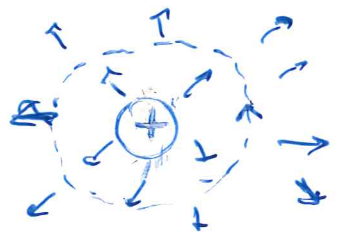
KOSTERLITZ-THOULESS TRANSITION

2D XY model : $\mathcal{H} = -J\sum_{\langle ij\rangle} \vec{\sigma}_i \cdot \vec{\sigma}_j$ $\qquad \sigma_i = (\cos\phi_i, \sin\phi_i)$

$Z_{2DXY} \simeq Z_{SW} \times \underline{Z_{\text{coulomb gas}}}$ ↗ neutral Coulomb gas of vortices / antivortices

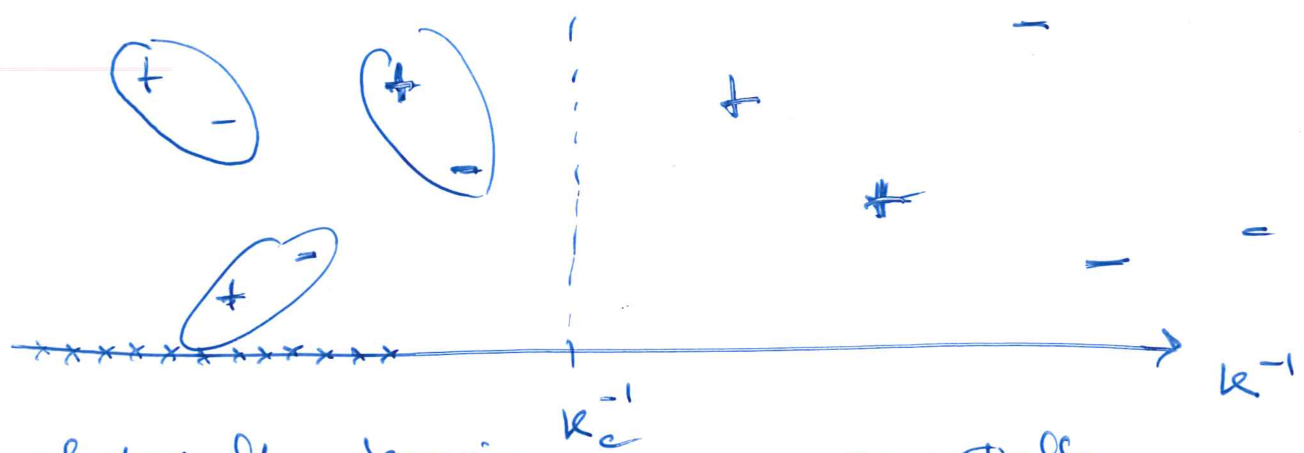


$q = +1$

$$Z_{CG} = \sum_N y_0^N \int \prod_{i=1}^{N} d^2 r_i \; e^{2\pi K \sum_{i<j} \ln\left(\frac{|\vec{r}_i - \vec{r}_j|}{a}\right)}$$

$N = 0, 2, 4, \ldots$

K = coupling constant $= \frac{J}{k_B T}$



algebraically decaying correlations

$\langle \vec{\sigma}_i \cdot \vec{\sigma}_j \rangle \sim \frac{1}{r^{\eta(T)}}$

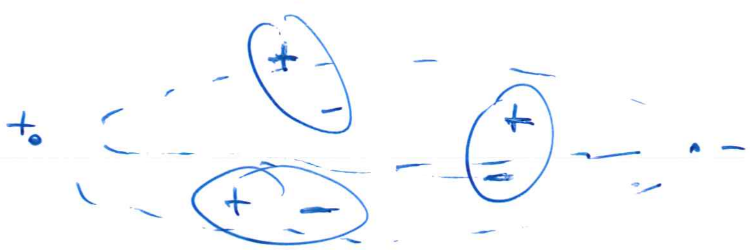
$\frac{1}{r}$ $\cancel{d}$ $2\pi\eta$

exponentially decaying correlations

$\langle \vec{\sigma}_i \cdot \vec{\sigma}_j \rangle \sim e^{-\frac{|\vec{r}_i - \vec{r}_j|}{\xi}}$

$\underline{K_c \simeq \frac{2}{\pi}}$

Dielectric response of the Coulomb gas



$$V_{eff}(\vec{r}_+ - \vec{r}_-) = -2\pi K_{eff} \log\left(\frac{|\vec{r}_+ - \vec{r}_-|}{a}\right)$$

$$K_{eff} = K - \frac{4\pi^3 K^2 y_0^{2}}{a^{4-2\pi K}} \int_a^\infty d\rho\, \rho^{3-2\pi K} + \mathcal{O}(y_0^4)$$

single dipoles $(N=2)$

ρ = length of the dipole

y_0 = fugacity $= e^{+\beta\mu} = e^{-\beta E_0}$

vortex-core energy

$y_0 \ll 1$

$\epsilon_r^{-1} = \frac{K_{eff}}{K} \to 0 \qquad \epsilon_r \to \infty$

perfect screening

$$\begin{matrix} + & - & & + & \\ & - & & + & - \\ - & & & & + \end{matrix}$$

K lowers, $T \nearrow$ $\qquad \int_a^\infty d\rho\, \rho^{3-2\pi K} \nearrow$

$3 - 2\pi K < 0$

$3 - 2\pi K < -1$ for the integral to be convergent

$2\pi K = 4 \qquad K = K_c = \frac{2}{\pi}$

KOSTERLITZ'S RG

$$K_{eff}^{-1} = K^{-1} + \underbrace{\frac{4\pi^3 y_0^2 K}{a^{4-2\pi K}} \int_a^\infty d\rho\, \rho^{3-2\pi K}}_{\int_a^{a e^l} + \int_{a e^l}^{\infty}} + \mathcal{O}(y_0^4)$$

