lower critical dimension $\quad d_{c_1}$:

Ising $\quad d_{c_1} = 1$

XY $\quad d_{c_1} = 2 \quad$ (transition at $d=2$, Kosterlitz - Thouless)

Heisenberg $\quad d_{c_1} = 2$

Ginzburg - Landau functional

$$\beta \Phi [\vec{\phi}(\vec{r})] = \beta \int d^d r \left[c |\vec{\nabla} \vec{\phi}|^2 + a(T-T_c) |\vec{\phi}|^2 + u |\vec{\phi}|^4 \right] \qquad \vec{\phi} = \vec{m}$$

Dimensional analysis : measure how small (or how big) $u|\vec{\phi}|^4$ is compared to the quadratic terms.

$$= 2\cancel{\beta c}^{\,1} \int d^d r \left[\frac{1}{2} |\vec{\nabla} \vec{\phi}|^2 + \underbrace{\frac{a(T-T_c)}{2c}}_{\frac{1}{2} r_0} |\vec{\phi}|^2 + \underbrace{\frac{u}{\beta c}}_{\frac{u_0}{4} = \frac{u}{4\beta c^2}} |\vec{\phi}|^4 \right] \qquad r_0 = \frac{a(T-T_c)}{c}$$

$$|\vec{\phi}|^4 \Rightarrow \frac{|\vec{\phi}|^4}{4\beta^2 c^2}$$

$$\vec{\phi} \to \frac{\vec{\phi}}{\sqrt{2\beta c}}$$

$$\Big[\int \underbrace{d^d r}_{\ell^d} \underbrace{|\vec{\nabla} \vec{\phi}|^2}_{\ell^{-2}} \Big] = 1 \qquad \boxed{[\phi] = \ell^{\frac{2-d}{2}} = \ell^{1-d/2}} \qquad \text{canonical dimensions}$$

$$\Big[\int \underbrace{d^d r}_{\ell^d} \underbrace{r_0 |\phi|^2}_{\ell^{2-d}} \Big] = 1 \qquad [r_0] = \ell^{-2} \qquad \boxed{[r_0^{-1/2}] = \ell} \sim \xi_{MF}$$

$$\Big[\int d^d r \; u_0 |\vec{\phi}|^4 \Big] \qquad \boxed{[u_0] = \ell^{d-4}}$$

$$\ell^d \qquad \ell^{2(2-d)}$$

$$\ell^{4-d}$$

$$\frac{\vec{\phi}}{(r_0^{-1/2})^{1-d/2}} = \vec{\varphi} \qquad \frac{\vec{r}}{(r_0^{-1/2})} = \vec{x}$$

$$\int d^d r \; u_0 \vec{\phi}^4 = \int d^d x \; r_0^{-d/2} \, u_0 \, (r_0^{-1/2})^{4-2d} \varphi^4$$

$$= u_0 \, r_0^{-\frac{4-d}{2}} = u_0 \, (r_0^{-1/2})^{4-d}$$

$$= \bar{u}_0$$

$$\beta \Phi[\vec{\varphi}] = \int d^d x \left[\frac{1}{2} |\vec{\nabla}_{\vec{x}} \cdot \vec{\varphi}|^2 + \frac{1}{2} |\vec{\varphi}|^2 + \frac{\bar{u}_0}{4} |\vec{\varphi}|^4 \right]$$

$$\bar{u}_0 = u_0 \, (r_0^{-1/2})^{4-d} \sim u_0 \, (T-T_c)^{\frac{d-4}{2}} \xrightarrow[d<4]{T \to T_c} \infty$$

