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Phase Transitions and Critical Phenomena

The Superfluid Transition in Liquid Helium 4

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Abstract

At $T_\lambda = 2.17$K, helium 4 undergoes a phase transition from quantum origin. This transition is signed by a specific heat singularity called the lambda point. The existing phase above T_λ is named **He I** or *normal fluid helium*, whereas helium under this point is called **He II** or *superfluid helium*. Superfluid helium exhibits some interesting properties such as an apparent absence of resistance to flow, which makes it worthy of studying theoretically and experimentally. As for today, He I and He II lack a precise microscopic description. However, phenomenology has yielded a satisfying description of the transition, with good agreement with experiments. In this essay I will first describe helium II through an experimental approach, before exploring the usual models for helium II such as Bose-Einstein Condensation, the elementary excitations and the two-fluid model. In a third time I will present the phenomenology of the transition, and finally exhibit some of the consequences of this phase transition.

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1 Experimental approach of the properties of helium II

1.1 Phase diagrams of helium

(FIG)

There are only two stable isotopes of helium: helium 3 and helium 4. Those two isotopes have the lowest boiling points among all known substances : $T_b = 4.21$K for helium 4, $T_b = 3.19$K for helium 3. Another particularity is that the solid phases of both isotopes are only stable at pressure above 30bar, even at low temperatures ; thus the liquid phases are the only stable phases down to absolute zero for usual pressure conditions.

Under the boiling point, liquid helium can go through a new phase transition. This transition occurs at $T_\lambda = 2.17$K for helium 4 and is characterized by a singularity of the specific heat. Helium 4 above T_λ is named **He I** or *normal liquid helium*, whereas helium under this point is called **He II** or *superfluid helium*. Helium II can be observed experimentally by cooling normal liquid using a simple evaporating technique : a pump lowers the pressure of a helium I bath at equilibrium with its vapor. As the pressure goes lower and lower the system at equilibrium follows the liquid-gas coexistence line on the phase diagram, thus lowering the temperature of the gas and liquid. The transition can be observed either by measuring the specific heat anomaly or simply by looking at the liquid : helium I is in permanent ebullition as the gas is pumped out, but the ebullition stops as soon as T_λ is reached thanks to the particular heat transfer mechanism of helium II, which will be described in a later paragraph.

Helium 3 undergoes a similar transition, but for a much lower lambda point : $T_\lambda = 2.49$mK for helium 3. Such temperatures cannot be reached using evaporating techniques : it requires more advanced technology such as optical or magnetic traps. Thus the superfluidity of helium 3 was only discovered in the 1970s by Lee, Osheroff and Richardson, whereas Kapitsa discovered superfluid helium 4 in 1937 [3]. For this reason and others that will be depicted in the next paragraphs, this essay will focus on superfluid helium 4 only.

1.2 Experimental observations

1.2.1 Viscosity

Gas and liquid helium are highly interesting fluids for hydrodynamics experiments and especially turbulence, as they present a strong dependence of viscosity with temperature. As a consequence, helium is frequently used when an experiment needs a wide range of controllable viscosity. When the techniques for obtaining helium II were fully developed, hydrodynamics and turbulence experiments followed naturally.

Two types of experiments were built up for the measurement of helium II viscosity : via measure of its resistance to flow (velocity measurement) or via measure of the drag exerted on a object (effort measurement). Those two types of experiment led to apparently contradictory results, and to the elaboration of the most used models for the description of helium II.

Resistance to flow is quantified by forcing helium II to flow through small capillaries. This was achieved as soon as 1939 by Allen & Misener [5] by emptying a vessel of helium II through a porous material and measuring the liquid's velocity. Results showed no viscous resistance to flow for small enough capillaries, suggesting a zero viscosity of helium II.

The other type of experiment consists in forcing an object to move in superfluid helium and measuring the drag that it suffers from the liquid. In 1938 Keesom and MacWood [6] measured the viscous drag exerted by helium II on rotating discs : they obtained a small but finite value for the superfluid viscosity, comparable to that of helium I just under the boiling point. This result being in contradiction with the one from flow resistance measurement, it led to the hypothesis that helium II is actually composed of two inseparable liquid phases : the *superfluid* phase of zero viscosity and the *normal fluid* phase of finite viscosity. This assumption is the start of the two-fluid model which will be discussed later on.

