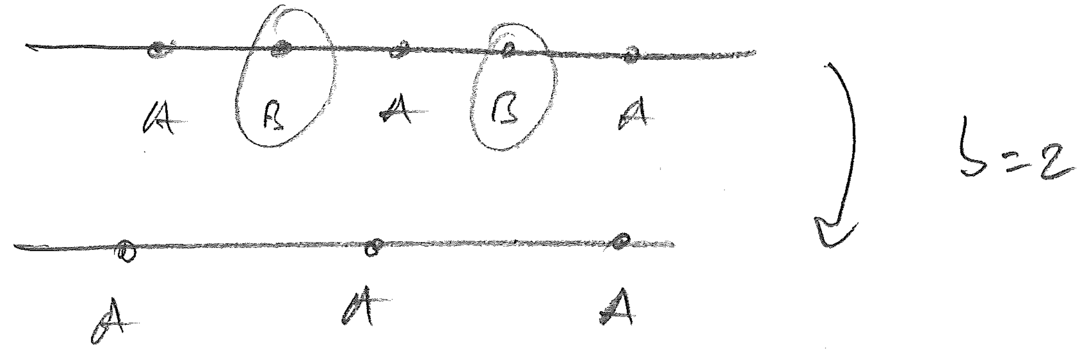
RENORMALIZATION GROUP in REAL SPACE

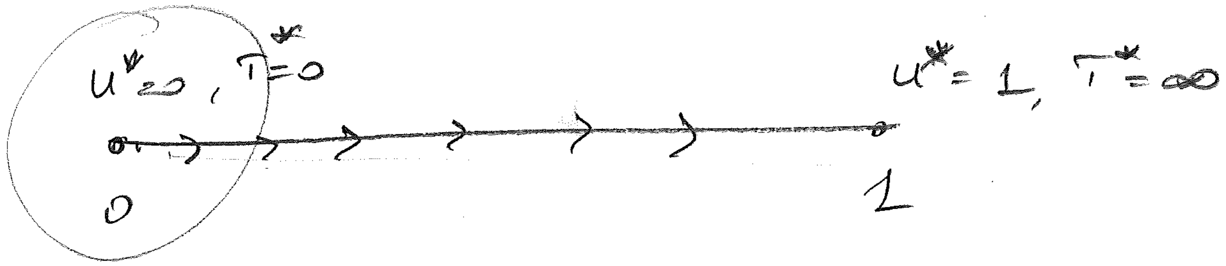
1d Ising model : $\mathcal{H} = -K \sum_{\langle ij \rangle} \sigma_i \sigma_{i+1}$



$\mathcal{H}[K'] = -K' \sum_{\langle ij \rangle} \sigma_i' \sigma_j'$

$K' = \frac{1}{2} \log (\cosh 2K)$ exact RG transformation

$K \in [0, \infty]$ $\quad \underline{u = e^{-2K} = e^{-\frac{2J}{k_BT}}}$ $\quad u' = \frac{2u}{1+u^2}$ $\nearrow u^* = 0 \quad T^* = 0$; $\searrow u^* = 1 \quad T^* = \infty$ FP points



$b=2$

$\underline{u^* = 0}$ $\quad \cancel{u^*} + \delta u \to \cancel{u^*} + \delta u' = \frac{2\delta u}{1+(\delta u)^2} \simeq 2\delta u = b^{y_u} \delta u$

