$[\hat{A},\hat{B}]=0$ A, B observables compatibles



$[\hat{A},\hat{B}]=0 \iff$ $\hat{A}$ et $\hat{B}$ admettent une base commune de vecteurs propres

$\exists\, |\chi_i\rangle$ : $\hat{A}|\chi_i\rangle = a_i|\chi_i\rangle$

$\mathbb{1} = \sum_i |\chi_i\rangle\langle\chi_i|$ $\hat{B}|\chi_i\rangle = b_i|\chi_i\rangle$

$\langle\chi_i|\chi_j\rangle = \delta_{ij}$

$\Leftarrow$

$$\hat{A}\hat{B}|\chi_i\rangle = \hat{A}(b_i|\chi_i\rangle) = b_i\hat{A}|\chi_i\rangle = b_i\underbrace{a_i|\chi_i\rangle}_{\hat{A}|\chi_i\rangle}$$

$$= \hat{B}\hat{A}|\chi_i\rangle$$

$\forall |\chi_i\rangle \Rightarrow \hat{A}\hat{B} = \hat{B}\hat{A}$

$\Rightarrow$ $\hat{A}$ a des valeurs propres <u>non dégénérées</u>

$$\hat{A}|a_i\rangle = a_i|a_i\rangle$$

$$\hat{B}\hat{A}|a_i\rangle = a_i(\hat{B}|a_i\rangle) = \hat{A}(\hat{B}|a_i\rangle)$$

$$\hat{B}|a_i\rangle = b_i|a_i\rangle$$

$\hat{A}$ admets des valeurs propres dégénérées

$$\hat{A}|a_p;\ell\rangle = a_p|a_p;\ell\rangle \qquad \ell = 1 \dots g_p$$

$$\hat{B}\,\hat{A}|a_p;\ell\rangle = a_p\left(\hat{B}|a_p;\ell\rangle\right) = \hat{A}\left(\hat{B}|a_p;\ell\rangle\right)$$

$\hat{B}|a_p;\ell\rangle$ est vecteur propre de $\hat{A}$ associé à $a_p$

$$\hat{B}|a_p;\ell\rangle = \sum_{\ell'} \beta^{(p)}_{\ell\ell'}|a_p;\ell'\rangle$$

$$\langle a_{p'};\ell'|\hat{B}|a_p;\ell\rangle \to \left(\begin{array}{ccc} \boxed{\begin{matrix}\beta^{(1)}_{11} & \beta^{(1)}_{12} & \cdots & \beta^{(1)}_{1g_1}\\ \vdots & & & \\ \beta^{(1)}_{g_11} & - & \cdots & \beta^{(1)}_{g_1g_1}\end{matrix}} & & \\ & \boxed{\begin{matrix}\beta^{(2)}_{11} & \cdots & \beta^{(2)}_{1g_2}\\ \vdots & & \vdots \\ \beta^{(2)}_{g_21} & - & \beta^{(2)}_{g_2g_2}\end{matrix}} & \\ & & \boxed{\phantom{\beta\beta\beta}} \\ & & & \ddots \end{array}\right) \begin{array}{l} p=1 \to g_p \times g_p \\ p=2 \\ p=3 \end{array}$$

$\sim \delta_{pp'}$

$|a_p;b_q\rangle$ : $\boxed{\hat{B}|a_p;b_q\rangle = b_q|a_p;b_q\rangle}$

$$|a_p;b_q\rangle = \sum_{\ell'} \gamma^{(p)}_{\ell',q}|a_p;\ell'\rangle \qquad \hat{A}|a_p;b_q\rangle = a_p|a_p;b_q\rangle$$

1) Si $b_q$ ne sont pas dégénérées → en physique : mesure de A et B spécifie d'une façon unique l'état

2) Si <u>$b_q$ est à la fois dégénérée</u> : $\hat{C}$ : $[\hat{A},\hat{C}]=0$, $[\hat{B},\hat{C}]=0$ $\Rightarrow$ $|a_p,b_q,c_m\rangle$

