Disorder-Induced Metal-Insulator Transition: The Anderson Transition

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Abstract

We start by understanding the phenomenon of Anderson localization through qualitative arguments and a simple one-dimensional model. We then focus on the transition between a metallic phase and a localized, insulating phase. The basic properties of the transition are explored, in analogy with the thermal ferromagnetic transition. Then, we sketch a scaling theory which allows us to derive a scaling relation between the critical exponents, to predict in which dimensions the transition occur and to roughly estimate the critical exponents. The lower critical dimension is identified with weak localization corrections.
1 Introduction

A crystal undergoes a metal-insulator transition under the modification of a parameter (be it the Fermi energy, the doping, the lattice constant, etc) if its electronic structure changes from an insulating phase, exhibiting a vanishing electrical conductivity at zero temperature, to a metallic phase, in which the conductivity remains finite even at zero temperature. A very well-known example of such a transition is the Mott transition, driven by the competition between the repulsive Coulomb electron-electron interaction and the energy gain of electron delocalization into Bloch waves. The first remarkable thing about this phenomenon is that, unlike the thermal ferromagnetic transition studied in class, the transition can occur at zero temperature. Therefore, thermal fluctuations are not expected to play a key role and we are faced with a quantum phase transition. Secondly, the physical picture looks very different from one side of the Mott transition to the other. On the metallic side, we have delocalized electrons propagating in Bloch waves, while on the insulating side they appear very much localized on individual atoms and the only thing propagating are spin waves. The two structures are nothing alike, which partly explains why the Mott transition remains a difficult theoretical problem. And third, though the two phases are very different, no broken symmetry presents itself. This prevents the problem from being tackled in a straightforward analogy with the thermal ferromagnetic transition.

But tuning up the Coulomb interaction between electrons is not the only way one can bring about a metal-insulator transition. Instead, one can introduce disorder. In that context, disorder means the presence of defects or impurities distributed randomly in the crystal lattice. Such defects interact with itinerant electrons as an additional random external potential. V. Dobrosavljevic notes in [1] that for a low concentration of impurities, electrons simply scatter on the defects as described by the semi-classical Drude-Sommerfeld picture. The static conductivity is then given by the Drude formula:

$$\sigma = \frac{n e^2 \tau}{m}$$

in which $n$ is the electron concentration, $m$ the electron mass and $\tau$ is the average time in which an electron scatters off an impurity. At finite temperature, electrons also scatter with lattice phonons and other electrons, and $\tau$ is reduced accordingly. The classical behaviour is a diffusive one, with a diffusion constant

$$D = \frac{v_F l}{d}$$

in which $d$ is the number of dimensions, $v_F$ is the Fermi velocity and $l = v_F \tau$ is the mean free path.

As discussed in [1], if the disorder is sufficiently high as to create random potential wells in which electrons at the Fermi energy can get trapped, the situation is different. The simplest approach, ignoring the quantum interference processes, distinguishes two types of regions: metallic regions in which the Fermi energy is higher than the random potential, and insulating region in which it is lower. Starting from a Fermi energy everywhere lower that the potential, tuning it up creates randomly distributed and expanding metallic regions. The system becomes metallic as soon as one side of the crystal is connected to the other by a metallic path: this is a percolation problem, leading to a disorder-induced metal-insulator transition. This picture is valid if the random potential correlation lengths are much larger than the phase coherence.
lengthscale. A qualitative argument for this criterion will be discussed in the next section.

If the phase coherence length scale is larger than the random potential correlation lengthscale, a fully quantum description is necessary. One could expect that in this framework, tunneling effects would favor a metallic behaviour, but P. W. Anderson suggested in 1958 in [2] that phase coherence effects in the presence of disorder could actually localize electrons into bound states. At strong enough disorder, all electrons are therefore localized and the crystal is insulating. This phenomenon, known as Anderson localization, is the mechanism for an metal-insulator transition known as Anderson transition. This is the topic that I chose for the present essay, with a focus on the theoretical aspects of the problem.

2 Anderson Localization

2.1 Understanding the localization

First, let us understanding qualitatively how quantum behaviours lead localization in the presence of disorder, i.e. the physical meaning of Anderson localization.