$\underline{y_u = 1 > 0}$

$\xi(T) = \underline{\xi(u) = b\, \xi(u')}$ $\quad \xi(u') = \frac{\xi(u)}{b}$

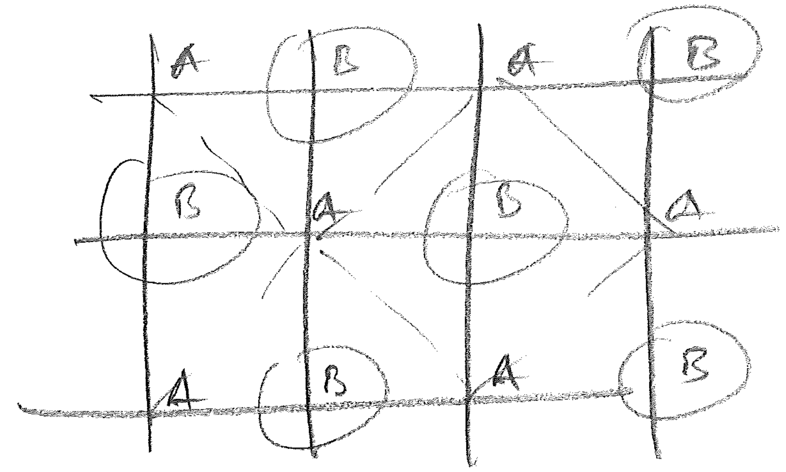
$u \approx u^* = 0$

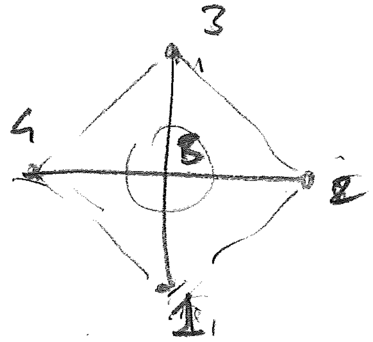
$y_u = 1$

$= b\, \xi(b^{y_u} u)$

$= b\, \xi(bu) = u^{-1} [\underbrace{bu\, \xi(bu)}_{F_\xi(bu)}] \xrightarrow{u \to 0} u^{-1} \overset{\xi_0}{F_\xi(0)} = \underline{\xi_0 e^{2J/k_BT}}$

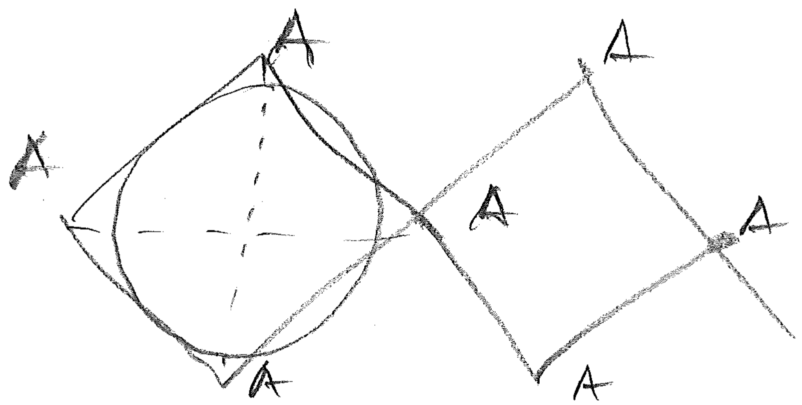
2d Ising model : $\mathcal{H} = -K\sum_{\langle ij\rangle}\sigma_i\sigma_j$





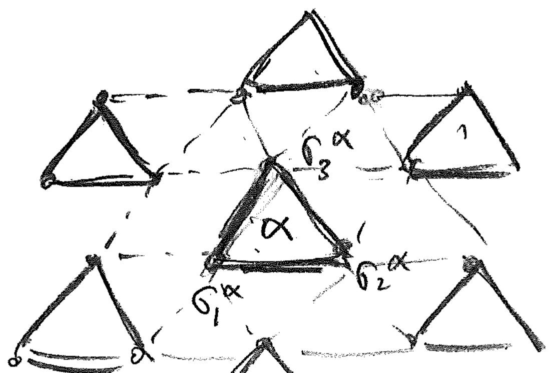
$$\sum_{\sigma_B=\pm1} e^{K\sigma_B(\sigma_1+\sigma_2+\sigma_3+\sigma_4)} = A\, e^{K'(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_4+\sigma_4\sigma_1)}\, e^{K'(\sigma_1\sigma_3+\sigma_2\sigma_4)}\, e^{K_4\sigma_1\sigma_2\sigma_3\sigma_4}$$

symmetric under any exchange of indices 1, 2, 3, 4



"proliferation of coupling constants"

