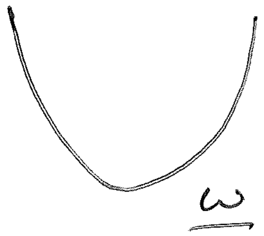
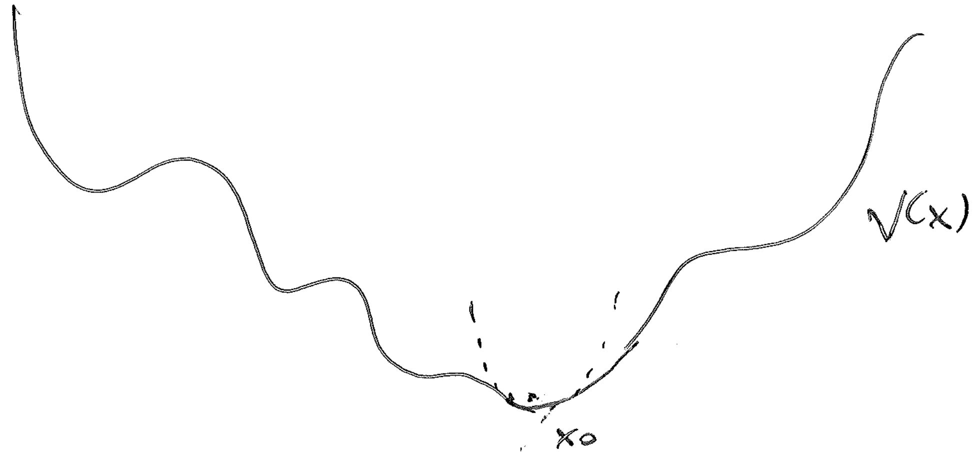
OSCILLATEUR HARMONIQUE



$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 x^2$$



$$V(x) \simeq V(x_0) + \frac{1}{2} \underbrace{V''(x_0)}_{m\omega^2} (x - x_0)^2$$

$\hbar\omega$

$$\frac{\hat{H}}{\hbar\omega} = \frac{\hat{p}^2}{2m\hbar\omega} + \frac{1}{2} \frac{m\omega}{\hbar} \hat{x}^2$$

$$p_{ho} = \sqrt{m\hbar\omega} \qquad a_{ho} = \sqrt{\frac{\hbar}{m\omega}}$$

$$\hat{X} = \frac{\hat{x}}{a_{ho}} \qquad \hat{P} = \frac{\hat{p}}{p_{ho}}$$

$$[\hat{X}, \hat{P}] = i\hbar \underbrace{\left(\sqrt{m\hbar\omega}\sqrt{\frac{\hbar}{m\omega}}\right)^{-1}}_{\sim \hbar^{-1} \sim} = i$$

$$\frac{\hat{H}}{\hbar\omega} = \frac{1}{2}\left[\hat{X}^2 + \hat{P}^2\right] \qquad x^2 + y^2 = (x+iy)(x-iy)$$

$$\begin{cases} \hat{a} = \frac{1}{\sqrt{2}}(\hat{X} + i\hat{P}) \\ \hat{a}^+ = \frac{1}{\sqrt{2}}(\hat{X} - i\hat{P}) \end{cases}$$

$\hat{a}$ → opérateur de destruction (descente)

$\hat{a} \neq \hat{a}^+ \qquad \hat{a}^+\hat{a} \neq \mathbb{1}$

$\hat{a}^+$ → opérateur de création (montée)

$$[\hat{a}, \hat{a}^+] = \frac{1}{2}\left[\hat{X} + i\hat{P}, \hat{X} - i\hat{P}\right] = \frac{1}{2}\left[(-i)i + i(-i)\right] = 1$$

$$\boxed{[\hat{a}, \hat{a}^+] = 1}$$

$$\hat{a}^+\hat{a} = \frac{1}{2}(\hat{X}-i\hat{P})(\hat{X}+i\hat{P}) = \frac{1}{2}\Big(\hat{X}^2+\hat{P}^2+i\underbrace{(\hat{X}\hat{P}-\hat{P}\hat{X})}_{i}\Big) = \frac{1}{2}(\hat{X}^2+\hat{P}^2-1)$$