$b = e^l$

$$k_{eff}^{-1} = \underbrace{K^{-1} + \frac{4\pi^3 y_0^2 K}{a^{4-2\pi K}} \int_a^{ae^l} d\rho \, \rho^{3-2\pi K}}_{(K')^{-1}} + (\quad) \int_{ae^l}^{\infty} d\rho \, (\ldots) + o(y_0^4)$$

$$\boxed{(K')^{-1} = K^{-1} + \frac{4\pi^3 y_0^2 K}{a^{4-2\pi K}} \int_a^{ae^l} d\rho \, \rho^{3-2\pi K} + o(y_0^4)}$$

$$(K')^{-1} - K^{-1} = o(y_0^2)$$

$$k_{eff} = (K')^{-1} + \frac{4\pi^3 y_0^2 K'}{a^{4-2\pi K'}} \int_{ae^l}^{\infty} d\rho \, \rho^{3-2\pi K'} + o(y_0^4)$$

$$\rho \Rightarrow \rho' = \rho/e^l \qquad \rho = e^l \rho'$$

$$= (K')^{-1} + \frac{4\pi^3 y_0^2 (e^l)^{4-2\pi K'}}{a^{4-2\pi K'}} K' \int_a^{\infty} d\rho' (\rho')^{3-2\pi K'} + o(y_0^4)$$

$$(y_0')^2 = y_0^2 (e^l)^{4-2\pi K'}$$

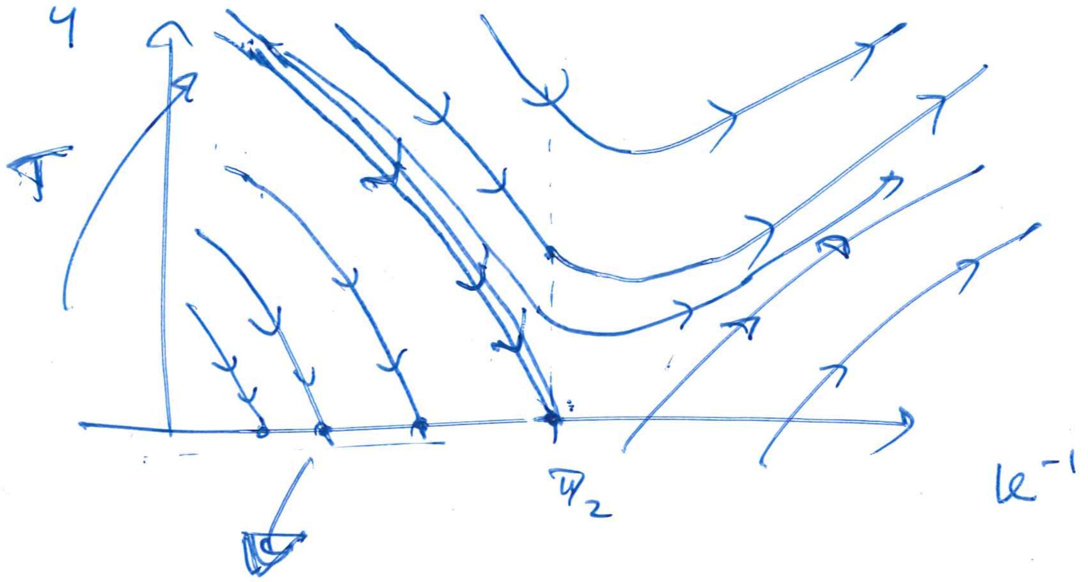
$$\boxed{y_0' = y_0 \, e^{(2-\pi K')l} + o(y_0^4)}$$

$$\boxed{\begin{aligned} \frac{dy}{dl} &= (2-\pi K) y + o(y^3) \\ \frac{d(K^{-1})}{dl} &= 4\pi^3 y^2 + o(y^4) \end{aligned}}$$

Kosterlitz's equations

$$\frac{d(\delta \vec{\mu})}{dl} = M \, \delta \vec{K}$$

$K(\ell) > K_c = \frac{2}{\pi}$ $\qquad$ $K \leftarrow K'(\ell)$



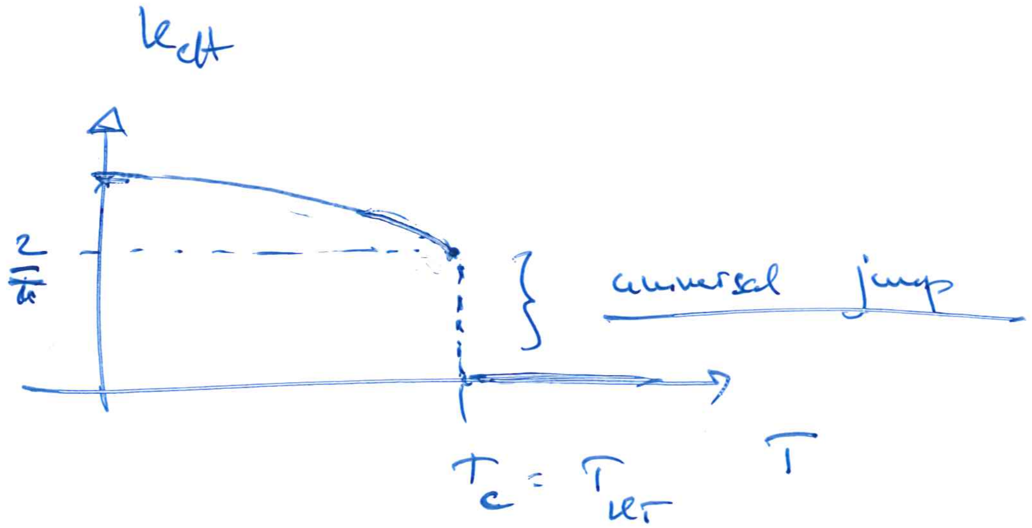
flow to $\boxed{y^* = 0}$ $\Rightarrow$ no vortices