Triangular lattice Ising model : Niemeijer – van Leeuwen approach



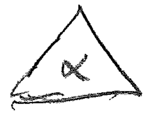
$$\mathcal{H} = -K\sum_{\langle ij\rangle}\sigma_i\sigma_j$$

$$= \underbrace{-K\sum_\Delta\sum_{i,j\in\Delta}\sigma_i\sigma_j}_{\mathcal{H}_0}\ \underbrace{-K\sum_{\langle\Delta\Delta'\rangle}\sum_{i\in\Delta}\sum_{j\in\Delta'}\sigma_i\sigma_j}_{V}$$

Block variable : $\sigma_\alpha = \text{sign}(\sigma_1^\alpha + \sigma_2^\alpha + \sigma_3^\alpha)$ majority rule

$$\boxed{e^{-\mathcal{H}[\sigma_\alpha]} = \sum_{\substack{\{\sigma_i\}\text{ compatible}\\ \text{with } \{\sigma_\alpha\}}} e^{-(\mathcal{H}_0[\sigma_i] + \mathcal{V}[\sigma_i])}}$$

if $\mathcal{V}=0$



$$\mathcal{H}_0[\sigma_1^\alpha, \sigma_2^\alpha, \sigma_3^\alpha] = -K(\sigma_1^\alpha\sigma_2^\alpha + \sigma_1^\alpha\sigma_3^\alpha + \sigma_2^\alpha\sigma_3^\alpha) = \begin{cases} -3K & \overbrace{\uparrow\uparrow\uparrow,\ \downarrow\downarrow\downarrow}^{2} \\ +K & \underbrace{\uparrow\uparrow\downarrow,\ \downarrow\uparrow\uparrow, \text{ and permutations}}_{6} \end{cases}$$

$$e^{-\mathcal{H}[\sigma_\alpha]} = \begin{cases} e^{3K} + 3e^{-K} & \sigma_\alpha = 1 \quad \uparrow\uparrow\uparrow,\ \uparrow\uparrow\downarrow,\ \uparrow\downarrow\uparrow,\ \downarrow\uparrow\uparrow \\ e^{3K} + 3e^{-K} & \sigma_\alpha = -1 \end{cases} = \text{const}$$

Cumulant expansion

$$\mathcal{H}[\sigma_\alpha] = -\log\left[\sum_{\substack{\{\sigma_i\}\text{ comp. with}\\ \{\sigma_\alpha\}}} e^{-(\mathcal{H}_0[\sigma_i] + \mathcal{V}[\sigma_i])}\right]$$

$$= -\log\left[\underbrace{\frac{1}{Z_0}\left(\sum e^{-(\mathcal{H}_0+\mathcal{V})}\right)}\right] + \log Z_0$$

$$Z_0^{[\sigma_\alpha]} = \sum_{\substack{\{\sigma_i\}\text{ comp.}\\ \text{with } \{\sigma^\alpha\}}} e^{-\mathcal{H}_0[\sigma_i]} = \left(e^{3K} + 3e^{-K}\right)^{N/3}$$

$$\langle e^{-\mathcal{V}[\sigma_i]}\rangle_0 = \frac{1}{Z_0} \sum_{\substack{\{\sigma_i\}\text{ comp. with}\\ \{\sigma^\alpha\}}} e^{-\mathcal{V}}\, e^{-\mathcal{H}_0}$$

$$= -\log\langle e^{-\mathcal{V}[\sigma_i]}\rangle_0 + \log Z_0$$

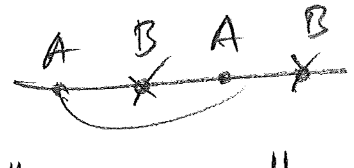
$$\log \langle e^{+\lambda A} \rangle_0 = \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} c_n \qquad \boxed{c_n = \frac{\partial^n \log \langle e^{+\lambda A} \rangle_0}{\partial \lambda^n}\Big|_{\lambda=0}} \quad \text{n-th cumulant}$$

