Exercice 1.  

**Jackson’s Networks**

A Jackson’s network is a set of $N$ queues of type $G/M/c_p$ with service rates $\mu_p(n_p) = \mu_p \cdot \min(c_p, n_p)$ where $n_p$ is the number of guys in queue $p$ and $c_p$ is the number of parallel servers for this queue. These queues are connected together such that the output of queue $p$ may enter queue $q$ with probability $r_p,q$ or leave the system forever with probability $r_p,N+1$.

From now, we will suppose that $\mu_p(n_p) > 0$ as soon as $n_p > 0$ and that $r_p,p = 0$ for commodity. The exogeneous arrivals (i.e. arriving from outside the system) in queue $p$ follow a Poisson process of intensity $\lambda_p$. We admit that the vector $X_t$ of the number of guys in each queue is a continuous time Markov chain.

1. Draw some figure representing the features of this system and describe the infinitesimal generator of $X_t$.

   Hint: The space of states is $N^N$, not $N$ as usual.

2. Suppose that the system is in stationary state. Describe the equilibrium equations over all the rates in the network.

3. Show that if the network has no deadlock, i.e every guy has a non null probability to leave the system, then the preceding set of equations over the rates admits an unique positive and finite solution.

4. Show the following theorem: given a continuous time irreducible Markov chain with infinitesimal generator $Q = q_{ij}$, if there exists a distribution $\pi$ and positive numbers $\tilde{q}_{ij}$, $i \neq j$, such that $\pi_i \tilde{q}_{ij} = \pi_j q_{ji}$ and $\sum_{j \neq i} \tilde{q}_{ij} = -q_{ii}$, then $\pi$ is the invariant distribution of the chain.

5. Suppose that each queue is $M/M/c$ and independent from the others. What is the invariant distribution for $X_t$?

6. Show that the previous answer is actually the invariant distribution of the system (although the queues are not independent). Explain why Jackson’s network are sometimes called *product-form networks*.

Exercice 2.  

**The bus paradox**

A bus company assures that the arrival time of the buses to a bus stop is modeled by a Poisson process $(X_n)_n$ of rate $\lambda = 0.1$ bus per minute and, thus, the average time between two buses is 10 minutes.

A client, Mister Relou, takes the bus every day at time $t \in \mathbb{R}$ after the beginning of the service and notes every day his waiting time at the bus stop. Mister Relou considers that he should wait in average 5 minutes “because” the average time between two buses is 10 minutes. He notices that he is waiting 10 minutes in average and complains to the bus company.

1. Calculate the probability that Mister Relou misses the $n$-th bus but catches the $(n+1)$-th bus.

2. Calculate the probability that he misses the $n$-th bus and has to wait the $(n+1)$-th bus for at least $s$ minutes.

3. Calculate the probability that he has to wait the next bus for at least $s$ minutes.

4. Deduce the average waiting time and answer to Mister Relou.

Exercice 3.  

**Isn’t this traffic light a bit too long?**

On an avenue in Monaco, the car traffic is modelled by a Poisson process of rate $\lambda = 2$ cars per minute.

1. What is the law of the arrival time $X_n$ of the $n$-th car?

2. With the Central Limit Theorem, give a Gaussian approximation of the law of $X_n$.

Because of a traffic light, the flow of cars is regularly stopped for $T$ minutes. We assume that when the traffic light turns green, all the waiting cars can go through the intersection. We also assume that all the cars in Monaco are limousines that occupy 10 meters while stopped.
3. For how long can we stop the traffic if we want that the resulting queue reaches a length of 250 meters only with a probability 0.2?

Note: The value of $x$ for which $\Pr(N(0, 1) < x) = 0.2$ is $x \approx -0.85$.

**Exercice 4.**  

L’important c’est de participer (from J.-F. Delmas)

At the London 2012 Olympic Games, France got 34 medals including 11 gold medals, over a total number of 962 medals including 302 gold medals. The earth population is roughly $6 \times 10^9$ people with 60 millions in France. Can we say the elite sportsmen are uniformly distributed inside the earth population?