$$\xrightarrow[d=4]{} \text{const}$$

$d < d_{c_2} = 4$ quartic term "diverges" when $T \to T_c$

$d > d_{c_2}$ $\qquad \underline{\bar{u}_0 \to 0}$ $\qquad T \to T_c$

<u>Ginzburg criterion reloaded</u>

MF theory becomes asymptotically correct when <u>$T \to T_c$</u> above d_{c_2}

$$\bar{u}_0 \to 0 \quad T \to T_c$$

$$Z = \int \mathcal{D}[\vec{\varphi}] \, e^{-\int d^d x \left[\frac{1}{2} |\vec{\nabla} \vec{\varphi}|^2 + |\vec{\varphi}|^2 \right]} \qquad \sum_{n=0}^{\infty} \frac{(-1)^n \bar{u}_0^n}{n!} \left(\int d^d r \, |\vec{\varphi}|^4 \right)^n \sim n!$$

the series is <u>divergent</u> : it is a sum of sign alternating and growing (with n) contributions

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Dimensional analysis

$$[\vec{\phi}\,] \simeq \ell^{1-d/2}$$

$$\underbrace{\langle \vec{\phi}(0)\cdot\vec{\phi}(\vec{r}\,)\rangle}_{\frac{1}{\ell^{d-2}}} \overset{T\simeq T_c}{\sim} \frac{e^{-\frac{|\vec{r}|}{\xi}}}{|\vec{r}\,|^{d-2+\eta}}$$

$\eta = 0$ MF ✓

$\eta = \frac{1}{4}$ Ising 2d ✗

$$[\xi] = \ell \qquad \xi \sim \frac{1}{|T-T_c|^{-\nu}}$$

$$\xi \sim r_0^{-1/2} \sim \frac{1}{|T-T_c|^{1/2}} \qquad \nu = \tfrac{1}{2}$$

MF ✓

$\nu = 1$ Ising 2d ✗

$$\chi = \beta \int d^d r\, \langle \vec{\phi}(0)\cdot\vec{\phi}(\vec{r}\,)\rangle \sim \beta \int d^d r\, \frac{e^{-|\vec{r}|/\xi(T)}}{|\vec{r}\,|^{d-2+\eta}} = \beta\, \underbrace{\xi^{d}\, \xi^{2-d-\eta}}_{\xi^{2-\eta}} \underbrace{\int d^d y\, \frac{e^{-|\vec{y}|}}{|\vec{y}\,|^{d-2+\eta}}}_{I}$$

$$\vec{y} = \frac{\vec{r}}{\xi}$$

$$\boxed{\chi \sim \xi^{2-\eta}}$$

$$\chi \sim |T-T_c|^{-\nu(2-\eta)} \quad T \to T_c$$

$$\boxed{\gamma = \nu(2-\eta)}$$ scaling relation (Fisher's)

$\gamma = 1$ MF ✓

$\gamma = 7/4$ Ising 2d ✓

$\eta \neq 0 \Rightarrow$ the order parameter at $T \simeq T_c$ has an effective "anomalous dimension"

as if "$[\phi] \sim \ell^{1-d/2 - \eta/2}$"

$\to$ anomalous dimension

Dimensional analysis

$$[m] \sim \ell^{1-d/2-\eta/2} \sim \xi^{1-d/2-\eta/2} \sim (T_c - T)^{\underbrace{\frac{\nu}{2}(2-d-\eta)}_{\beta}}$$

Ising 2d

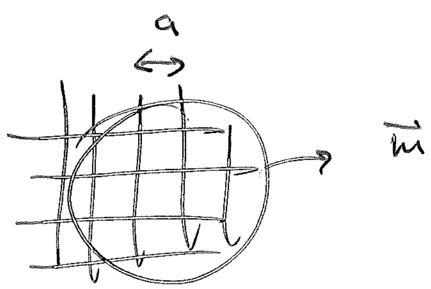
$\delta = 15$

$$\boxed{\beta = \frac{\nu}{2}(d-2+\eta)}$$ ✓

Scaling relation ("hyperscaling": involves the # of dimensions d)