$$\boxed{\frac{\hat{H}}{\hbar\omega} = \hat{a}^+\hat{a}+\frac{1}{2}}$$

$$\hat{a}^+\hat{a}\,|\nu\rangle = \nu\,|\nu\rangle \qquad \langle\nu|\nu\rangle = 1$$

$$\langle\nu|\hat{a}^+\hat{a}|\nu\rangle = \|\hat{a}|\nu\rangle\|^2 = \underline{\nu \geq 0}$$

$$\hat{H} \to \hbar\omega\left(\nu+\frac{1}{2}\right)$$

$\hat{a}|\nu\rangle$

$\hat{a}^+|\nu\rangle$

$\hat{a}^+\hat{a}\;\hat{a}|\nu\rangle$

$$\begin{cases}[\hat{a}^+\hat{a},\hat{a}] = [\hat{a}^+,\hat{a}]\,\hat{a} = -\hat{a}\\ [\hat{a}^+\hat{a},\hat{a}^+] = \hat{a}^+[\hat{a},\hat{a}^+] = \hat{a}^+\end{cases}$$

$$\hat{a}^+\hat{a}\;\hat{a}|\nu\rangle = \hat{a}\,(\hat{a}^+a-1)\,|\nu\rangle$$
$$= (\nu-1)\;\hat{a}|\nu\rangle$$

$$(\hat{a}^+\hat{a})\hat{a} = \hat{a}(\hat{a}^+\hat{a})-\hat{a} = \hat{a}\,(\hat{a}^+a-1)$$
$$(\hat{a}^+a)\hat{a}^+ = \hat{a}^+(a^+a)+a^+ = \hat{a}^+(\hat{a}^+\hat{a}+1)$$

$$\hat{a}^+\hat{a}\;\hat{a}^+|\nu\rangle = \hat{a}^+(\hat{a}^+\hat{a}+1)\,|\nu\rangle$$
$$= (\nu+1)\;\hat{a}^+|\nu\rangle$$

$\hat{a}|\nu\rangle$ vecteur propre de $\hat{a}^+\hat{a}$ avec valeur propre $\underline{\nu-1}$

$\hat{a}^+|\nu\rangle$ " " " " " " " $\underline{\nu+1}$

$$\hat{a}^+\hat{a}\;\hat{a}^2|\nu\rangle = \hat{a}\,(\hat{a}^+\hat{a}-1)\,\hat{a}\,|\nu\rangle = \hat{a}^2\,(\hat{a}^+\hat{a}-2)\,|\nu\rangle = (\nu-2)\,\hat{a}^2|\nu\rangle$$

$$\hat{a}^+\hat{a}\;\hat{a}^p|\nu\rangle = \quad - - \quad = (\nu-p)\;\hat{a}^p|\nu\rangle \qquad p\in\mathbb{N}$$

$\nu-p<0$ $\qquad$ $\nu-p+1>0$

$$\|\hat{a}^{p+1}|\nu\rangle\|^2 = \langle\nu|(\hat{a}^+)^{p+1}\,\hat{a}^{p+1}|\nu\rangle$$
$$= \langle\nu|(\hat{a}^+)^p(\hat{a}^+\hat{a})\,\hat{a}^p|\nu\rangle = \nu-p<0$$

$\boxed{v = n \in \mathbb{N}}$ $\quad p = n \quad$ $\hat{a}^p|n\rangle \sim |0\rangle$

