# TUTORIAL IX

## 1 Error correcting VS error detecting

Show that the following statements are equivalent for a code linear code $C$:

1. $C$ has minimum distance $\geq d \geq 2$.
2. If $d$ is odd, $C$ can correct $(d-1)/2$ errors.
3. If $d$ is even, $C$ can correct $d/2-1$ errors.
4. $C$ can detect $d-1$ errors.
5. $C$ can correct $d-1$ erasures (in the erasure model, the receiver know where the errors have occurred).

## 2 Generalized Hamming bound

Prove the following bound: for any $(n,k,d)$ code $C \subseteq (\mathbb{F}_q)^n$,

$$k \leq n - \log_q \left( \sum_{i=0}^{\left\lfloor \frac{(d-1)}{2} \right\rfloor} \binom{n}{i} (q-1)^i \right)$$

## 3 Hamming riddle

There are $n$ people in a room, each of whom is given a black/white hat chosen uniformly at random (and independent of the choices of all other people). Each person can see the hat colour of all other people, but not their own. Each person is asked if (s)he wishes to guess their own hat colour. They can either guess, or abstain. Each person makes their choice without knowledge of what the other people are doing. They either win collectively, or lose collectively. They win if all the people who don't abstain guess their hat colour correctly and at least one person does not abstain. They lose if all people abstain, or if some person guesses their colour incorrectly. The goal below is to come up with a strategy that will allow the $n$ people to win with pretty high probability

1. Argue that the $n$ people can win with probability at least $\frac{1}{2}$
2. Lets say that a directed graph $G$ is a subgraph of the $n$-dimensional hypercube if its vertex set is $\{0,1\}^n$ and if $u \to v$ is an edge in $G$, then $u$ and $v$ differ in at most one coordinate. Let $K(G)$ be the number of vertices of $G$ with in-degree at least one, and out-degree zero. Show that the probability of winning the hat problem equals the maximum, over directed subgraphs $G$ of the $n$-dimensional hypercube, of $K(G)/2^n$
3. Using the fact that the out-degree of any vertex is at most $n$,show that $K(G)/2^n$ is at most $\frac{n}{n+1}$ for any directed subgraph $G$ of the $n$-dimensional hypercube.
4. Show that if $n = 2^r - 1$, then there exists a directed subgraph $G$ of the $n$-dimensional hypercube with $K(G)/2^n = \frac{n}{n+1}$.
   Hint:This is where the Hamming code comes in.