Identified one characteristic length $r_0^{-1/2}$

second natural length: MICROSCOPIC LENGTH a lattice spacing



$$\Lambda \simeq \frac{\pi}{a} < \infty$$

$$C(\vec r) = \underbrace{\langle \vec\Phi(0)\cdot\vec\Phi(\vec r)\rangle}_{\ell^{2-d}} \underset{r\to\infty}{\overset{r\ll\xi}{\simeq}} A \frac{e^{-r/\xi}}{r^{d-2+\eta}} = \frac{1}{r^{d-2}} F\left(\frac{r}{\xi}, \frac{a}{r}\right)$$

$[A] = \ell^{\eta}$

F is dimensionless

$F(x,y) \xrightarrow{y\to 0}$ const be replaced by $F(x,0)$

$r \to \infty$

$\underline{r \gg a}$

$$\boxed{C(\vec r) \sim \int_{|q|\le\Lambda} d^dq \, \frac{e^{i\vec q\cdot\vec r}}{\xi^{-2}+q^2}}$$

Gaussian theory

$\underline{\eta = 0}$

$$\underset{a/r\to 0}{\simeq} \frac{1}{r^{d-2}} \exp\left(-\frac{r}{\xi}\right)\left(\frac{a}{r}\right)^{\eta}$$

$$\boxed{\xi = r_0^{-1/2} F_\xi\left(a r_0^{1/2}\right)} \xrightarrow{T\to T_c} r_0^{-1/2}\left(a r_0^{1/2}\right)^{-2\vartheta}$$

$a r_0^{1/2} \to 0$

$T \to T_c$

$$\sim |T-T_c|^{\underbrace{-\frac{1}{2}-\vartheta}} a^{-2\vartheta}$$

$F_\xi(x) \cancel{\xrightarrow[x\to 0]{}}$ const $< \infty$

$$\boxed{\nu = \frac{1}{2} + \vartheta}$$

$\vartheta > 0$

$\nu \geq \frac{1}{2}$

$$\boxed{C(\vec r) = \frac{1}{r^{d-2}} F\left(\frac{r}{\xi}, \frac{a}{r}\right)}$$

<u>Scaling Ansätze</u>

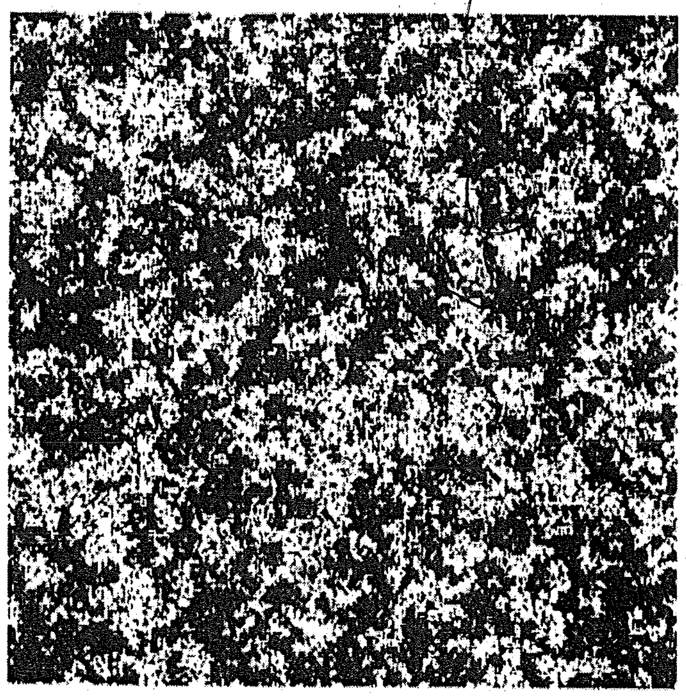
$C(\vec r) = \int d^dq \; e^{i\vec q\cdot\vec r} \underline{S(\vec q)}$