$\hat{a}^{p+1}|n\rangle = 0$ vecteur nul

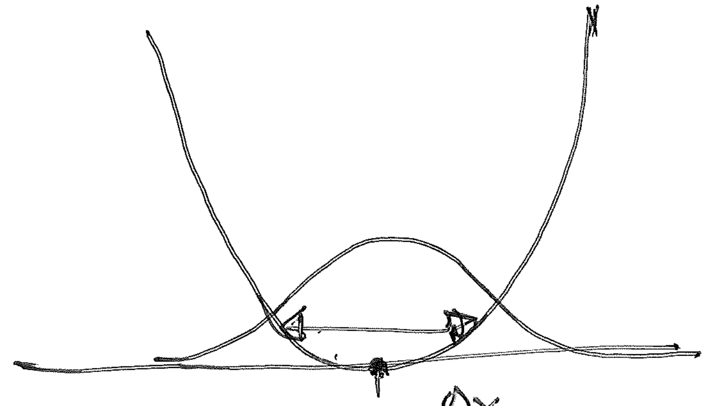
$\hat{a}^+\hat{a}$ = opérateur "nombre" $= \hat{n}$

valeurs propres de $\hat{H}$

$$\hat{H} = \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right) \qquad E_n = \hbar\omega\left(n + \frac{1}{2}\right) \qquad n = 0, 1, 2, \dots, \infty$$

$$E_0 = \frac{\hbar\omega}{2} \neq 0$$

Energie de point zéro



$\hookrightarrow \Delta p = \hbar/\Delta x$

$|n\rangle$ états propres non dégénérés

Fondamental est non dégénéré : $|0\rangle$ $\qquad \langle n|n'\rangle = \delta_{nn'}$ $\qquad |n\rangle$

Récurrence $\rightarrow$ $|1\rangle \sim \hat{a}^+|0\rangle$, $|2\rangle \sim (\hat{a}^+)^2|0\rangle$, ...

$|n\rangle$ soit état propre non-dégénéré associé à n pour $\hat{a}^+\hat{a}$

$\Rightarrow$ $|\phi_{n+1}\rangle$: $\hat{a}^+\hat{a}|\phi_{n+1}\rangle = (n+1)|\phi_{n+1}\rangle$

$$\hat{a}^+\hat{a}\left(\hat{a}|\phi_{n+1}\rangle\right) = n\,\hat{a}|\phi_{n+1}\rangle \quad \Rightarrow \quad \hat{a}|\phi_{n+1}\rangle = c|n\rangle$$

$(n+1)|\phi_{n+1}\rangle = c\,a^+|n\rangle$ $\qquad$ $|\phi_{n+1}\rangle$ est unique

$$|\phi_{n+1}\rangle = \frac{c}{n+1}\,a^+|n\rangle$$

$\hat{a}^+|n\rangle \rightarrow$ $\cancel{(\hat{a}^+\hat{a})\,\hat{a}^+|n\rangle = (n+1)\,\hat{a}^+|}$

$[\hat{a},\hat{a}^+] = 1$

$$\|\hat{a}^+|n\rangle\|^2 = \langle n|\underbrace{\hat{a}\hat{a}^+}_{(\hat{a}^+\hat{a}+1)}|n\rangle = (n+1)$$

$$\hat{a}^+|n\rangle = \sqrt{n+1}\;|n+1\rangle$$

$$\|\hat{a}|n\rangle\|^2 = \langle n|\hat{a}^+\hat{a}|n\rangle = n$$

$$\hat{a}|n\rangle = \sqrt{n}\;|n-1\rangle$$

$$\boxed{\begin{aligned}\hat{a}^+|n\rangle &= \sqrt{n+1}\;|n+1\rangle \\ \hat{a}|n\rangle &= \sqrt{n}\;|n-1\rangle\end{aligned}}$$

Fonctions propres de l'oscillateur harmonique

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2\right)\psi_n(x) = \hbar\omega\left(n+\frac{1}{2}\right)\psi_n(x)$$

$u = \dfrac{x}{a_{ho}}$

$\phi_n(u) = \psi_n(x)$

$$\left(-\frac{d^2}{du^2} + u^2\right)\phi_n(u) = (2n+1)\,\phi_n(u)$$

équation d'Hermite

$$\hat{a}|0\rangle = 0 \qquad \hat{a} = \frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{a_{ho}} + i\frac{\hat{p}}{p_{ho}}\right)$$

$$\langle x|\hat{a}|0\rangle = \frac{1}{\sqrt{2}}\left(\frac{x}{a_{ho}} + \frac{\hbar}{p_{ho}}\frac{d}{dx}\right)\psi_0(x) = 0$$