$(K^{-1})^* \neq 0$

$(K^{-1})^* < \frac{\pi}{2}$

$$\boxed{K^* = K'(\ell = \infty) = K_{eff}}$$

$K(\ell=0) = K = \frac{1}{k_B T}$ $\qquad$ $y(\ell=0) = y_0 = e^{-\beta E_0}$



$T = T_c$ $\qquad$ $K_{eff} \to K_c = \frac{2}{\pi}$

$x = K^{-1} - \pi/2 \ll 1$ $\qquad$ $K \simeq 2/\pi$

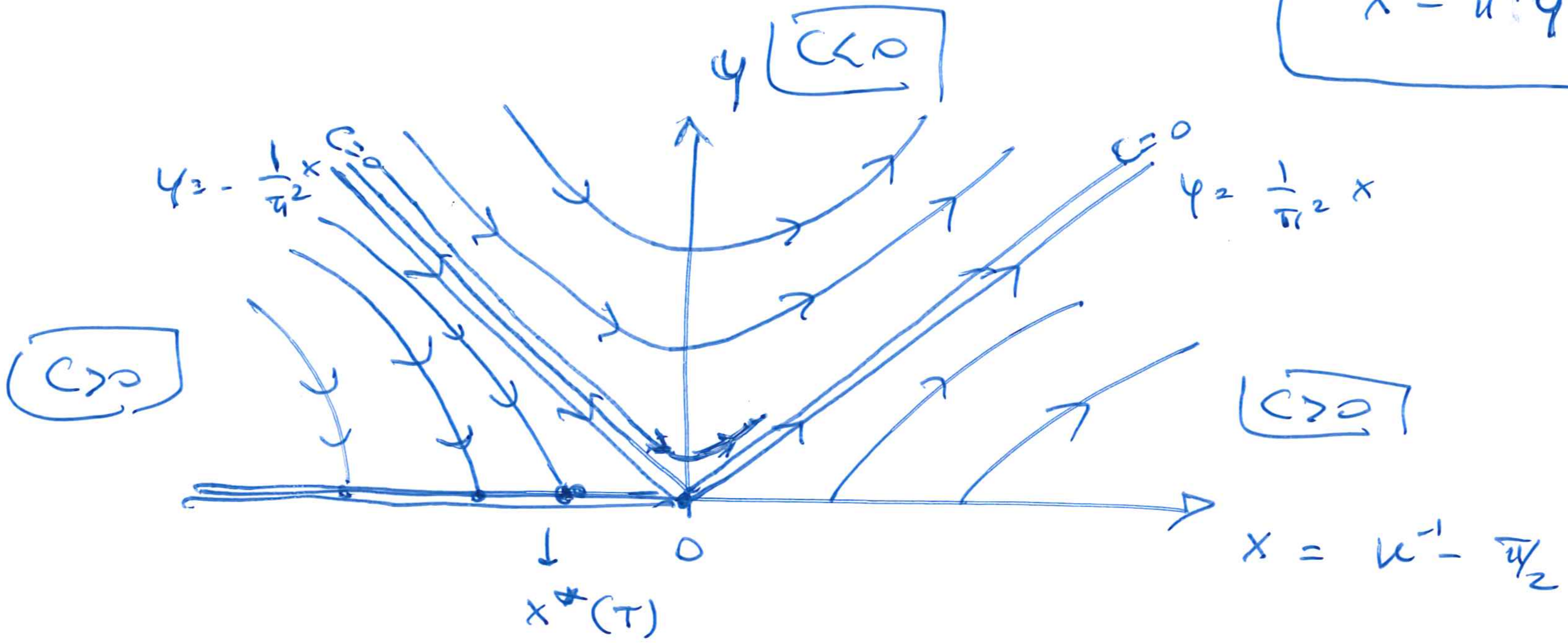
$$\boxed{\frac{dx}{d\ell} = 4\pi^3 y^2 + \mathcal{O}(y^3) \qquad \frac{dy}{d\ell} = \frac{4}{\pi} xy + \mathcal{O}(x^2)}$$

$x, y \ll 1$

$$\frac{dx^2}{dl} \qquad \frac{dy^2}{dl} \quad \rightarrow \quad \frac{dx^2}{dl} - \pi^4 \frac{dy^2}{dl} = 0 \qquad \frac{d}{dl}\left(x^2 - \pi^4 y^2\right) = 0$$

$$\boxed{x^2 - \pi^4 y^2 = \text{const.} = c}$$

hyperbola



$c = 0 \qquad y = \pm \frac{1}{\pi^2}|x|$

$c = 0 \qquad T = T_c$

$c > 0 \qquad c = b^2 (T_c - T)$

$$(x^*(T))^2 - \pi^4 \cancel{(y^*)}^{\,0} = b^2 (T_c - T)$$

$$x^* = b\sqrt{T_c - T}$$

$$\kappa_{\text{eff}}^{-1}(T) = \pi/2 + b\sqrt{T_c - T}$$

$$\kappa_{\text{eff}}(T) = \frac{2}{\pi} - b\sqrt{T_c - T} + O(T_c - T)$$

$$\boxed{T < T_c}$$

$\boxed{T > T_c}$

$$x^2 - \pi^4 y^2(\ell) = b^2 (T_c - T) \qquad y^2 = \frac{1}{\pi^4}\left(x^2 - b^2 (T_c - T)\right)$$

$$\frac{dx}{d\ell} = 4\pi^3 y^2 = 4\pi^3 \left(\quad \right)$$

$$= \frac{4}{\pi}\left[x^2 + b^2 (T - T_c) \right]$$

$$\frac{dx}{x^2 + b^2 (T - T_c)} = \frac{4}{\pi} d\ell$$

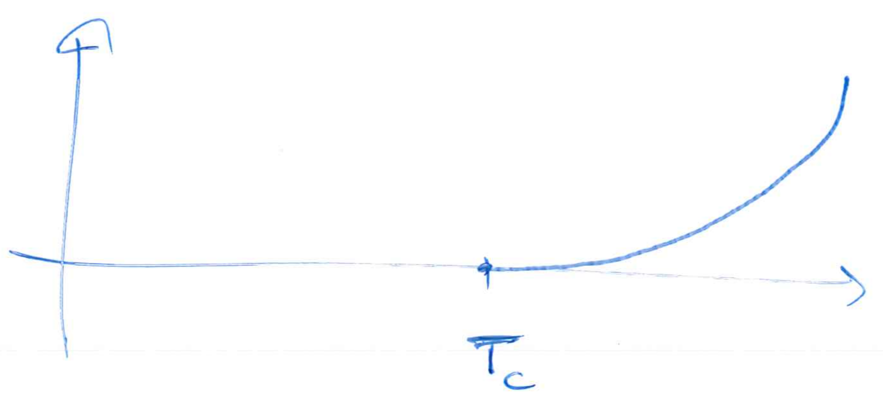
$$\frac{1}{b\sqrt{T-T_c}} \underbrace{\arctan\left(\frac{x}{b\sqrt{T-T_c}}\right)}_{< \pi/2} \Bigg|_{x(0)}^{x(\ell)} = \frac{4}{\pi}\ell$$