$$c_1 = \langle A \rangle_0 \qquad c_2 = \langle A^2 \rangle_0 - \langle A \rangle_0^2 \quad \dots$$

$$A = -\mathcal{V}$$

$$\mathcal{H}[\sigma_\alpha] = \underbrace{+\log Z_0}_{\mathcal{O}(\mathcal{V}^0)} + \langle \mathcal{V} \rangle_0 + \frac{1}{2}\left[\langle \mathcal{V}^2 \rangle_0 - \langle \mathcal{V} \rangle_0^2\right] + \dots$$

Renormalization group program



1) define geometrically an RG transformation ("decimation", block variables, ...)
2) write down the transformation $\vec{K}' = R[\vec{K}]$
3) search for the fixed points
4) linearize R around a critical fixed point $\Rightarrow$ y_σ scaling dimensions

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Field theoretical approach to RG : momentum-space RG

Real-space RG : Kadanoff (1966)
⋮

Momentum-space RG : Kenneth Wilson (1970) & co.

Ginzburg - Landau functional

$$\Phi[\vec{\phi}(\vec{r})] = \int d^d r \left[\frac{1}{2}|\vec{\nabla}\vec{\phi}|^2 + \frac{r_0}{2}|\vec{\phi}|^2 + \frac{u_0}{4}|\vec{\phi}|^4 \right]$$

$$[\Phi] = 1$$

$$Z = \int \mathcal{D}[\vec{\phi}]\, e^{-\Phi}$$

$$\vec{\phi}(\vec{r}) = \frac{1}{V} \sum_{|\vec{q}| \le \Lambda} e^{i\vec{q}\cdot\vec{r}}\, \vec{\phi}(\vec{q})$$

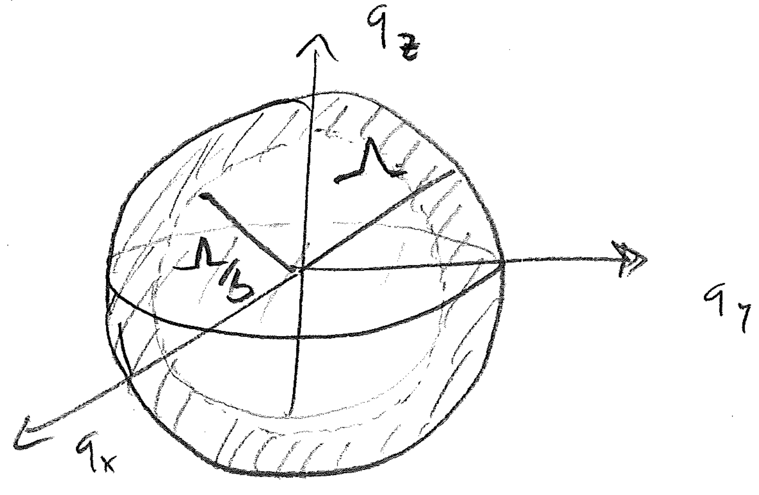
$$\Lambda \sim \frac{\pi}{a}$$

$V \to \infty$ $\quad \frac{1}{V}\sum_{\vec{q}} \to \boxed{\int_{|\vec{q}| \le \Lambda} \frac{d^d q}{(2\pi)^d} = \int_{\vec{q}}}$

$$\boxed{\begin{aligned} \Phi[\vec{\phi}(\vec{q})] = {} & \int_{\vec{q}} \frac{1}{2}(r_0 + \overbrace{q^2}^{|\vec{q}|^2})|\vec{\phi}(\vec{q})|^2 \\ & + \frac{u_0}{4}\int_{\vec{q}_1}\int_{\vec{q}_2}\int_{\vec{q}_3}\int_{\vec{q}_4} (2\pi)^d\, \delta(\vec{q}_1+\vec{q}_2+\vec{q}_3+\vec{q}_4) \\ & \quad \left(\vec{\phi}(\vec{q}_1)\cdot\vec{\phi}(\vec{q}_2)\right)\left(\vec{\phi}(\vec{q}_3)\cdot\vec{\phi}(\vec{q}_4)\right) \end{aligned}}$$