$\psi_0(x) = \langle x|0\rangle$

$$\Rightarrow \left(u + \frac{d}{du}\right)\phi_0(u) = 0$$

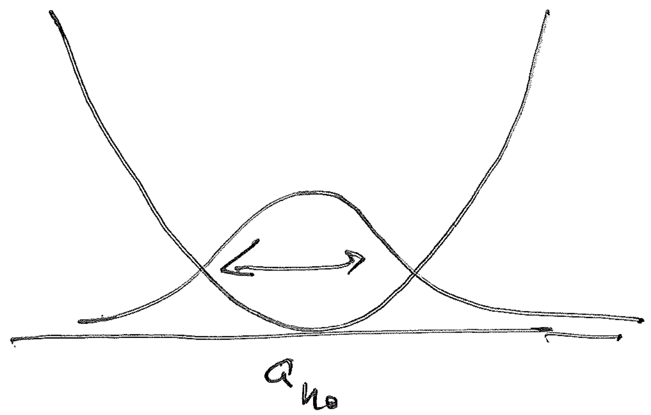
$$\frac{d}{du}\phi_0(u) = -u\,\phi_0(u)$$

$$\boxed{\phi_0(u) = A\,e^{-u^2/2}}$$

Solution unique

$$\int_{-\infty}^{+\infty} dx\, e^{-x^2/a_{ho}^2} = a_{ho}\int_{-\infty}^{+\infty} du\, e^{-u^2} = \sqrt{\pi}\, a_{ho}$$

$$\boxed{\psi_0(x) = \frac{1}{(a_{ho}\sqrt{\pi})^{1/2}}\, e^{-x^2/2a_{ho}^2}}$$



$$|1\rangle = \hat{a}^+|0\rangle \Rightarrow \psi_1(x) = \frac{1}{\sqrt{2}}\left(\frac{x}{a_{ho}} - \frac{\hbar}{p_{ho}}\frac{d}{dx}\right)\psi_0(x)$$

$$= \frac{1}{\sqrt{2}}\left(u - \frac{d}{du}\right)\phi_0(u)$$

$$\hat{a}^+|0\rangle = \sqrt{1}\,|1\rangle$$

$$(\hat{a}^+)^2|0\rangle = \sqrt{1}\sqrt{2}\,|2\rangle$$

$$(\hat{a}^+)^3|0\rangle = \sqrt{1}\times\sqrt{2}\times\sqrt{3}\,|3\rangle$$

$$(\hat{a}^+)^n|0\rangle = \sqrt{n!}\,|n\rangle$$

$$\boxed{|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^+)^n|0\rangle}$$

$$\langle x|n\rangle = \frac{1}{\sqrt{n!}}\frac{1}{\sqrt{2^n}}\left(u - \frac{d}{du}\right)^n \phi_0(u) = \phi_n(u)$$