A first way of describing it, as do M. V. Sadovskii in [3] and V. Dobrosavljevic in [1], is simply to invoke Fermi’s Golden Rule: for one electron of energy \( E \) to tunnel through a potential barrier, there needs to be a state of energy \( E \) on the other side. More precisely, the tunneling escape rate \( \tau_{\text{esc}} \) is given by:

\[
\frac{\hbar}{\tau_{\text{esc}}} = t^2 \rho_{\text{loc}}(E)
\]

in which \( t \) is a matrix element between the bound state and its environment and \( \rho_{\text{loc}}(E, \vec{r}) \) is the local density of state (a very brief and elementary but crystal clear introduction to which can be found in [4]) at the energy \( E \) and location \( \vec{r} \) of the bound state. \( \rho_{\text{loc}}(E, \vec{r}) \) is basically proportional to the sum of the wave function amplitudes at \( \vec{r} \) of the states \( \alpha \) of energy \( E \):

\[
\rho_{\text{loc}}(E, \vec{r}) = \sum_{\alpha} |\psi_E(\vec{r})|^2
\]

Now, if these states \( \alpha \) are themselves randomly localized elsewhere in the system, their wave function amplitude at \( \vec{r} \), \( |\psi_E(\vec{r})|^2 \), will likely be small, and the tunneling rate is then suppressed, localizing the bound state. In this situation, the local density of state typically looks like Fig 1. Notice that for a state to be localized, it requires that the other states of the same energy are themselves localized. As I see it, this provides a glimpse of how, despite the absence of electron-electron interactions, Anderson localization emerges as a kind of collective phenomenon among electronic states, and thus can be characterized as a phase transition.

By this mechanism, all the electrons in a partially filled band can be localized, and the system becomes insulating. Let us consider a band of width \( V \), and a disorder of energy scale \( W \). As P. W. Anderson showed in [2], there is a critical value of the ratio \( \frac{W}{V} \), \( \left( \frac{W}{V} \right)_c \), above which the entire band is localized. Below \( \left( \frac{W}{V} \right)_c \), only the edges of the band are localized (see Fig 2), due to their lower density of states. The center of the band, between the ‘mobility edges’ \( E'_c \) and \( E_c \), remains delocalized. Therefore, to induce a metal-insulator transition, one can either lower the Fermi energy to cross the
Figure 1: From [1]. Experimental local density of states for light in photonic crystals.

Figure 2: From [3]. Grey areas are localized, white are not. $E_c'$ and $E_c$ are called 'mobility thresholds' or 'mobility edges'.
mobility edge $E'_c$, or increase the disorder strength $W$ for the localization to engulf the filled part of the band. In the case of a half-filled band, the threshold value is $(W/V)_c$.

Another way of understanding Anderson localization is to examine the backscattering processes, following P. Wölfle and D. Vollhardt in [5] as well as C. A. Müller and D. Delande in [6]. Let us just consider the probability for an electron at the position $\vec{r}$ to end up at a position $\vec{r}'$ after scattering off two impurities at $\vec{r}_1$ and $\vec{r}_2$. To calculate it, we have to sum the probability amplitudes associated with the two possible paths sketched in Fig 3 ($\vec{r}_1$ and then $\vec{r}_2$ or vice-versa), and then take the square modulus. This yields an interference term. If, as in the left side of Fig 3, the end point is different from the starting point, the phase shift between the two paths is nonzero and the interference term depends on $\vec{r}_1$ and $\vec{r}_2$. But remember, we are in the presence of a random disorder, therefore the result should be averaged over $\vec{r}_1$ and $\vec{r}_2$. In that case, the averaging kills the interference term. If, on the contrary, $\vec{r} \approx \vec{r}'$ as in the right side of Fig 3, then the phase shift vanishes (provided that time-reversal symmetry holds) and the constructive interference term remains unaffected by the averaging. Therefore, this kind of interference processes favour backscattering, a phenomenon called ‘coherent backscattering’, or, in this context, ‘weak localization’. This qualitative argument allows us to see how quantum phase effects can keep the electrons localized. Moreover, if the spatial correlation lengthscale of the disorder is not small compared to the phase coherence lengthscale $L_{\Phi}$, the paths followed by the electrons in Fig 3 are at least comparable to $L_{\Phi}$, and when reaching $\vec{r}'$ the electrons have lost their coherence. In this case, the interference term vanishes no matter what and localization is suppressed: Anderson localization breaks down and we are back to the percolation picture. Now, we understand why the criterion for the validity of this picture is that the disorder correlation lengthscale is not small compared to $L_{\Phi}$ (which depends mainly on the temperature).

### 2.2 One-dimensional disorder

In order to understand more precisely how localization emerges and to get some quantitative results, it is useful to study a simple, solvable model. Following C. A. Müller
Figure 4: From [6]. Relations between the incoming and outgoing wave-function amplitudes around a defect.