$\hat{A}, \hat{B}, \hat{C}, \ldots$ qui commutent entre eux (compatibles) tel que l'état du système est uniquement déterminé par les valeurs propres des opérateurs $\Rightarrow$ ensemble complet d'opérateurs compatibles (ECOC)



$\hat{A}$ a valeurs propres dégénérées

$$|\psi\rangle \rightarrow \boxed{\hat{A}} \rightarrow |\psi'\rangle = \frac{\sum_\ell \hat{P}(|a_p, \ell\rangle)|\psi\rangle}{\| \qquad \|}$$

$$\downarrow a_p$$

$$P(a_i) = \sum_\ell |\langle a_p; \ell | \psi\rangle|^2$$

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## Evolution des états en MQ

Norme des états. $\|\psi\|^2 = \langle\psi|\psi\rangle = 1$

$$P(a_i) = |\langle a_i|\psi\rangle|^2 \qquad \sum_i P(a_i) = 1 = \langle\psi|\psi\rangle$$

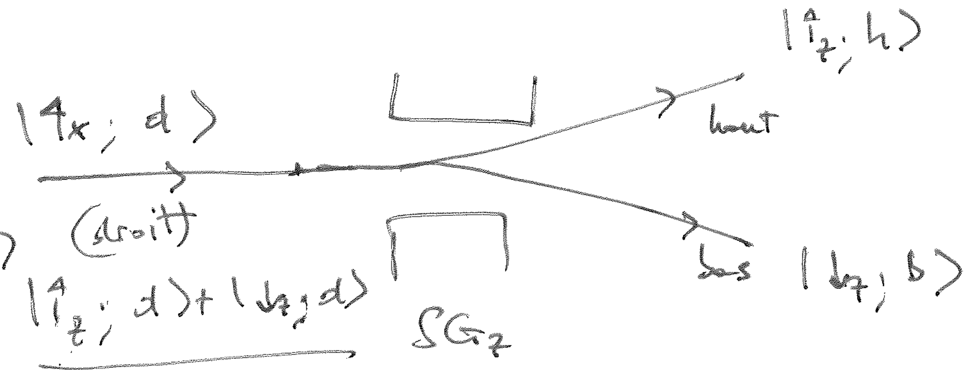
$$|\psi\rangle \rightarrow |\psi'\rangle \ : \ \langle\psi'|\psi'\rangle = 1$$

Transformation physique de l'état :

- évolution
- changement de référentiel
- [ • mesure ]

Stern - Gerlach






$|\psi\rangle = \dfrac{|\uparrow_z; d\rangle + |\downarrow_z; d\rangle}{\sqrt{2}}$

séparateur de faisceau / polariseur

$|\psi'\rangle = \dfrac{|\uparrow_z; h\rangle + |\downarrow_z; b\rangle}{\sqrt{2}}$

entrée $\qquad$ sortie

$|\uparrow_z; d\rangle \rightarrow |\uparrow_z; h\rangle = \hat{U}|\uparrow_z; d\rangle$

$|\downarrow_z; d\rangle \rightarrow |\downarrow_z; b\rangle = \hat{U}|\downarrow_z; d\rangle$

$$|\psi'\rangle = \hat{U}|\psi\rangle = \frac{\hat{U}|\uparrow_z; d\rangle + \hat{U}|\downarrow_z; d\rangle}{\sqrt{2}}$$

$\hat{U}$ opérateur d'évolution linéaire

$$|\psi'\rangle = \hat{U}|\psi\rangle \Rightarrow \langle\psi'|\psi'\rangle = \langle\psi|\hat{U}^+\hat{U}|\psi\rangle = \langle\psi|\psi\rangle = 1$$

$\hat{U}^+\hat{U} = \mathbb{1} \Rightarrow$ UNITAIRE

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$$|\psi\rangle = \alpha_1|\psi_1\rangle + \alpha_2|\psi_2\rangle \rightarrow \boxed{\hat{A}} \rightarrow |\psi'\rangle = \underbrace{\Sigma_\ell\left(\alpha_1\langle a_p;\ell|\psi_1\rangle + \alpha_2\langle a_p;\ell|\psi_2\rangle\right)|a_p;\ell\rangle}$$