$\eta \neq 0$ experimentally

numerically

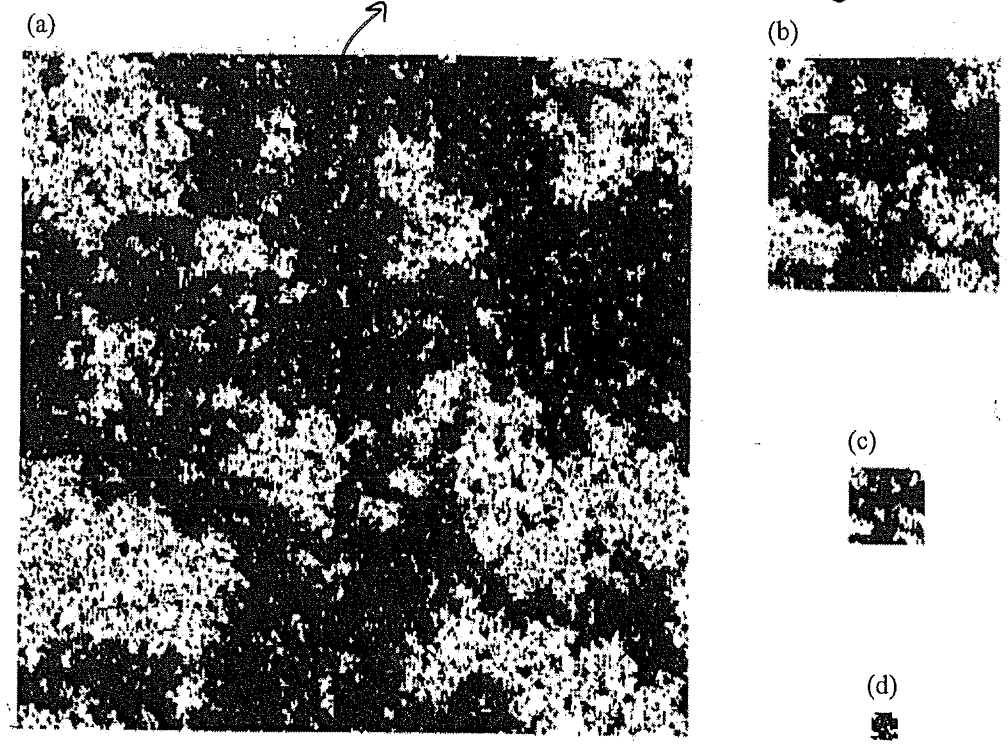
Two-dimensional Ising model

ξ



$T > T_c$

critical cluster $\xi \simeq L$



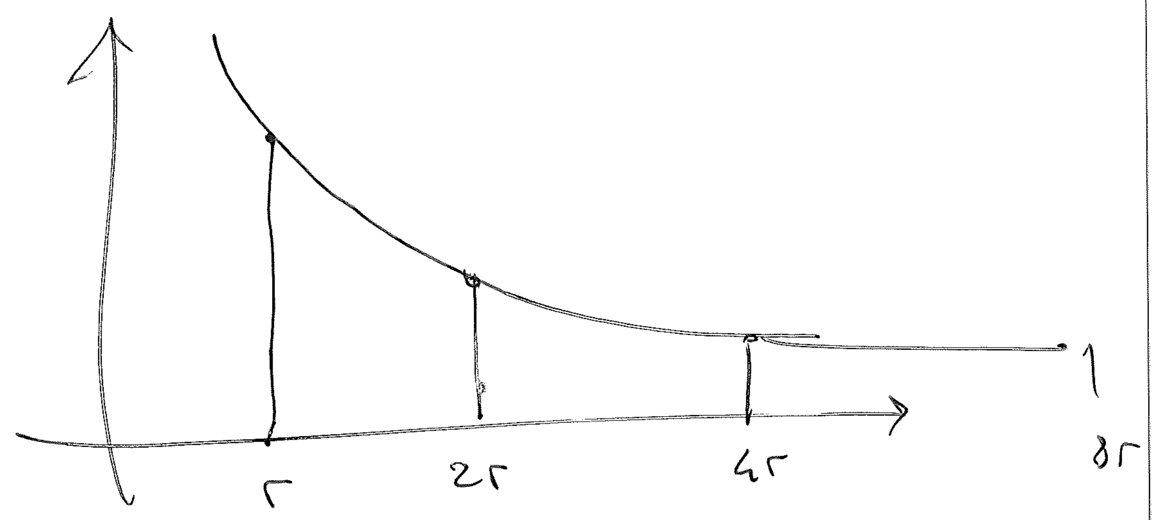
$T = T_c$

majority rule on 3x3 blocks

FRACTALITY and SELF-SIMILARITY at $T=T_c$

Scale invariance or self-similarity

$$C(r) \overset{T=T_c}{\sim} \frac{1}{r^{d-2+\eta}} \qquad \frac{C(2r)}{C(r)} \overset{\forall r}{=} \frac{1}{2^{d-2+\eta}}$$



Fractals : critical cluster is a fractals

at $T=T_c$ $\quad \langle M \rangle = \langle \sum_{i=1}^{N} \sigma_i \rangle = 0$

critical cluster $\sim L^{d_f}$ $\qquad d_f < d$ $\qquad$ d_f is fractional

fraction occupied by the critical cluster $\sim L^{d_f - d} \to 0$ $\qquad$ magnetization per spin coming from the critical cluster $\sim L^{d_f - d}$

$$\frac{M}{L^d} = m \sim |T-T_c|^\beta \sim \xi^{-\beta/\nu} \underset{\xi \simeq L}{\simeq} L^{-\beta/\nu}$$

$$d_f - d = -\beta/\nu$$

$$\boxed{d_f = d - \beta/\nu}$$

MF $\Rightarrow$ $d_f = d-1$ $\qquad$ 2d Ising $\quad d_f = 2 - \frac{1}{8} = \frac{15}{8}$

SCALING RELATIONS and SCALING HYPOTHESIS

$\gamma = \nu(2-\eta)$ Fisher's

$\frac{2\beta}{\nu} = d-2+\eta$ "hyperscaling"

