $\chi(G) = \ell - 1$.
- Given a coloring of $G$ with $\ell - 1$ colors:
  - **singleton**: a vertex in $N$ unique with its color.
  - **doubleton**: two vertices (and only them) in $N$ sharing the same color.
- Consider a coloring of $G$ maximizing number of singletons (noting that $N$ must have all $\ell - 1$ colors).
Proof of $\chi(G) \geq 3.54t + 4 \iff G \geq_i K_t$.

There is a Kempe chain between any **singleton–singleton**:

![Singleton-Singleton Diagram]

For any **singleton–doubleton**, there are two cases:

![Singleton-Doubleton Case 1]

or

![Singleton-Doubleton Case 2]
Proof of $\chi(G) \geq 3.54t + 4 \implies G \geq_i K_t$.

For any doubleton–doubleton, there are 3 cases:
Proof of $\chi(G) \geq 3.54t + 4 \iff G \geq_i K_t$.

- Split off these chains to get new edges.

- Consider the graph $H$ with:
  - vertex set: $S \cup D_1 \cup D_2$
  - edge set: new edges.

- Immerse from $H$ a very dense graph $H'$ on $S \cup D_1$ by splitting off paths and swapping some primary–secondary pairs.

- Obtain $H'$ with density $\approx 0.7$, noting that $|S \cup D_1| \geq 2.6t$.

- Apply Lemma 2 to get a $K_t$ immersion. $\square$
Thank you.