$\boxed{\hat{A}} \downarrow a_p$

$$|\psi_{1,2}\rangle \rightarrow \underbrace{\Sigma_\ell \langle a_p\,\ell|\psi_{1,2}\rangle|a_p;\ell\rangle}$$

$$\neq \underbrace{\Sigma_\ell\langle a_p;\ell|\psi_1\rangle|a_p;\ell\rangle} + \underbrace{\Sigma_\ell\langle a_p;\ell|\psi_2\rangle|a_p;\ell\rangle}$$

mesure n'est pas linéaire

## Evolution

$$|\psi(t)\rangle \to |\psi(t+\Delta t)\rangle = \hat{U}(t, t+\Delta t)\,|\psi(t)\rangle \qquad \hat{U} \text{ unitaire}$$

$$|\psi(t+\Delta t)\rangle \underset{\Delta t\to 0}{=} |\psi(t)\rangle + \Delta t \frac{d}{dt}|\psi(t)\rangle + O(\Delta t^2) = \left(\mathbb{1} + \Delta t \frac{d}{dt} + o(\Delta t^2)\right)|\psi(t)\rangle$$

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Opérateurs unitaires : exponentiels d'opérateurs hermitiens

$$\hat{U}^+ \hat{U} = \hat{\mathbb{1}}$$

$$\exp[\alpha\hat{A}] = \widehat{\exp[\alpha A]} =: \sum_{n=0}^{\infty} \frac{1}{n!}\alpha^n \hat{A}^n \qquad \alpha \in \mathbb{C}$$

$$\exp(\hat{A})\exp(-\hat{A}) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{1}{n!\,m!} \underbrace{\hat{A}^n \hat{A}^m}_{\hat{A}^{n+m}} (-1)^m = \hat{\mathbb{1}}$$

$\exp(x)\exp(-x)$

$$\exp(\alpha\hat{A})\exp(-\alpha\hat{A}) = \hat{\mathbb{1}}$$

$$\exp(\hat{A})\exp(\hat{B}) \neq \exp(\hat{A}+\hat{B})$$

sauf si $[\hat{A},\hat{B}]=0$

$$\exp(-\alpha\hat{A}) = (\exp(\alpha\hat{A}))^+$$

$$(\exp(\alpha\hat{A}))^+ = \sum_{n=0}^{\infty}\frac{1}{n!}(\alpha^*)^n(\hat{A}^+)^n = \exp(\alpha^*\hat{A}^+)$$

$$f^*(\hat{A}) = f(\hat{A}^+)$$

$\hat{A} = \hat{A}^+$ $\hat{A}$ hermitien

$\alpha = i\phi, \; \phi \in \mathbb{R}$ $\alpha^* = -\alpha$

$\exp(\alpha \hat{A})^+ = (\exp(i\phi \hat{A}))^+ = \exp(-i\phi \hat{A})$

$(\exp(i\phi \hat{A}))^+ \exp(i\phi \hat{A}) = \exp(-i\phi \hat{A}) \exp(i\phi \hat{A}) = \mathbb{1}$

$\boxed{\hat{U} = \exp(i\phi \hat{A})}$ à chaque $\hat{U}$ on associe un $\hat{A}$ HP que et un $\phi \in \mathbb{R}$

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$\hat{U}(t, t+\Delta t) = e^{-i\phi(t, t+\Delta t)\hat{A}(t, t+\Delta t)} \simeq \left(\mathbb{1} - i\Delta t \hat{A} + O(\Delta t^2)\right)$

$\Delta t \to 0$ $\phi(t, t+\Delta t) \to 0$

$\boxed{\phi = \Delta t}$

$\hat{U}(t, t+\Delta t)|\psi(t)\rangle = \left(\mathbb{1} + \Delta t \frac{d}{dt} + O(\Delta t^2)\right)|\psi(t)\rangle$