1.2.2 Thermal properties

As mentioned previously, the superfluid transition is marked by an anomaly of the fluid's specific heat called the lambda point. This anomaly leads to very high values of heat capacity for helium II under the lambda point : more than $5 \text{kJ}\cdot\text{mol}^{-1}\cdot\text{K}^{-1}$ close to the transition.

High values of thermal conductivity are also observed and can be interpreted thanks to the two-fluid model. The model predicts a new heat transport mechanism for helium II : heat is transported by convection of the normal fluid phase, with superfluid current in the opposite direction. Such convection is called a *counter-flow* and leads to the propagation of temperature in a wave-like manner called *second sound* rather than the usual diffusive or convective mechanism. This new mechanism makes the thermal conductivity of helium II unusually high ; it is described in 4.

2 Models of superfluid

This section presents the main models of superfluid used to describe helium II. In 1938 London [7] proposed the application of the Bose-Einstein Condensation theory to helium II. In 1941 Landau [8] proposed a dispersion curve for the excitations in helium II which led to the phonon-roton model. From those two models can be derived the two-fluid model, yet they have been perceived as rivals until 1947 when Bogoliubov's theory of the weakly interacting Bose gas which proved them to be compatible.

2.1 Bose-Einstein Condensation

The theory of Bose-Einstein condensation describes the behaviour of an ideal gas of bosons at small temperatures. Under some transition temperature T_b the fundamental energy level becomes significantly populated, because of bosons ability to occupy a same quantum state. The gas can the be described as the union of two phases : the condensate that is composed of all particles in the fundamental state, and the remaining particles that are still in excited levels of energy. As the temperature gets smaller and smaller, the fundamental level is increasingly occupied and the condensate proportion grows. At absolute zero, the whole gas is in the condensate. As the condensate is contained in one single-particle state, it is reasonable to assume that it can be described by a single wave function that shall be called the *macroscopic wave function*. The existence of such a wave function suggests a certain order in the condensate phase : that is why Bose-Einstein condensation is considered as an order-disorder phase transition.

Let us recall that it can be assumed that helium II is actually constituted of a superfluid phase with no viscosity and a normal viscous fluid phase. The superfluid phase only appears under T_λ and its proportion grows as T decreases. These observations make it hard not to think of the superfluid phase transition as an example of Bose-Einstein condensation. Indeed it should be notice that atoms of helium 4 are composite bosons, thus should be able to go through a Bose-Einstein condensation. This was the first idea for modeling helium 4 by London and Tisza in 1938. It identifies the superfluid phase with a Bose condensate and the normal phase with particle in excited states.

It can then be assumed than the superfluid phase can be described with a macroscopic wave function $\psi(\mathbf{x}, t)$, and that this wave function obeys a Schrödinger equation:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V(\mathbf{x})\psi \tag{1}$$

Under the Madelung transformation : $\psi = \psi_0\, \mathrm{e}^{i\phi}$ with ψ_0 and ϕ reals, the real and imaginary part of this equation lead respectively to a Bernoulli equation and a mass conservation equation :

$$\frac{\hbar}{m}\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(\nabla\phi\right)^2 + \frac{V}{m} - \frac{\hbar^2}{2m}\frac{\Delta\psi_0}{\psi_0} = 0 \tag{2}$$

$$\frac{\partial}{\partial t}\psi_0^2 + \frac{\hbar}{m}\nabla\left(\nabla\phi\right) = 0 \tag{3}$$

for an ideal irrotational fluid of density ψ_0^2, which current derives from a potential : $\mathbf{v} = \frac{\hbar}{m}\nabla\phi$. The condensate phase can thus be described a non viscous fluid, which current is created by the gradient in the phase of the macroscopic wave function. The movement of condensate particles issuing from the phase of one single wave function strengthen the order-disorder picture for the superfluid phase transition of helium 4.

Let's note that this interpretation in terms of a Bose-Einstein condensation is possible because helium 4 is a composite boson. Helium 3 on the other hand is a composite fermion, so it should not allow such an interpretation. Actually the superfluid transition in helium 3 can be interpreted in a Bose-Einstein condensation with a pairing mechanism like Cooper pairs in superconductors. That is a completely different mechanism form the one described previously, and this is an other reason for this essay to focus on helium 4 only.