polynômes d'Hermites $H_n(u)$ polynôme d'ordre n

$$\left(u - \frac{d}{du}\right)^n e^{-u^2/2} = H_n(u)\, e^{-u^2/2}$$

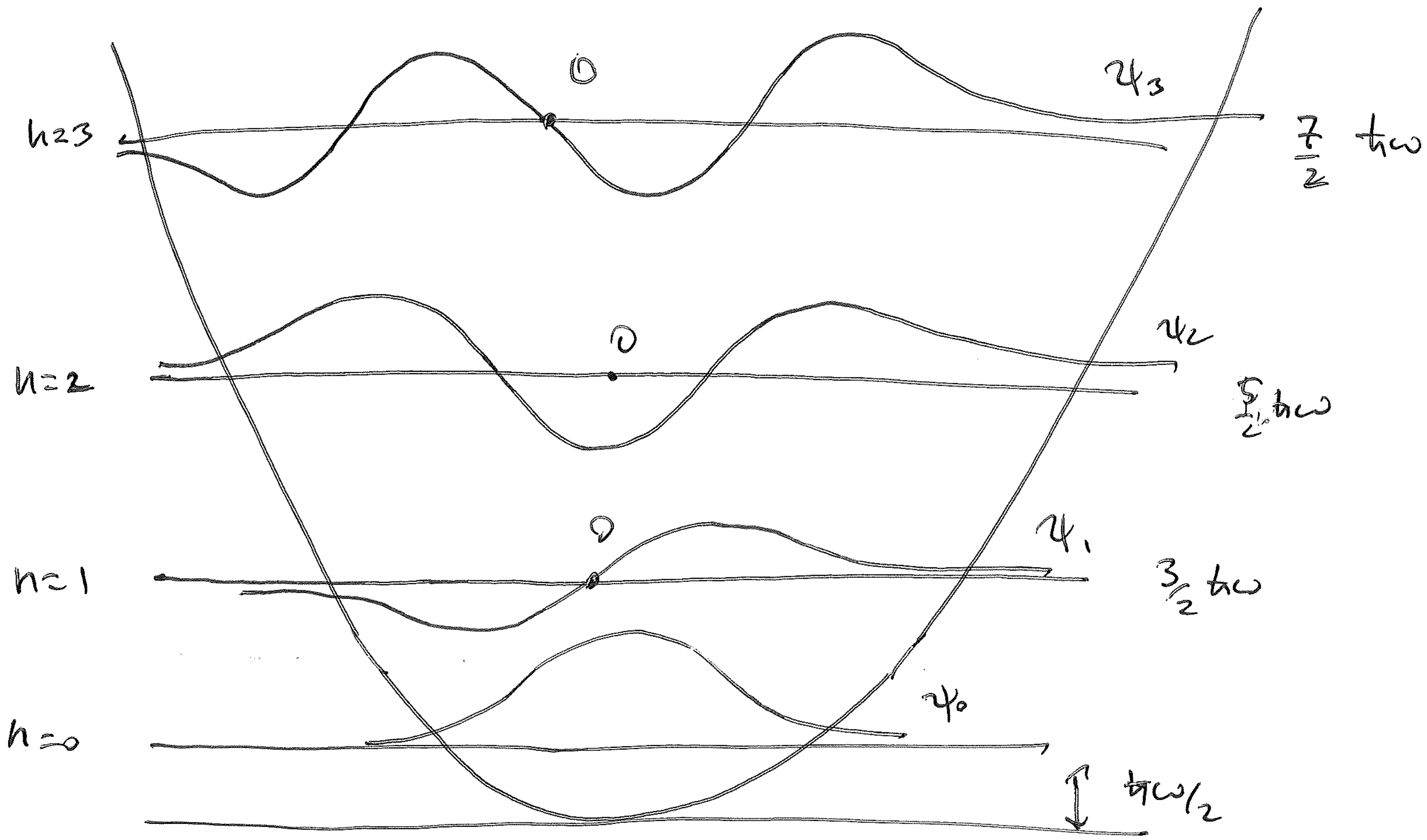
$$\boxed{\psi_n(x) = \frac{1}{(\sqrt{\pi}\, 2^n\, n!\, a_{ho})^{1/2}}\, H_n\!\left(\frac{x}{a_{ho}}\right) e^{-x^2/2a_{ho}^2}}$$

$H_0(u) = 1$

$H_1(u) = 2u$

$H_2(u) = 4u^2 - 2$

....



