**Exercise 1.**

Let us recall the (useful) Foster theorems:

**Theorem (First Foster theorem).** Let \((X_n)\) be a homogeneous irreducible Markov chain of general term \(p_{i,j}\) on a countable set \(E\). If there exists a function \(h : E \to \mathbb{R}^+\), a finite set \(F\) and a constant \(\epsilon > 0\) such that:

\[
\mathbb{E}(h(X_1)|X_0 = i) = \sum_{k \in E} p_{ik} h(k) < \infty \quad \text{for all } i \in F
\]

\[
\mathbb{E}(h(X_1) - h(X_0)|X_0 = i) = \sum_{k \in E} p_{ik} h(k) - h(i) \leq -\epsilon \quad \text{for all } i \notin F,
\]

then \((X_n)\) is positive recurrent.

**Theorem (Second Foster theorem).** Let \((X_n)\) be a homogeneous irreducible Markov chain of general term \(p_{i,j}\) on a countable set \(E\). If there exists a function \(h : E \to \mathbb{R}^+\) and a finite set \(F\) such that:

\[
\mathbb{E}(h(X_1) - h(X_0)|X_0 = i) = \sum_{k \in E} p_{ik} h(k) - h(i) < +\infty \quad \text{for all } i \in E
\]

\[
\mathbb{E}(h(X_1) - h(X_0)|X_0 = i) = \sum_{k \in E} p_{ik} h(k) - h(i) \geq 0 \quad \text{for all } i \notin F
\]

\[
h(j_0) > \max_{i \in F} h(i) \quad \text{for some } j_0 \notin F
\]

then \((X_n)\) is not positive recurrent.

Aloha is a communication protocol on a canal shared by several stations unaware of each other. Transmissions and retransmissions can only start at times of type \(k\Delta\) with \(k\) integer and \(\Delta > 0\) the width of a slot. When two stations simultaneously try to transmit messages, they interfere each other and detect the conflicts and both delay the transmissions. This means that there is at most 1 message in the whole system is transmitted at each step. The protocol is the following:

- Fresh messages systematically try to pass right after their arrival.
- In case of conflict, each concerned station independently tries to retransmit its message at the next slot with probability \(0 < \nu < 1\).

We denote by \(A_n\) the r.v. representing the number of fresh messages arriving to the system at the beginning of step \(n\). We assume that \(A_n\) are i.i.d. and we set \(a_i = \mathbb{P}(A_n = i)\), \(\lambda = \mathbb{E}(A_n) = \sum_{i=0}^{\infty} i a_i\). We also denote by \(X_n\) the r.v. representing the number of messages delayed at the beginning of step \(n\).

**Aloha Instability:**

1. Compute the probability \(b_i(k)\) that if \(k\) stations are in conflict, \(i\) stations among them will try to retransmit in the next step.

We will assume to simplify that the retransmission of a message depends only on the message itself and not on its station. This has a bad consequence that two messages from the same station can conflict.

2. Compute the probability \(p_{k,l}\) that the number of delayed messages in the system is shift from \(k\) to \(l\) after one step.

3. Show that this protocol is unstable (i.e., \((X_n)\) is not positive recurrent).

4. What does likely happen in this system in the long run?
**B - Aloha Stabilization**

Instead of using a retransmission policy with $\nu$ fixed, we will try to reach stability using $\nu(k)$ depending on the number of delayed messages. We will show that the following condition implies stability.

$$\lambda < \liminf_{k \to +\infty} (b_1(k)a_0 + b_0(k)a_1)$$

It is equivalent to the existence of $\varepsilon > 0$ and a finite set $F \subset \mathbb{N}$ such that

$$\lambda < b_1(k)a_0 + b_0(k)a_1 - \varepsilon \quad \text{for all } k \notin F.$$

5. Under this assumption, prove the stability of the protocol.

6. Find the maximum of $g_k(\nu) = (1 - \nu)^k a_1 + k\nu(1 - \nu)^{k-1} a_0$.

7. Noticing that $\left(\frac{k-1}{k-n_1/n_0}\right)^{k-1} \xrightarrow{k \to \infty} \exp(\frac{a_1}{a_0} - 1)$, give a sufficient stability condition.

8. Explicit this condition when $A_n$ follows a Poisson distribution.

9. What is the drawback of this policy?

**Exercice 2.**

Isotropic 1D, 2D, 3D random walks

The isotropic random walk over $\mathbb{Z}^d$ is an HMC, with probability $1/2d$ to jump towards any of the $2d$ neighbors in the grid. It is clearly irreducible. Check for $d = 1, 2, 3$ whether it is recurrent? positive recurrent?

Realization in $\mathbb{Z}^2$ (10000 steps, Wikipedia)  
Realization in $\mathbb{Z}^3$ (10000 steps, Wikipedia)

**Exercice 3.**

1. A drunk robot is stuck in a corridor with a wall at one end and an exit at the other end. Every second, he makes one move: with probability $1/2$ he makes a step towards the wall and with probability $1/2$ he makes a step towards the exit. At the beginning, he is $N_0$ steps away from the wall and $N_1$ steps away from the exit. What is the probability that the robot eventually finds the exit rather than crashes into the wall?

2. The same robot is now stuck in a semi-infinite corridor with a wall at one end and an infinite path on the other way. He starts at $N_0$ steps away from the wall. What is the probability that the robot eventually crashes into the wall? Assuming that he crashes, what is the average time until the crash?

3. Two drunk robots are standing in an infinite corridor (in both directions). They are $N_0$ steps away from each other. What is the probability that they will eventually crash? They crash if they cross or end at the same place.

**Exercice 4.**

Uniform random walks on graphs

Let $G = (V,E)$ be a undirected connected finite graph, we study the random walk which starts a path from a vertex and extend it by choosing at each step a vertex uniformly from the neighborhood of the last vertex (coming back to vertex already visited is allowed). Show that this Markov chain is homogeneous, irreducible, positive recurrent and describe precisely its stationary distribution. What if the graph is directed?