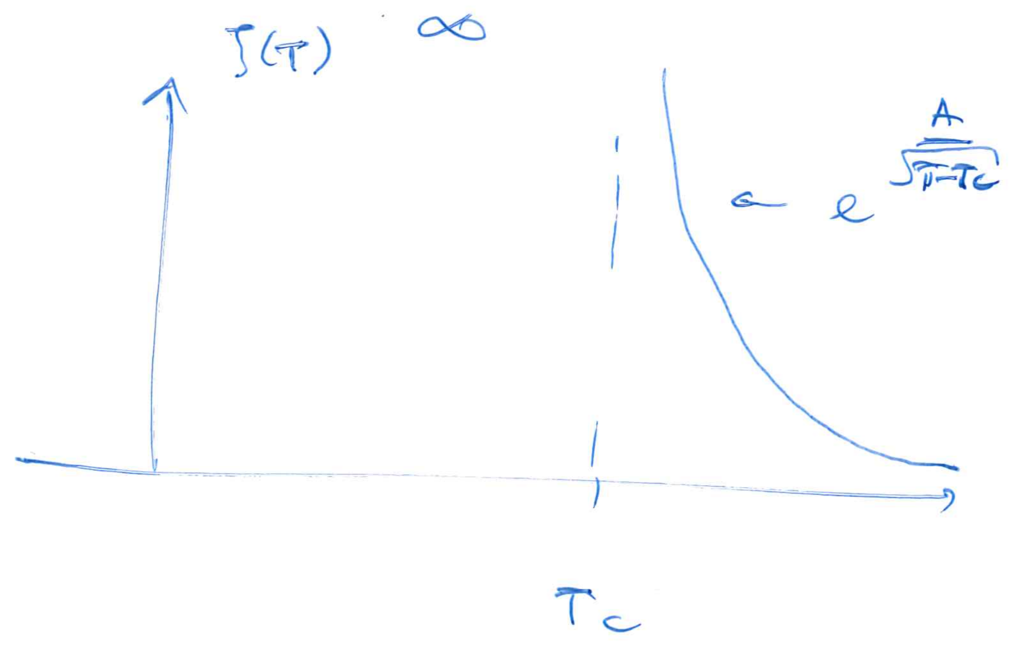
$$\frac{4}{\pi}\ell\, b\sqrt{T-T_c} < \frac{4}{\pi}\ell^* b\sqrt{T-T_c} = \pi$$

$$\ell^* = \frac{\pi^2}{4b}\frac{1}{\sqrt{T-T_c}} \Rightarrow \boxed{\xi = a e^{\ell^*} = a e^{\frac{\pi^2}{4b}\frac{1}{\sqrt{T-T_c}}}}$$

$$f_s \sim \xi^{-2} \sim \exp\left(-\frac{2A}{\sqrt{T-T_c}}\right)$$

essential singularity





QUANTUM PHASE TRANSITIONS (QPT)

$\boxed{T=0}$ $$\hat{\mathcal{H}}(g) = \hat{\mathcal{H}}_0 + g\hat{V} \qquad (\text{typically } [\hat{\mathcal{H}}_0, \hat{V}] \neq 0)$$

change g : phase transition " the ground state of $\mathcal{H}$

$g \to g_c$

$\Downarrow$

SSB ?

$$\hat{\mathcal{H}} \to \hat{\mathcal{H}} - h\hat{O}$$

$$\hat{\mathcal{H}}|\psi_0(g)\rangle = E_0(g)|\psi_0(g)\rangle$$

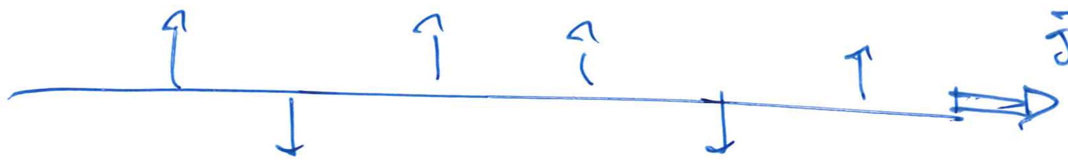
SSB: $\lim\limits_{h\to 0} \lim\limits_{N\to\infty} \langle \hat{O} \rangle \neq 0$

$\hat{O}$ does not share the same symmetries as $\hat{\mathcal{H}}$

$$\hat{U}^+\hat{\mathcal{H}}\hat{U} = \hat{\mathcal{H}} \qquad \hat{U}^+\hat{O}\hat{U} = -\hat{O}$$

$$\hat{\mathcal{H}} = \overbrace{-J\sum_{\langle ij\rangle} \sigma_i^z\sigma_j^z}^{\hat{\mathcal{H}}_0} - g\overbrace{\sum_i \sigma_i^x}^{\hat{V}} - h\sum_i \sigma_i^z \qquad [\sigma_i^\alpha, \sigma_j^\beta] = i\sum_\gamma \epsilon_{\alpha\beta\gamma}\,\sigma_i^\gamma\,\delta_{ij}$$

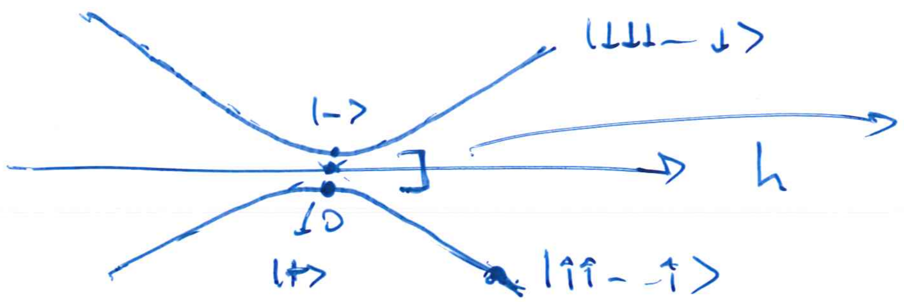
$\boxed{g \ll J}$



$\vec{J} = g\vec{e}_x$

$\hat{O} = \sum_i \sigma_i^z$

$$|\psi_0\rangle \simeq \frac{|\uparrow\uparrow\uparrow\uparrow\ldots\uparrow\rangle + |\downarrow\downarrow\downarrow\downarrow\ldots\downarrow\rangle}{\sqrt{2}} = \text{Schrödinger's cat}$$