2.2 The phonon-roton model

The Bose-Einstein condensation picture was first criticized by Landau because of its assumptions. Indeed, it lays on the model of the ideal gas, that is to say without interactions : this picture certainly isn't compatible with the liquid nature of helium II, in which interactions play an important role. For Landau, those interactions are essential to the superfluidity of helium 4, as they allow collective excitations : Landau proved that superfluidity requires thermal excitations with a particular shape of dispersion curve [8]. His proposition for the dispersion relation is what we call the *phonon-roton* model.

First, it is clear that the dispersion relation should present a phonon branch as longitudinal phonon waves are allowed to propagate in liquids. Landau then added another prescription : the dispersion curve should exhibit a local energy minimum for a non-zero wave-number value. Those well-defined excitations are called rotons and are a necessary condition to superfluidity. This dispersion relation has been widely confirmed by experiments such as neutron diffusion in helium II. The roton minimum has been precisely measured. The accepted values for the energy and wave number of this minimum are $\epsilon_0/k_B = 8.65\text{K}, k_0 = 19.1\text{nm}^{-1}$.

So as to make this point, let's imagine a solid body moving in a helium II bath. The object suffers a viscous drag from the liquid if and only if it is able to exchange momentum and energy with the bath. This energy-momentum exchange is performed by excitation emission in the fluid and thus is only allowed for excitations verifying the dispersion relation. Let's write the momentum and energy conservation laws for the interaction:

$$\frac{1}{2}MV^2 = \frac{1}{2}MV'^2 + \epsilon \tag{4}$$

$$M\mathbf{V} = M\mathbf{V}' + \hbar\mathbf{k} \tag{5}$$

with M the mass of the object, $\mathbf{V}$ and $\mathbf{V}'$ the original and final speed of the object respectively, and ϵ and $\mathbf{k}$ the energy and wave-number of the excitation respectively, which have to verify the dispersion relation to exist. The final speed can be eliminated thanks to the second equation, and then the first one gives :

$$\frac{1}{2}MV^2 = \frac{1}{2}M\left(\mathbf{V} - \frac{\hbar}{M}\mathbf{k}\right) + \epsilon \tag{6}$$

which gives, for $\hbar k \ll MV$:

$$V \geq \frac{\epsilon}{\hbar k} \tag{7}$$

which is the condition for the emission of an excitation in the liquid, and therefore for a drag to exist. As a consequence, if the object velocity is smaller than the minimum value of $\frac{\epsilon}{\hbar k}$ allowed by the dispersion relation, it doesn't suffer any viscous drag : the liquid is therefore superfluid. It can be easily proven that the existence of such a minimum of $\frac{\epsilon}{\hbar k}$ is equivalent to the existence of a roton minimum. The accepted values of the roton minimum give $59.5\text{m}\cdot\text{s}^{-1}$ for the upper velocity bound of the superfluid domain. At those speeds though, this model may not hold, and interactions between excitations should be held into account.

3 The lambda point transition

The superfluid transition stills lack a precise microscopic description : although the Bose-Einstein condensation model is promising, especially in the Bogolyubov interacting picture, it still fails to describe the properties of the transition. The heat capacity anomaly is not well-described by the Bose-Einstein model for example. This section first presents the successes and fails of the statistical mechanics of Bose-Einstein condensation applied to helium II, and then exhibits the phenomenology and measured of the lambda transition.

3.1 Success and fail of the Bose-Einstein picture

So as to calculate characteristics of the lambda transition, let's start from the Bose-Einstein statistics. A state of energy ϵ_i is, at temperature T, populated in average by $< n(\epsilon_i, T) >$ particles. The Bose-Einstein distribution function gives:

$$< n(\epsilon_i, T) >= \frac{1}{\exp[(\epsilon_i - \mu)/k_B T] - 1} \tag{8}$$

$$\sum_i < n(\epsilon_i, T) >= N \tag{9}$$

For a system of N particles in a large volume V, we can treat the energy levels as a continuum and rewrite the normalization relation:

$$N =< n(0, T) > + \int_0^\infty \mathcal{D}(\epsilon) < n(\epsilon, T) > \mathrm{d}\epsilon \tag{10}$$

with $\mathcal{D}(\epsilon) = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \epsilon^{1/2}$ the density of states of the system. The number of particles in the condensate is given by $N_0(T) =< n(0, T) >$, the integral $N'(T)$ gives the number of particles in excited states. This integral can be calculated numerically [3]. An upper bound $N'_m(T)$ is given for $\mu = 0$:

$$N'_m(T) = 2.612V \left(\frac{mk_B T}{2\pi\hbar^2}\right)^{3/2} \tag{11}$$

At the transition point $T = T_\lambda$, $N'_m(T_\lambda) = N$, which gives an expression of T_λ:

$$T_\lambda = \frac{2\pi\hbar^2}{mk_B T} \left(\frac{N}{2.612V}\right)^{2/3} \tag{12}$$