$$\varepsilon_n = \hbar\omega\left(n+\frac{1}{2}\right)$$

$n \gg 1$

$$\begin{cases} \hat{x} = \sqrt{\dfrac{\hbar}{2m\omega}}\left(\hat{a}+\hat{a}^+\right) \\ \hat{p} = \sqrt{\dfrac{m\hbar\omega}{2}}\ \dfrac{\left(\hat{a}-\hat{a}^+\right)}{i} \end{cases}$$

$$\langle \hat{x} \rangle_n = \sqrt{\frac{\hbar}{2m\omega}}\langle n|\left(\hat{a}+\hat{a}^+\right)|n\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\ \langle n|\left(\sqrt{n}\,|n-1\rangle + \sqrt{n+1}\,|n+1\rangle\right) = 0$$

$$\langle \hat{p} \rangle_n = 0$$

$$\langle \hat{x}^2\rangle_n = \frac{\hbar}{2m\omega}\langle n|(\hat{a}+\hat{a}^+)(\hat{a}+\hat{a}^+)|n\rangle = \frac{\hbar}{2m\omega}\langle n|(\hat{a}^+)^2+\hat{a}^2+\hat{a}^+\hat{a}+\underbrace{\hat{a}\hat{a}^+}_{\hat{a}^+\hat{a}+1}|n\rangle$$

$$= \frac{\hbar}{2m\omega}(2n+1) = \boxed{\frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right) = \langle x^2\rangle_n}$$

$\langle \hat{p}^2\rangle_n$

$$H = \frac{1}{2}m\omega^2x^2+\frac{p^2}{2m} \qquad \langle H\rangle_n = \hbar\omega\left(n+\frac{1}{2}\right) = \underbrace{\frac{1}{2}m\omega^2\frac{\hbar}{m\omega}}_{\hbar\omega}\left(n+\frac{1}{2}\right)+\frac{\langle p^2\rangle_n}{2m}$$

$$\frac{1}{2}m\omega^2\langle x^2\rangle_n = \frac{1}{2}\hbar\omega\left(n+\frac{1}{2}\right)$$

$$\frac{\langle p^2\rangle_n}{\cancel{2}m} = \frac{1}{\cancel{2}}\hbar\omega\left(n+\frac{1}{2}\right) \qquad \boxed{\langle p^2\rangle_n = \hbar m\omega\left(n+\frac{1}{2}\right)}$$

$$(\Delta x)_n(\Delta p)_n = \sqrt{\langle x^2\rangle_n}\sqrt{\langle p^2\rangle_n} = \sqrt{\frac{\hbar}{m\omega}*\hbar m\omega}\left(n+\frac{1}{2}\right)$$

$$= \hbar\left(n+\frac{1}{2}\right)$$

$n=0$ $\boxed{(\Delta x)_0(\Delta p)_0 = \frac{\hbar}{2}}$ $\Delta x\,\Delta p \geq \frac{\hbar}{2}$

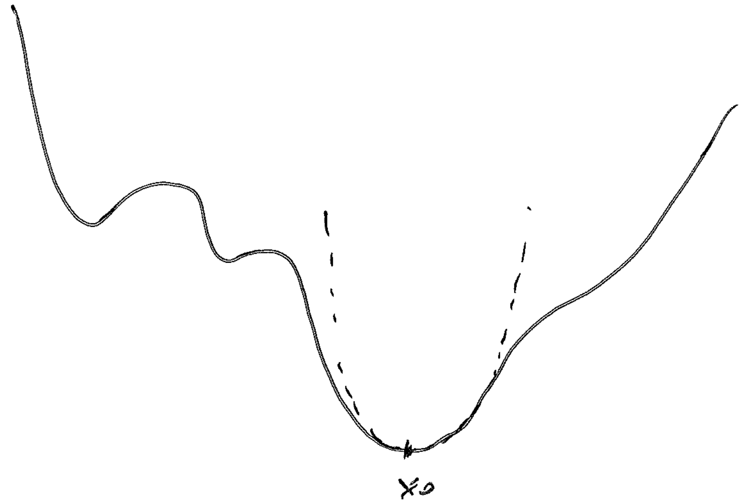
THEORIE DES PERTURBATIONS

$$\hat{\mathcal{H}} = \underset{\substack{\uparrow \\ \text{exactement résolvable}}}{\hat{\mathcal{H}}_0} + \lambda \hat{V}$$