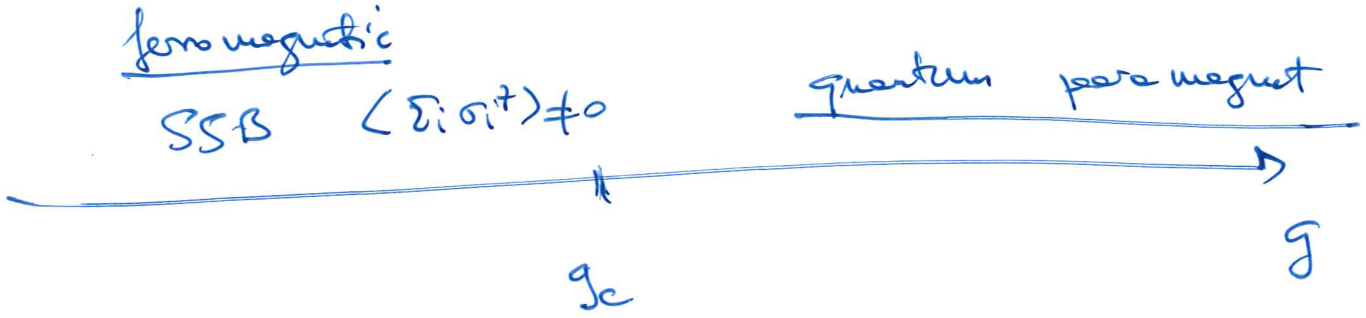


$$\sim\left(\frac{h}{J}\right)^N = \exp\left(-\left|\log\left(\frac{h}{J}\right)\right| N\right)$$

$\boxed{g \gg J}$

$\rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \Longrightarrow g$

$$\langle \hat{S} \rangle = \langle \Sigma_i \sigma_i^z \rangle = 0$$

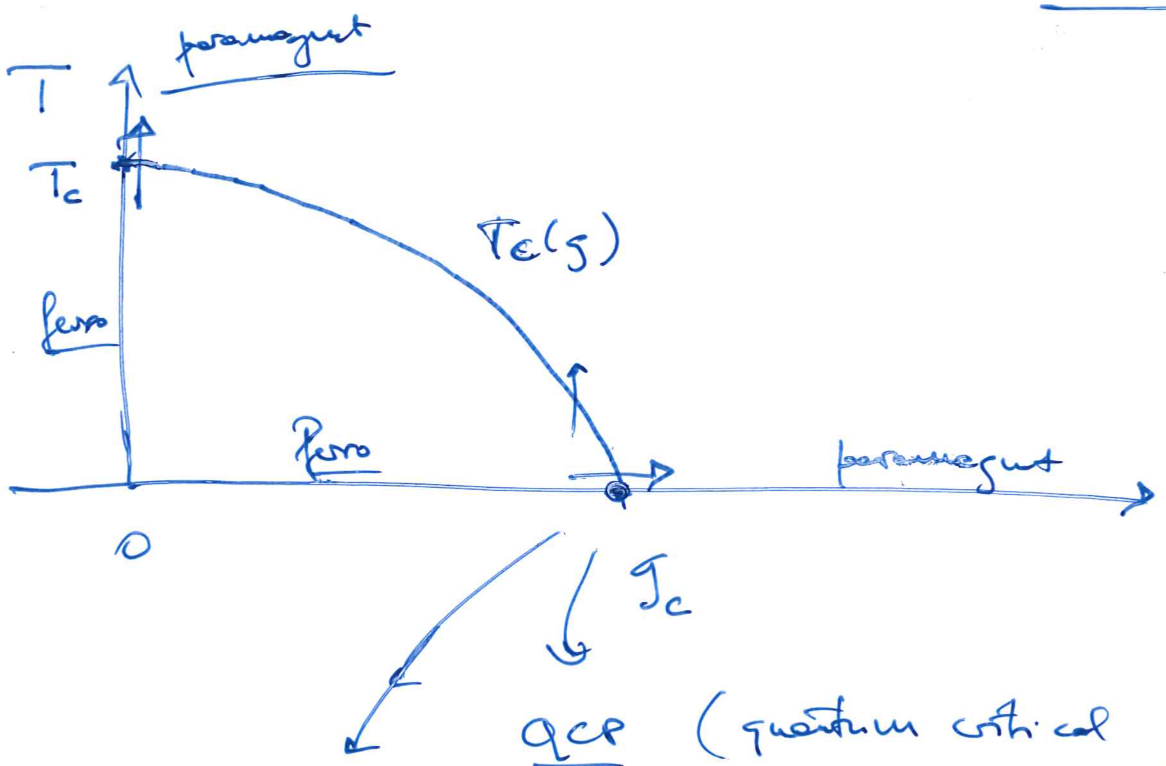


$T=0$!

"Quantum fluctuations" $\equiv$ Heisenberg's uncertainty

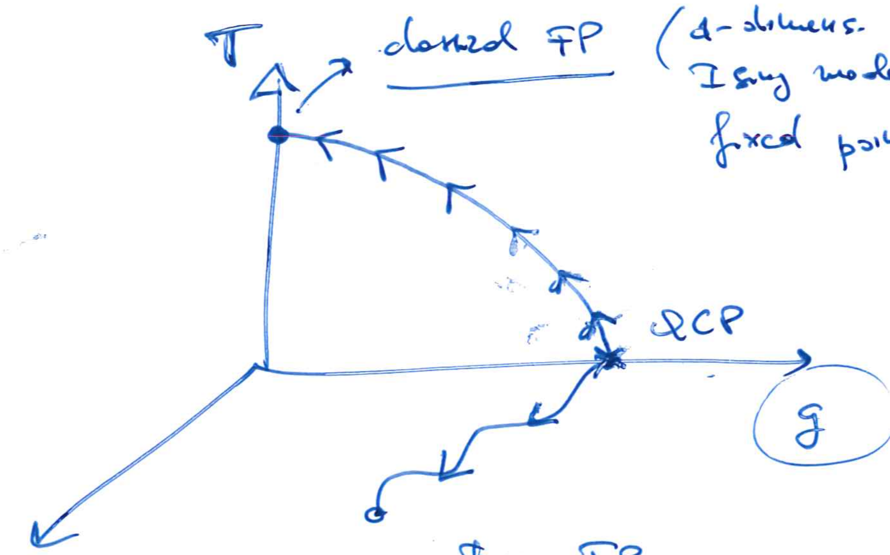
$$[\hat{H}_0, \hat{V}] \neq 0$$

$d = 2, 3, \ldots$ Ising model has also a thermal (classical) transition



QCP (quantum critical point)