The numerical calculation gives $T_\lambda = 3.1$K, which is qualitatively close to the experimental value $T_\lambda = 2.17$K. The combination of equations (11) and (12) gives the temperature dependance of $N'(T)$:

$$\frac{N'(T)}{N} \propto \left(\frac{T}{T_\lambda}\right)^{3/2} \tag{13}$$

which gives $N_0(T) \propto 1 - \left(\frac{T}{T_\lambda}\right)^{3/2}$, which gives for small values of $T - T_\lambda$: $N_0(T) \propto (T_\lambda - T)$. However, the temperature dependance observed experimentally is $N_0(T) \propto (T_\lambda - T)^{2/3}$ with a good precision : this is an important difference that marks the first fail of the Bose-Einstein model at depicting the lambda transition. An other point that this model fails to describe properly is the temperature dependance of the heat capacity. The Bose-Einstein model does not describe the lambda anomaly but rather a cusp singularity [3].

These fails of the Bose-Einstein picture show that it isn't a satisfying microscopic description. Here we only considered the ideal Bose gas, but even the Bogolyubov interacting model fails to describe the lambda transition. As for today there is no satisfying microscopic description for the lambda transition, which would allow calculations of its critical exponents.

3.2 Phenomenology and measures of the lambda transition

The main phenomenological model for studying the lambda transition is the Ginzburg-Landau picture [4]Tilley. In this model we consider an order parameter ψ describing the transition. As seen previously, observations are in favor of describing the lambda transition as an order-disorder transition. Because of the quantum nature of the problem, the order parameter ψ is assumed to be complex : we'll see that it plays a role similar to that of the macroscopic wave function in the Bose-Einstein picture.

In the Ginzburg-Landau model it is assumed that in a region where ψ is small, the free energy density can be developed in even powers of the order parameter and its gradient:

$$f(\mathbf{x}) = f_1 + \alpha|\psi(\mathbf{x})|^2 + \frac{1}{2}\beta|\psi(\mathbf{x})|^4 + \frac{\hbar^2}{2m}|\nabla\psi(\mathbf{x})|^2 \tag{14}$$

with f_1 the free energy density of helium I. So as to have an ordered phase for $T < T_\lambda$, the simplest expression of α and β are $\alpha = A(T - T_\lambda)$ with A positive and β a constant. To know the stable state of the system for a given temperature, we have to minimize the total free energy :

$$F = \int f(x)\mathrm{d}x \tag{15}$$

which is a functional of the order parameter ψ. Its minimization with respect to ψ leads to Euler-Lagrange equations for the Lagrangian density $f(\mathbf{x})$. Those are the following :

$$0 = -\frac{\hbar^2}{2m}\Delta\psi + \alpha(T)\psi + \beta|\psi|^2\psi \tag{16}$$

which is a Ginzburg-Landau equation. Solving this equation would lead to the shape of the order parameter $\psi(\mathbf{x})$ in the stable state at a given temperature. One can notice the resemblance of this equation with a Schrödinger equation. It seems like we obtain a stationary Schrödinger equation with a linear potential term $\alpha\psi$ and a non-linear potential term $|\psi|^2\psi$. The linear potential $\alpha(T)$ only depends on temperature : it is a control parameter that will give the stable state for temperature T. The nonlinear term can be interpreted by identifying the order parameter with the macroscopic wave function of the Bose-Einstein condensate. In fact in the Bogolyubov interacting picture, the Hamiltonian of the Bose gas contains a pair potential term that describes the interaction. By choosing a local interaction, that is to say a pair potential that is a Dirac distribution, the equation of motion that derives from such a Hamiltonian is precisely equation (16). Such an equation is called in the instationnary case the local Gross-Pitaevskii equation [1].

In 1958 Pitaevskii added a kinetic energy term to the Ginzburg-Landau free energy density and derived equations of motion for a two-fluid model close to the lambda point. In 1969 Khalatnikov used the minimization of Pitaevskii's total free energy to describe the attenuation of first and second sound close to the lambda point. He obtained temperature dependances of $(T_\lambda - T)^{-1}$ and $(T_\lambda - T)^{-1/3}$ respectively, with good agreement with experiments [3].