λ paramètre

$[\hat{V}, \hat{\mathcal{H}}_0] \neq 0$

$\lambda \ll 1$ perturbation



$$V(x) \simeq V(x_0) + \frac{V''(x_0)}{2}(x-x_0)^2 + \frac{V'''(x_0)}{6}(x-x_0)^3$$

Hypothèse fondamentale

$$\hat{\mathcal{H}}_0 |\psi_a^{(0)}\rangle = E_a^{(0)} |\psi_a^{(0)}\rangle$$

$\hat{\mathcal{H}}_0$ ↓ Hamiltonien imperturbé

a indice

$$\hat{\mathcal{H}} |\psi_a\rangle = E_a |\psi_a\rangle$$

$$E_a^{(\lambda)} = E_a^{(0)} + \lambda \epsilon_a^{(1)} + \lambda^2 \epsilon_a^{(2)} + \dots$$

$$|\psi_a\rangle = |\psi_a^{(0)}\rangle + \lambda |\phi_a^{(1)}\rangle + \lambda^2 |\phi_a^{(2)}\rangle + \dots$$

$E_a^{(0)}$ valeur propre imperturbée

E_a " " perturbé

$|\psi_a^{(0)}\rangle \Rightarrow$ vecteur propre imperturbé

$|\psi_a\rangle$ " " perturbé

cas non-dégénéré

$$\boxed{\begin{array}{l} |\psi_a^{(0)}\rangle = |E_a^{(0)}\rangle \to |\phi_0\rangle \qquad |\phi_a^{(n)}\rangle \to |\phi_n\rangle \\ \epsilon_a^{(n)} \to \epsilon_n \\ E_a^{(0)} \to \epsilon_0 \end{array}}$$

NOTATION

$$(\hat{\mathcal{H}}_0 + \lambda \hat{V})(|E_a^{(0)}\rangle + \lambda |\phi_1\rangle + \lambda^2 |\phi_2\rangle + \dots)$$

$$= (\epsilon_0 + \lambda \epsilon_1 + \lambda^2 \epsilon_2 + \dots)(|E_a^{(0)}\rangle + \lambda |\phi_1\rangle + \lambda^2 |\phi_2\rangle + \dots)$$

$$a_0 + a_1\lambda + a_2\lambda^2 + \dots = b_0 + b_1\lambda + b_2\lambda^2 + \dots \qquad a_0 = b_0 \qquad a_1 = b_1 \quad \dots$$

$$\hat{H}_0|E_a^{(0)}\rangle = E_a^{(0)}|E_a^{(0)}\rangle$$ ordre zero

$$\hat{H}_0|\phi_1\rangle + \hat{V}|E_a^{(0)}\rangle = \epsilon_0|\phi_1\rangle + \epsilon_1|\underbrace{E_a^{(0)}}_{|\phi_0\rangle}\rangle$$

$$\rightarrow \hat{H}_0|\phi_1\rangle + \hat{V}|\phi_0\rangle = \epsilon_0|\phi_1\rangle + \epsilon_1|\phi_0\rangle$$ ordre 1

$$\rightarrow \hat{H}_0|\phi_2\rangle + \hat{V}|\phi_1\rangle = \epsilon_0|\phi_2\rangle + \epsilon_1|\phi_1\rangle + \epsilon_2|\phi_0\rangle$$ ordre 2

$$\begin{cases} (\hat{H}_0 - \epsilon_0)|\phi_1\rangle + (\hat{V} - \epsilon_1)|\phi_0\rangle = 0 \\ (\hat{H}_0 - \epsilon_0)|\phi_2\rangle + (\hat{V} - \epsilon_1)|\phi_1\rangle - \epsilon_2|\phi_0\rangle = 0 \\ \dots \end{cases}$$