$= \left(\mathbb{1} - i\Delta t \hat{A} + O(\Delta t^2)\right)|\psi(t)\rangle$

$\boxed{i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle}$

équation de Schrödinger

$i\frac{d}{dt}|\psi(t)\rangle = \hat{A}|\psi(t)\rangle$

$E = \hbar\omega$

opérateur énergie $\hat{H}$

opérateur Hamiltonien

$\hat{A} = \hat{H}/\hbar$

$$\boxed{\hat{U}(t,\ t+\Delta t) \underset{\Delta t \to 0}{=} e^{-\frac{i}{\hbar}\hat{H}(t)\Delta t}}$$

$$\hat{H} = \hat{H}^{+}$$



Exemple : spin $\frac{1}{2}$

$\Uparrow \vec{B}(t)$

$\hat{\vec{S}} \to \hat{\vec{m}} = \frac{g\mu_B}{\hbar}\hat{\vec{S}}$

$$E = -\vec{m}\cdot\vec{B} \Rightarrow \boxed{\begin{aligned}\hat{H}(t) &= -\frac{g\mu_B}{\hbar}\hat{\vec{S}}\cdot\vec{B}(t) \\ &= -\frac{g\mu_B}{\hbar}\left[B^x\hat{S}^x + B^y\hat{S}^y + B^z\hat{S}^z\right]\end{aligned}}$$

$\vec{B} = B\,\vec{e}_\alpha \qquad E = \pm \frac{g\mu_B}{2} B \qquad \hat{S}^\alpha \to \pm\frac{\hbar}{2}$

$$\boxed{i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)\,|\psi(t)\rangle}$$

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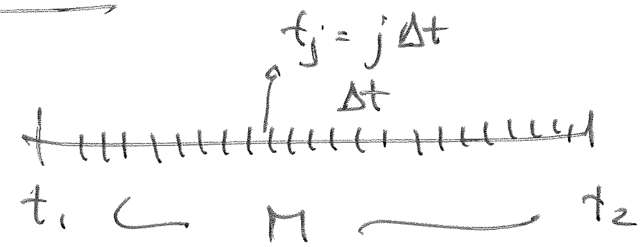
$\hat{H}$ indépendant du temps

$\hat{H}(t) = \hat{H}$ const

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}}$$
$$\downarrow$$
$$[\hat{A},\hat{B}] = 0$$

$$\hat{U}(t_1,t_2) \cong \prod_{j=1}^{M} e^{-\frac{i}{\hbar}\hat{H}(t_j+t_1)\Delta t}$$




$$= e^{-\frac{i}{\hbar}\sum_{j=1}^{M}\hat{H}\Delta t} = e^{-\frac{i}{\hbar}\hat{H}(t_2-t_1)}$$

$t_2 = M\Delta t + t_1$

$$\boxed{\hat{U}(t_1,t_2) = e^{-\frac{i}{\hbar}\hat{H}(t_2-t_1)}}$$

Valeurs propres et vecteur propres de $\hat{H}$

$\hat{H} | a_n \rangle = E_n | a_n \rangle$ $\quad E_n$ valeur possible pour l'énergie

$t_1 = 0 \quad t_2 = t$

$$\hat{U}(t) \, | a_n \rangle = e^{-i/\hbar \hat{H} t} | a_n \rangle = \sum_{m=0}^{\infty} \left( \frac{-it}{\hbar} \right)^m \frac{1}{m!} \underbrace{\hat{H}^m | a_n \rangle}_{(E_n)^m}$$

$$= \exp\left( \frac{-i}{\hbar} E_n t \right) | a_n \rangle$$

Si $|a_n\rangle$ est vecteur propre de $\hat{A}$ avec valeur propre $a_n$

$f(\hat{A}) | a_n \rangle = f(a_n) | a_n \rangle$

$|a_n\rangle$ : état stationnaire de l'évolution

$\underline{t=0}$

$|\psi(0)\rangle = \sum_n \psi_n^{(0)} | a_n \rangle$ $\quad P(E_n, t=0) = | \langle a_n | \psi \rangle |^2 = | \psi_n(0) |^2$