$\vec{\phi}(\vec{q}) \in \mathbb{C}$

$\vec{\phi}(\vec{q}) = \vec{\phi}^*(-\vec{q})$



$\Lambda \quad , \quad \Lambda/b$ $\qquad$ $\underline{b > 1}$

1) $\Lambda/b \le |\vec{q}| \le \Lambda$ $\quad$ fast components

$$\vec{\phi}(\vec{q}) \to \vec{\psi}(\vec{q})$$

2) $0 \le |\vec{q}| \le \Lambda/b$ $\quad$ slow components $\quad \vec{\phi}(\vec{q})$

$$\vec{\phi}(\vec{r}) = \underbrace{\int_{0\le|\vec{q}|\le\Lambda/b}}_{\text{slow}} e^{i\vec{q}\cdot\vec{r}}\,\vec{\phi}(\vec{q}) + \int_{\frac{\Lambda}{b}\le|\vec{q}|\le\Lambda} e^{i\vec{q}\cdot\vec{r}}\,\vec{\psi}(\vec{q})$$

Gaussian model $\quad u_0 = 0$

$$\Phi[\vec{\phi}(\vec{q}),\vec{\psi}(\vec{q})] = \underbrace{\int_{0\le|\vec{q}|\le\Lambda/b} \frac{1}{2}(r_0+q^2)\,|\vec{\phi}(\vec{q})|^2}_{\Phi_\phi} + \underbrace{\int_{\Lambda/b\le|\vec{q}|\le\Lambda} \frac{1}{2}(r_0+q^2)\,|\vec{\psi}(\vec{q})|^2}_{\Phi_\psi}$$

$$Z = Z_\phi\, Z_\psi = \underbrace{\int_{0\le|\vec{q}|\le\Lambda/b} \prod_{\alpha,\vec{q}} d\phi_\alpha(\vec{q})\, d\phi^*_\alpha(\vec{q})\, e^{-\Phi_\phi}}_{Z_\phi} \left(\underbrace{\int_{\Lambda/b\le|\vec{q}|\le\Lambda} \prod_{\alpha,\vec{q}} d\psi_\alpha(\vec{q})\, d\psi_\alpha(\vec{q}^*)\, e^{-\Phi_\psi}}_{Z_\psi}\right)$$

$$\Phi_\phi = \int_{0\le|\vec{q}|\le\Lambda/b} \frac{1}{2}(r_0+q^2)\,|\vec{\phi}(\vec{q})|^2 = \int_{0\le|\vec{q}'|\le\Lambda} \frac{d^dq'}{(2\pi)^d}\,\frac{1}{b^d\,b^2}\,\frac{1}{2}\left[r_0 b^2 + (q')^2\right]|\vec{\phi}(\vec{q})|^2 =$$

$$\left[\begin{array}{llll} \ell_0 \to & \underline{\ell_0 \times b = \ell_0'} & & q \to \underline{q' = bq} \\ \text{unit of length} & & & \\ q_0 \to & q_0' = q_0/b & & 0\le q\le\Lambda/b \quad q' \to \quad 0\le q'\le\Lambda \\ \text{unit of wavevector} & & & \end{array}\right]$$

$$= \int_{0 \le |\vec{q}'| \le \Lambda} \frac{d^d q'}{(2\pi)^d} \; \frac{1}{2} \left[r_0 b^2 + (q')^2 \right] \left| \frac{1}{b^{1+d/2}} \vec{\phi}(\vec{q}) \right|^2 =$$

$\Rightarrow$ same form as the original Gaussian model

$$\begin{cases} r_0' = b^2 r_0 \\ \phi'(\vec{q}') = \dfrac{\phi(\vec{q})}{b^{1+d/2}} \end{cases}$$

"wavefunction renormalization"

Dimensional analysis ("power counting")