$\nu, \gamma, \alpha, \beta, \ldots$



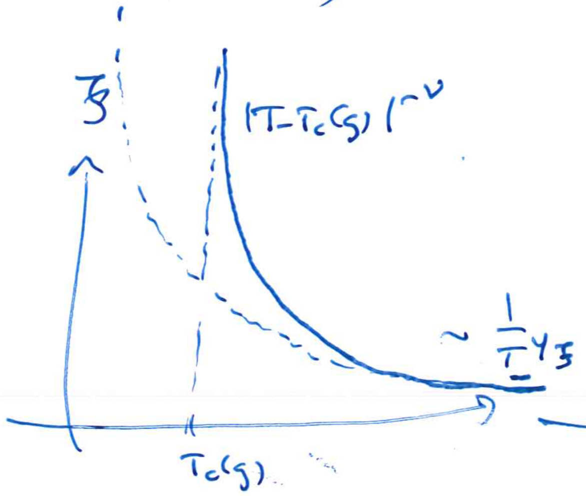
quantum FP

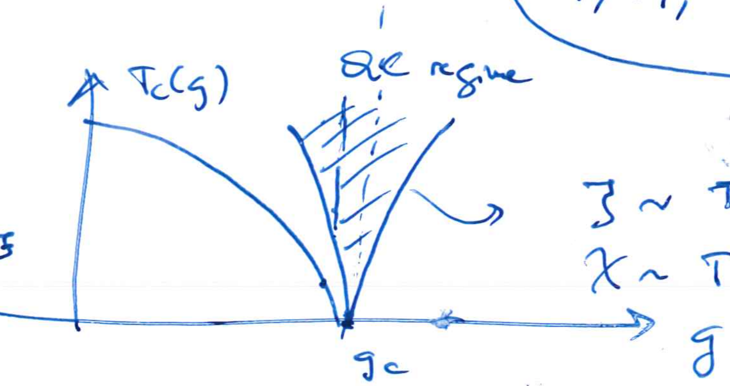
$\nu_q, \gamma_q, \alpha_q, \beta_q, \ldots$

power-law singularities

$$\xi \sim |g - g_c|^{-\nu_q} \qquad \chi \sim |g - g_c|^{-\gamma_q}$$

$\ldots$





$$\xi \sim T^{-y_\xi}$$
$$\chi \sim T^{-y_\chi}$$

Transverse field Ising model

$$\hat{\mathcal{H}} = -J \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z - g \sum_i \sigma_i^x$$

$$\hat{\mathcal{H}}_0 \qquad + g\hat{V}$$

$$\sigma_i^z |\sigma_i\rangle = \sigma_i |\sigma_i\rangle$$

$$\vec{\sigma} = (\sigma_1, \sigma_2, \ldots \sigma_N)$$

$$|\vec{\sigma}\rangle$$

$$Z = \mathrm{Tr}\left[e^{-\beta\hat{\mathcal{H}}} \right] = \sum_{\{\vec{\sigma}^{(1)}\}} \langle \vec{\sigma}^{(1)} | e^{-\beta\hat{\mathcal{H}}} | \vec{\sigma}^{(1)} \rangle$$

$$\vec{\sigma}^{(1)}$$

$$e^{-\beta\hat{\mathcal{H}}} = e^{-\beta(\hat{\mathcal{H}}_0 - g\hat{V})} = \lim_{M\to\infty} \left(e^{-\frac{\beta}{M}\hat{\mathcal{H}}_0} e^{+\frac{\beta}{M} g\hat{V}} \right)^M$$

Trotter-Lie formula

$$\sum_{\vec{\sigma}} |\vec{\sigma}\rangle\langle\vec{\sigma}| = \mathbb{1}$$

$$= \lim_{M\to\infty} \sum_{\vec{\sigma}^{(1)}, \vec{\sigma}^{(2)}, \ldots, \vec{\sigma}^{(M)}} \langle \vec{\sigma}^{(1)} | e^{+\frac{\beta}{M} g\hat{V}} | \vec{\sigma}^{(2)} \rangle \langle \vec{\sigma}^{(2)} | \ldots \langle \vec{\sigma}^{(M)} | e^{+\frac{\beta}{M} g\hat{V}} | \vec{\sigma}^{(1)} \rangle$$

$$\exp\left[+ \frac{J}{M} \sum_{\langle ij \rangle} \sum_{k=1}^{M} \sigma_i^{(k)} \sigma_j^{(k)} \right]$$

$$\langle \sigma_i^{(k)} | e^{\frac{g\sigma_i^x}{M}} | \sigma_i^{(k+1)} \rangle = \cosh\left(\frac{\beta g}{M}\right) \exp\left[\beta J_\tau \left(\sigma_i^{(k)} \sigma_i^{(k+1)} - 1 \right) \right]$$