$\epsilon_1, \epsilon_2, \dots$

$|\phi_1\rangle, |\phi_2\rangle \dots$

$E_k^{(0)}$

$$\langle E_k^{(0)}|(\hat{H}_0 - \epsilon_0)|\phi_1\rangle + \langle E_k^{(0)}|(\hat{V} - \epsilon_1)|\phi_0\rangle = 0$$

$$|\phi_0\rangle = |E_a^{(0)}\rangle$$

$$\epsilon_0 = E_a^{(0)}$$

$k = a$

$$\boxed{\langle E_a^{(0)}|\hat{V}|E_a^{(0)}\rangle = \epsilon_1}$$

$k \neq a$

$$(E_k^{(0)} - E_a^{(0)})\langle E_k^{(0)}|\phi_1\rangle + \langle E_k^{(0)}|\hat{V}|E_a^{(0)}\rangle - \epsilon_1 \cancelto{0}{\delta_{ka}} = 0$$

$$\langle E_k^{(0)}|\phi_1\rangle = \frac{\langle E_k^{(0)}|\hat{V}|E_a^{(0)}\rangle}{E_a^{(0)} - E_k^{(0)}}$$

$$|\phi_1\rangle = \sum_{k\neq a} \frac{\langle E_k^{(0)}|\hat{V}|E_a^{(0)}\rangle}{E_a^{(0)} - E_k^{(0)}}\, |E_k^{(0)}\rangle + \underset{\substack{\downarrow \\ 0}}{\overset{i\eta}{\langle E_a^{(0)}|\phi_1\rangle}}\, |E_a^{(0)}\rangle$$

$$\langle E_a|E_a\rangle = 1 = \left(\langle E_a^{(0)}| + \lambda\langle\phi_1| + \lambda^2\langle\phi_2| + \dots\right)\left(|E_a^{(0)}\rangle + \lambda|\phi_1\rangle + \lambda^2|\phi_2\rangle \dots\right)$$

$$= 1 + \lambda\,\underbrace{\left(\langle\phi_1|E_a^{(0)}\rangle + \langle E_a^{(0)}|\phi_1\rangle\right)}_{=0} + O(\lambda^2)$$

$$\langle\phi_1|E_a^{(0)}\rangle = -\langle\phi_1|E_a^{(0)}\rangle^* \qquad \langle E_a^{(0)}|\phi_1\rangle = i\eta \qquad \eta\in\mathbb{R}$$

$$|E_a\rangle \to e^{i\lambda\alpha}|E_a\rangle = \left(1 + i\lambda\alpha - \frac{1}{2}\lambda^2\alpha^2 + \dots\right)\left(|E_a^{(0)}\rangle + \lambda|\phi_1\rangle + \dots\right)$$

$$= |E_a^{(0)}\rangle + i\lambda\alpha|E_a^{(0)}\rangle + \lambda|\phi_1\rangle + \dots$$

$$\langle E_a^{(0)}|e^{i\lambda\alpha}|E_a\rangle = 1 + i\lambda\alpha + \lambda\,\cancel{\sum_{k\neq a}(\quad)}^{\,0} + i\eta\lambda + O(\lambda^2)$$

$$= 1 + i\lambda(\alpha + \eta) + \dots$$

$$\underline{\alpha = -\eta}$$

$$\boxed{|\phi_1\rangle = \sum_{k\neq a} \frac{\langle E_k^{(0)}|\hat{V}|E_a^{(0)}\rangle}{E_a^{(0)} - E_k^{(0)}}\, |E_k^{(0)}\rangle} \qquad \boxed{\epsilon_1 = \langle E_a^{(0)}|\hat{V}|E_a^{(0)}\rangle}$$

$$(\hat{H}_0 - \epsilon_0)\,|\phi_2\rangle + (\hat{V} - \epsilon_1)\,|\phi_1\rangle - \epsilon_2\,|\phi_0\rangle = 0$$

$$\langle E_a^{(0)} |\, \hat{V} \,| \phi_1 \rangle = \epsilon_2$$

$$\boxed{\epsilon_2 = \sum_{k \neq a} \frac{|\langle E_k^{(0)} | \hat{V} | E_a^{(0)} \rangle|^2}{E_a^{(0)} - E_k^{(0)}}}$$