$[r_0] = \ell^{-2}$

$[\phi(\vec{r})] = \ell^{1-d/2}$ $\qquad [\phi(\vec{q})] = \ell^{1+d/2}$

$r_0 \to r_0' = b^2 r_0$ $\qquad \phi(\vec{q}) \to \phi' = b^{-(1+d/2)} \phi(\vec{q})$

measured in units of $b\ell_0$

$[u_0] = \ell^{d-4}$ $\qquad u_0' = b^{4-d} u_0$

$r_0' = b^2 r_0$ RG transformation for the Gaussian model

fixed point $r_0^* = \infty$ $\qquad r_0^* = 0$ critical fixed point $\qquad r_0 \sim T - T_c \sim t = \dfrac{T-T_c}{T_c}$

$t' = b^{y_t}\, t$ $\quad \underline{y_t = 2} \quad \Rightarrow \quad \boxed{\nu = \frac{1}{y_t} = \frac{1}{2}}$ ✓

Add a field $\quad \Phi - \underbrace{\int d^d r\, \vec{\phi}(\vec{r}) \cdot \vec{h}} = \Phi - \vec{h} \cdot \vec{\phi}(\vec{q}=0)$

$= \Phi - \underbrace{\vec{h}\, b^{+(1+\frac{d}{2})}}_{\vec{h}'}\, \vec{\phi}'(\vec{q}'=0)$

$h' = b^{1+d/2}\, h$

$y_h = 1 + d/2$

$\Delta = 2 - \alpha - \beta = y_h / y_t = \frac{1}{2}\left(1 + d/2\right)$ $\quad \underline{2 - \alpha = \nu d}$ $\quad$ assume hyperscaling ?!

$\boxed{\beta = \frac{d-2}{4}}$ ✗ $\quad \left(\beta = 1/2\right)$

$\alpha + 2\beta + \gamma = 2$ $\quad \boxed{\gamma = 1}$ ✓ $\quad \gamma_\nu = (2 - \eta)$ $\quad \boxed{\eta = 0}$ ✓

$\beta\delta = \Delta = \beta + \gamma$ $\quad \boxed{\delta = \frac{d+2}{d-2}}$ ✗ $\quad \left(\delta = 1/3\right)$

RG picture for the Gaussian model

critical fixed point $r_0^* = 0$ $t^* = 0$

$(h^* = 0)$

$u_0 = 0$ $u_0^* = 0$

one direction of instability : r, t $\underline{y_t = 2 > 0}$

u_0 : $u_0' = b^{4-d} u_0$ $y_u = 4-d < 0$ $\underline{d > d_c = 4}$

stable direction / irrelevant perturbation in $\underline{d>4}$

(Ginzburg criterion)

$$f_s = \overbrace{f_s(t, h, u_0)}^{\vec{\mathcal{K}}}$$

$$= |t|^{\nu d} F\left(\frac{h}{|t|^{y_h/y_t}}, \frac{u_0}{|t|^{y_u/y_t}}\right)$$

$$m \sim \frac{\partial f_s}{\partial h} \simeq |t|^{\nu d - y_h/y_t} F_m\left(\frac{h}{|t|^{y_t/y_u}}, \underbrace{\frac{u_0}{|t|^{y_u/y_t}}}\right)$$

$$\sim \left(\frac{u_0}{|t|^{y_u/y_t}}\right)^{-\frac{1}{2}}$$

$u_0 \to 0 = u_0^*$

$$\beta = \nu d - y_h/y_t = \frac{d-2}{4}$$

$$m \sim \sqrt{\frac{a(T-T_c)}{u_0}}$$

$$m \sim |t|^{\nu d - y_h/y_t + y_u/2y_t} \, u_0^{-1/2} \, G\left(\frac{h}{|t|^{y_h/y_t}}\right)$$

$$\downarrow$$

$$\underline{\beta = 1/2}$$

$$m \underset{T=T_c}{\sim} \left(\frac{h}{4u_0}\right)^{1/3} \sim \underline{\underline{u_0^{-1/3}}}$$