$$|\psi(t)\rangle = \hat{U} \sum_n \psi_n^{(0)} | a_n \rangle = \sum_n \underbrace{\psi_n(0) \, e^{-i/\hbar E_n t}} | a_n \rangle$$

$$\psi_n(t) = \psi_n(0) e^{-i/\hbar E_n t}$$

$$P(E_n; t) = |\psi_n(t)|^2 = |\psi_n(0)|^2$$

$$\underline{\langle \hat{H} \rangle_t} = \langle \psi(t) | \hat{H} | \psi(t) \rangle = \sum_n E_n \, P_n(E_n; t) = \sum_n E_n \, P(E_n, t=0) = \underline{\langle \hat{H} \rangle_{t=0}}$$

<u>conservation de l'énergie</u>

Oscillations de Rabi (precession de Larmor)



$\vec B = B\vec e_x$ $\qquad |\uparrow_z\rangle = |\psi(0)\rangle \qquad t=0$

$$\mathcal H = -\underbrace{\boxed{\frac{g\mu_B}{\hbar} B}}_{\Gamma}\hat S^x$$

$\Gamma$ fréquence de Larmor

$$= \frac{g\mu_B B}{\hbar}$$

$$|\uparrow_z\rangle = \frac{|\uparrow_x\rangle - |\downarrow_x\rangle}{\sqrt2}$$

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}(-\Gamma\hat S^x)t}\,|\psi(0)\rangle$$

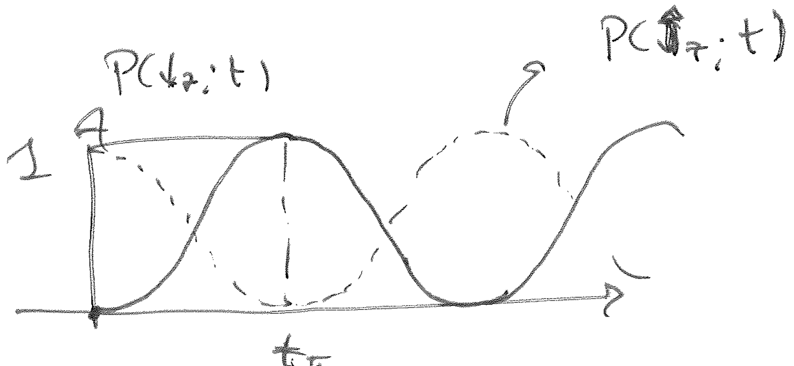
$$= \frac{e^{\frac{i}{\hbar}\Gamma\frac{\hbar}{2}t}|\uparrow_x\rangle - e^{-\frac{i\Gamma t}{2}}|\downarrow_x\rangle}{\sqrt2}$$

$$|\uparrow_x\rangle = \frac{|\uparrow_z\rangle+|\downarrow_z\rangle}{\sqrt2}$$

$$|\downarrow_x\rangle = \frac{-|\uparrow_z\rangle+|\downarrow_z\rangle}{\sqrt2}$$

$$= \frac{e^{i\frac{\Gamma}{2}t}\left(|\uparrow_z\rangle+|\downarrow_z\rangle\right) - e^{-i\frac{\Gamma t}{2}}\left(-|\uparrow_z\rangle+|\downarrow_z\rangle\right)}{2}$$

$$= \cos\left(\frac{\Gamma t}{2}\right)|\uparrow_z\rangle + i\sin\left(\frac{\Gamma t}{2}\right)|\downarrow_z\rangle$$




$$P(\downarrow_z;t) = \sin^2\left(\frac{\Gamma t}{2}\right)$$

$t_\pi \qquad \frac{\Gamma t}{2} = \frac{\pi}{2} \qquad \boxed{t_\pi = \frac{\pi}{\Gamma}}$