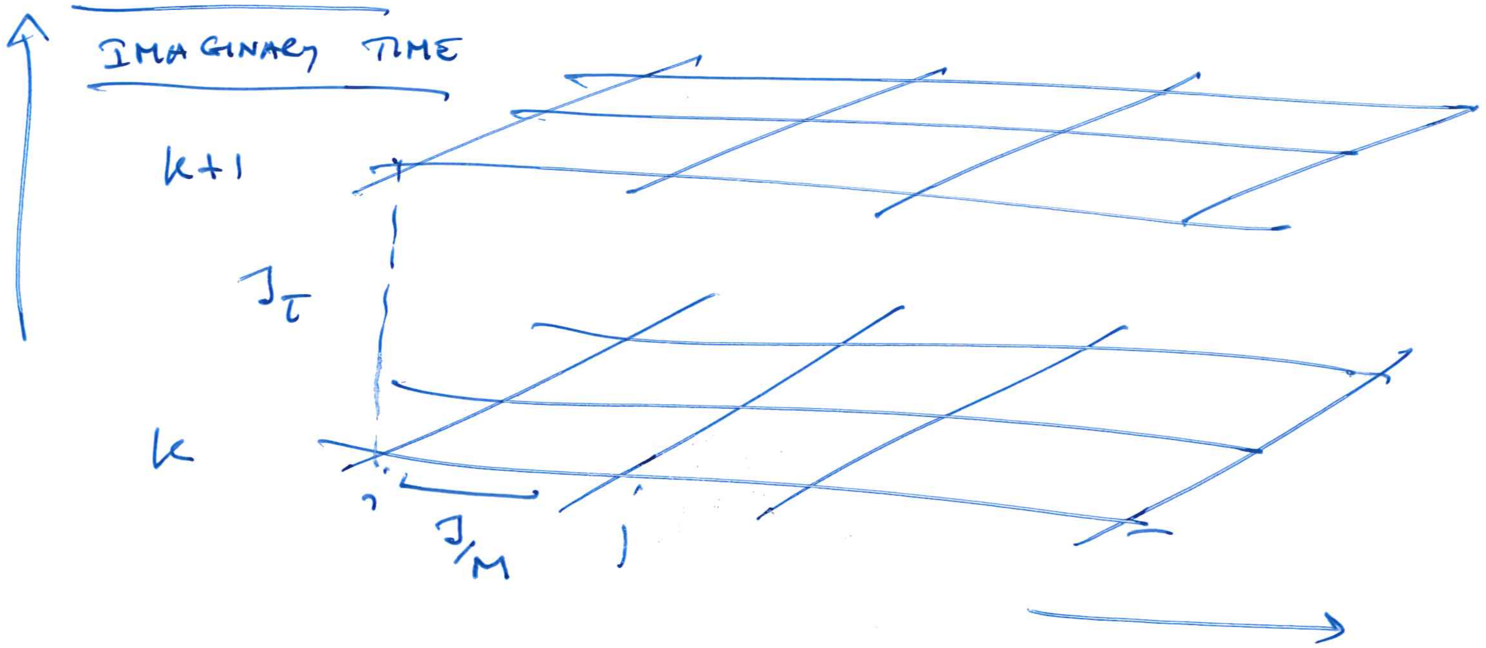
$$J_\tau = \frac{k_B T}{2} \left| \log\left(\tanh\left(\frac{\beta g}{M} \right) \right) \right|$$

$$= \lim_{M\to\infty} \sum_{\vec{\sigma}^{(1)} \ldots \vec{\sigma}^{(M)}} \exp\left[-\beta \mathcal{H}_{\mathrm{eff}} \right]$$

$$\mathcal{H}_{eff} = -\frac{J}{M} \sum_{\langle ij \rangle} \sum_{k=1}^{M} \sigma_i^{(k)} \sigma_j^{(k)} - J_\tau \sum_{k=1}^{M} \sum_{i=1}^{N} \left(\sigma_i^{(k)} \sigma_i^{(k+1)} \cancel{-1} \right)$$

extra dimension

IMAGINARY TIME

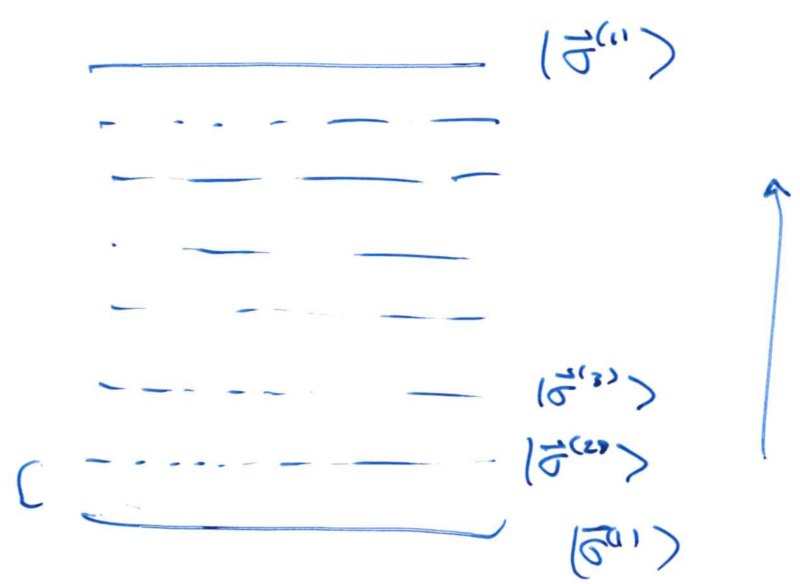


real space dimensions

quantum Ising model in d dimensions

$\Rightarrow$ classical anisotropic (d+1) dimensional Ising model

$$Z = \sum_{\vec{\sigma}^{(1)}} \langle \vec{\sigma}^{(1)} | e^{-\beta \mathcal{H}} | \vec{\sigma}^{(1)} \rangle$$



$$\Delta\tau = \frac{\beta}{M}$$

$$e^{-\beta \mathcal{H}} = e^{-\frac{i}{\hbar} \mathcal{H} t}$$

$$t = \left(-\frac{i}{\hbar} \tau \right)$$

$$\left(e^{-\frac{\beta}{M} (\hat{\mathcal{H}}_0 - g \hat{V})} \right)^M = \left(e^{-\frac{\beta}{M} \hat{\mathcal{H}}_0} e^{+g \frac{\beta}{M} \hat{V}} \right)^M + O\Big(\overset{\Delta\tau}{\frac{\beta}{M}} \Big)$$