4 Application to heat transport

As presented previously, the two fluid model predicts a new mechanism for heat transfer called second sound. This section aims at describing this phenomenon.

To explain second sound let's start with simplified two-fluid equations. The superfluid phase motion is driven by gradients of the chemical potential :

$$\frac{\partial \mathbf{v_s}}{\partial t} + (\mathbf{v_s} \cdot \nabla) \, \mathbf{v_s} = -\nabla \mu \tag{17}$$

The entropy density s is only transported by the normal phase, because of its thermal excitation nature:

$$\frac{\partial}{\partial t}(\rho s) + \nabla \cdot (\rho s \mathbf{v_n}) = 0 \tag{18}$$

The total mass current of helium II is the sum of currents of both phases :

$$\mathbf{j} = \rho_s \mathbf{v_s} + \rho_n \mathbf{v_n} \tag{19}$$

First, let's rewrite the gradient of chemical potential in terms of the temperature. Thermodynamical identities give $\mathrm{d}\mu = -s\mathrm{d}T + \frac{1}{\rho}\mathrm{d}P$. Assuming there is no pressure gradient we obtain $\nabla \mu = -s\nabla T$. Equation (17) becomes:

$$\frac{\partial \mathbf{v_s}}{\partial t} + (\mathbf{v_s} \cdot \nabla) \, \mathbf{v_s} = s\nabla T \tag{20}$$

Now let's take the system in a uniform equilibrium state ($T = T_0, \mathbf{v_s} = \mathbf{0}, s = s_0$) and make a small perturbation ($T_1, \mathbf{v_{s1}}, s = s_1$) around that equilibrium state. Assuming the absence of density fluctuations $\partial_t \rho = 0$, we can linearize the previous equations and yield :

$$\frac{\partial \mathbf{v_{s1}}}{\partial t} = s_0 \nabla T \tag{21}$$

$$\frac{\partial s_1}{\partial t} - s_0 \nabla \cdot (\mathbf{v_n}) \tag{22}$$

Assuming there is no local mass transport, we can put $\mathbf{j}$ to zero in equation (19). The normal velocity can the be eliminated from the last equation. In addition, the time variation of s_1 can be rewrote :

$$\frac{\partial s_1}{\partial t} = \frac{\partial T_1}{\partial t}\frac{\partial s_1}{\partial T_1} \approx \frac{c_p}{T_0}\frac{\partial T_1}{\partial t} \tag{23}$$

which finally gives :

$$\frac{\partial \mathbf{v_{s1}}}{\partial t} = s_0 \nabla T \tag{24}$$

$$\frac{c_p}{T_0}\frac{\partial T_1}{\partial t} - \frac{s_0 \rho_s}{\rho_n} \nabla \cdot (\mathbf{v_s}) \tag{25}$$

Those two last equations can be combined into :

$$\frac{\partial^2 T_1}{\partial t^2} - \frac{\rho_s}{\rho_n}\frac{s_0^2 T_0}{c_p} \Delta T_1 = 0 \tag{26}$$

which is a wave equation for T_1. Hence temperature propagates in a wave-like manner in helium II according to the two-fluid model. This mechanism is called second sound and its velocity is $c_2 = \sqrt{\frac{\rho_s}{\rho_n}\frac{s_0^2 T_0}{c_p}}$. Second sound as been widely observed in helium II and is very useful for the measurement of critical exponents. The critical exponents for superfluid density and heat capacity have been precisely measured by second sound experiments. As for today, the most precise measurement of the heat capacity critical exponent as been achieved for the lambda transition in a space shuttle so as to minimize pressure differences in the sample [10]Ferell2Tilley

Conclusion

The superfluid transition in helium 4 was discovered more than seventy years ago, yet it still imposes challenges on today's physicists. A satisfying microscopic description is still to be established, so as to justify theoretically the two-fluid model and to describe the transition more precisely. It is the bosonic nature of the problem that makes it so difficult to have this microscopic description, contrary to superconductivity or helium 3 superfluidity which are fermionic problems for which the microscopic description is well known. However, phenomenology has been proven to still be efficient for predicting critical exponents of the lambda transition with great agreement with experiments. Helium II also have a lot of other physics challenges that could not be mentioned in this essay, such as quantum turbulence.

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