TUTORIAL V

1 Homework 4

Given two channels $W^1_{Y_1|X_1}$ and $W^2_{Y_2|X_2}$ with input spaces $\mathcal{X}_1, \mathcal{X}_2$ and outputs spaces $\mathcal{Y}_1, \mathcal{Y}_2$. Consider the channel W^{12} defined on input space $\mathcal{X}_1 \times \mathcal{X}_2$ and output space $\mathcal{Y}_1 \times \mathcal{Y}_2$ and $W^{12}_{Y_1Y_2|X_1X_2}(y_1y_2|x_1x_2) = W^1_{Y_1|X_1}(y_1|x_1) \cdot W^2_{Y_2|X_2}(y_2|x_2)$.

1. Compute $\mathrm{C}(W^{12}) = \max_{P_{X_1X_2}} I(X_1, X_2 : Y_1, Y_2)$ (where Y_1Y_2 is the output of W^{12} when the input is X_1X_2) as a function $\mathrm{C}(W^1)$ and $\mathrm{C}(W^2)$.

2 Fun with Fano

1. Consider the two following pairs of correlated random variables:

   i. X is uniform on $\{0,1\}^n$, Y equals the first $n/2$ bis of X.

   ii. With probability $\alpha \in [0;1]$, X is uniform on $\{0;1\}^n$ and $Y = X$; and with probability $1-\alpha$, X is uniform on $\{0;1\}^n$ and Y is the all 0s string.

   Suppose we observe Y and estimate $\hat{X} = g(Y)$. What is the minimum possible value of $\mathbf{P}(\hat{X} \neq X)$ in the above two examples ? What lower bound does Fano's inequality give in the two examples ?

2. For two vectors $u, v \in \{0,1\}^n$, we denote by $\Delta(u,v)$ the following set: $\Delta(u,v) = \{j \in \{1,\ldots,n\} : u_j \neq v_j\}$. Suppose X and Y are two correlated random variables taking values in $\{0;1\}^n$.
   For $i \in \{0,\ldots,n\}$, we define $\theta_i = \mathbf{P}(\Delta(X,Y) = i)$. Prove that

$$H(X|Y) \leq \sum_{i=0}^{n} \theta_i \log_2 \left(\binom{n}{i} \frac{1}{\theta_i} \right)$$

   (Hint: Define the random variable $\Delta(X,Y)$ and mimic steps from the proof of Fano's inequality)

3 Expurgation

Let C be a code with 2^{nR} codewords that achieves an average probability of block error $P_{err,av}(C) = \delta$.

1. Show that you can find another code with almost same rate that achieve a maximal probability of error $\leq 2\delta$.