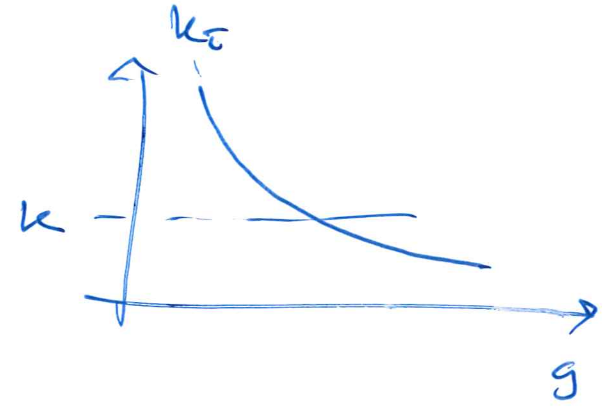
Trotter decomposition

Fix $\Delta\tau$ $\Rightarrow$ fixing the error in the Trotter decomposition

$\Delta\tau \to \quad M = \frac{1}{\Delta\tau\, k_B T}$

$$\boxed{K = \Delta\tau\, J = J\,\frac{\beta}{M}}$$

$$\boxed{K_\tau = \beta J_\tau = \frac{1}{2}\left|\log\left(\tanh(\Delta\tau\, g)\right)\right|}$$

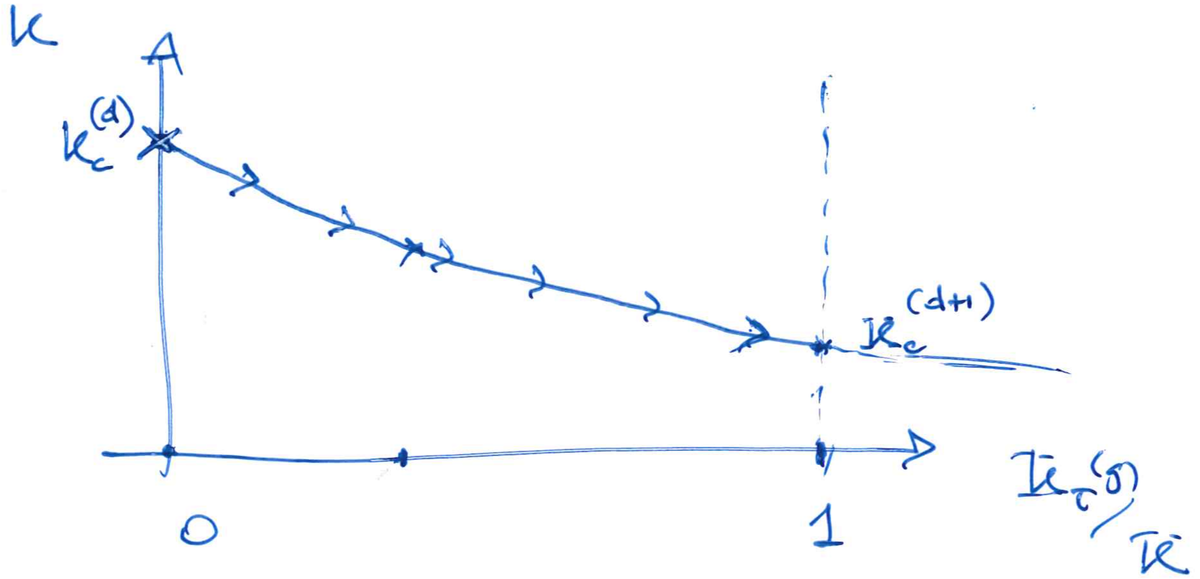


$$\boxed{\beta \mathcal{H}_{eff} = -K \sum_{\langle ij\rangle} \sum_k \sigma_i^{(k)} \sigma_j^{(k)} - K_\tau \sum_i \sum_k \sigma_i^{(k)} \sigma_i^{(k+1)} + \text{const.}}$$

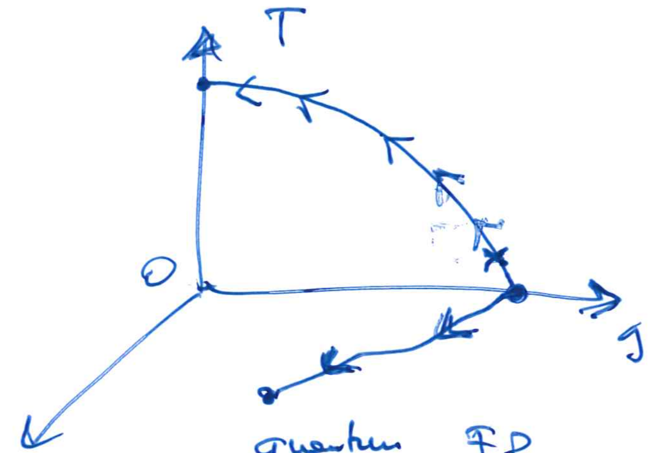
quantum critical point : $T \to 0$ at fixed $\Delta\tau$

$M \to \infty$

$N \to \infty$



$K_\tau = K$



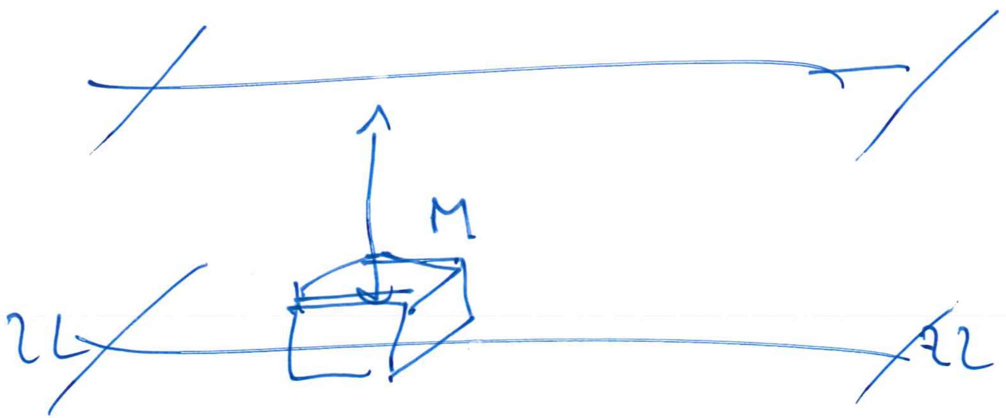
quantum FP
(d+1) dimensional
Ising model

$T > 0$ |

fixed $\Delta\tau \Rightarrow$ finite M $\quad M = \frac{1}{T\Delta\tau}$

$\nu_q,\ \gamma_q,\ \alpha_q,$ —

critical exponents of the
(d+1) dimensional Ising model



→ real dimension
$N \to \infty$