$$\begin{cases} \alpha + 2\beta + \gamma = 2 & \text{Rushbrooke's} \\ \beta\delta = \beta + \gamma & \text{Griffiths'} \end{cases}$$

~ 1960

$\alpha, \beta, \gamma, \delta, \eta, \nu \rightarrow$ only two are really independent

Widom (1965) $t = \frac{T-T_c}{T_c}$ $\tilde{h} = \frac{h}{k_B T_c}$ ferromagnetic models

Scaling hypothesis: free energy density $f = f(t, \tilde{h}) = \underbrace{f_s(t,h)}_{\text{singular derivatives as a function of } t \text{ and } \tilde{h}} + f_r(t,h)$

$\Downarrow$

$$f_s(t,\tilde{h}) = |t|^{2-\alpha} F(t,\tilde{h}) = |t|^{2-\alpha} F\left(\frac{\tilde{h}}{t^{\Delta}}\right)$$

$$c = -T\frac{\partial^2 f}{\partial T^2} \sim |t|^{-\alpha}$$

(8)

$$m = -\frac{\partial f_s}{\partial h}\Big|_{h=0} = |t|^{2-\alpha}\, \mathcal{F}'(0)\, \frac{1}{t^{\Delta}} \approx |t|^{2-\alpha-\Delta} = |t|^{+\beta}$$

$$2-\alpha-\Delta = \beta$$

$$\Delta = 2-\alpha-\beta$$

$$\chi = \frac{\partial m}{\partial h} = -\frac{\partial^2 f_s}{\partial h^2}\Big|_{h=0} \approx |t|^{2-\alpha-2\Delta} = |t|^{-\gamma}$$

$$\gamma = \alpha - 2 + 2\Delta$$

$$= \alpha - 2 + 2(2-\alpha-\beta)$$

$$= -\alpha + 2 - 2\beta$$

$$\boxed{\alpha + 2\beta + \gamma = 2}$$