and D. Delande in [6], we will now consider a one-dimensional, single-channel system of length $L$ in which $nL$ point-like, randomly distributed defects have been introduced. Each defect is characterized by the left transmission coefficient $t$, the left reflection coefficient $r$, the right transmission coefficient $t'$ and the right reflection coefficient $r'$. These coefficients determine the relation between the outgoing wave-function amplitudes $\psi_{\text{out}}^{L/R}$ and the incoming amplitudes $\psi_{\text{in}}^{L/R}$, as in Fig 4. The associated reflection and transmission probabilities are $R = |r|^2$, $T = |t|^2$, $R' = |r'|^2$ and $T' = |t'|^2$. Enforcing the flux conservation, $|\psi_{\text{out}}^{L}|^2 + |\psi_{\text{out}}^{R}|^2 = |\psi_{\text{in}}^{L}|^2 + |\psi_{\text{in}}^{R}|^2$ yields $R + T = 1$, $R = R'$, $T = T'$ and $r't' + tr' = 0$. Using these relations, we express the amplitudes on the right as functions of the amplitudes on the left:

$$(\begin{pmatrix} \psi_{\text{out}}^{R} \\ \psi_{\text{in}}^{R} \end{pmatrix}) = M \begin{pmatrix} \psi_{\text{in}}^{L} \\ \psi_{\text{out}}^{L} \end{pmatrix}$$

in which $M$ is called the transfer matrix and reads:

$$M = \begin{pmatrix} t & r' \\ -r & t' \end{pmatrix}$$

A very useful feature of the transfer matrix is that it can be chained over several defects to account for the total transmission and reflection of the system. For instance, in the presence of two defects 1 and 2 (in that order from left to right), the total transfer matrix of the system reads: $M_{12} = M_2 M_1$. It follows that the total transmission coefficient is:

$$t_{12} = \frac{t_1 t_2}{1 - r_1^* r_2}$$

which contains the multiple reflections terms. Note therefore that the coefficients contain the propagative phases between the defects. Taking the square modulus, the total transmission probability reads:

$$T_{12} = \frac{T_1 T_2}{|1 - \sqrt{R_1 R_2} e^{i \phi}|^2}$$

in which $\phi$ is the propagative phase acquired during an internal reflection.

The end goal here is to calculate the conductance $G(L)$ of the system. In quantum transport, the conductance (as opposed to the conductivity) is the most easily accessible quantity since it is given by the Landauer formula:

$$G(L) = G_Q \frac{T}{1 - T}$$

(2)
in which \( T \) is the total transmission probability of the system and \( G_Q = \frac{e^{2\pi}}{2\pi} \). This formula, discussed by Y. V. Nazarov and Y. M. Blanter in [7] and by P. Markos and C. M. Soukoulis in [8], does not take into account the contact resistances. We must therefore compute the total transmittance \( T \). The straightforward solution is to chain the \( nL \) transfer matrices of the \( nL \) defects, which is quite complicated. Moreover, the results would feature all kinds of propagative phases such as \( \phi \), and since the defects are randomly distributed, they should all be averaged over the range \([0, 2\pi]\). A more practical solution is to look for additive quantities.

A first way of doing it, which reduces the problem to classical transport, is to average \( \phi \) in \( T_{12} \) first, essentially killing the quantum interference processes:

\[
<T_{12}> = \frac{1}{2\pi} \int_0^{2\pi} \frac{T_1T_2}{|1 - \sqrt{R_1R_2e^{i\phi}}|^2} = \frac{T_1T_2}{1 - R_1R_2}
\]

in which the integral is solved using Bioche’s rules. The quantity \( \frac{1 - T_i}{T_i} \) is then additive:

\[
\frac{1 - T_{12}}{T_{12}} = \frac{1 - T_1}{T_1} + \frac{1 - T_2}{T_2}
\]

and so, generalizing to \( nL \) defects and using equation 2, we obtain:

\[
\frac{G_Q}{G(L)} = nL \frac{R_1}{T_1}
\]

We find that the resistance is proportional to \( L \): this is classical Ohmic behaviour.

But if we are to describe Anderson localization, we must preserve the quantum interference processes. The trick here is to take the logarithm of equation 1, then average over \( \phi \) and notice that the denominator vanishes:

\[
<\ln(|1 - \sqrt{R_1R_2e^{i\phi}}|^2) > = \frac{1}{2\pi} \int_0^{2\pi} \ln(1 - \sqrt{R_1R_2e^{i\phi}}) + \frac{1}{2\pi} \int_0^{2\pi} \ln(1 - \sqrt{R_1R_2e^{-i\phi}})
\]

and, since the complex logarithm has no pole outside of 0, each integral vanishes. We are left with:

\[
<T_{12}> = \ln(T_1) + \ln(T_2)
\]

Generalizing again to \( nL \) defects:

\[
T_\text{typ} \approx e^{<\ln(T)>} = e^{-\frac{L}{\xi}}
\]

in which \( T_\text{typ} \) is a typical value for \( T \) and \( \xi = \frac{1}{n||\ln(T_1)||} \) is the localization length. Using equation 2 yields:

\[
\frac{G_Q}{G(L)} = \frac{1}{e^{\frac{L}{\xi}} - 1}
\]

which for \( L >> \xi \) yields:

\[
\frac{G_Q}{G(L)} = e^{-\frac{L}{\xi}}
\]

We thus find the remarkable result that quantum interference processes localize the electronic wave functions exponentially with a typical localization lengthscale of \( \xi \), thus creating an insulating phase.
3 Anderson Transition

3.1 First properties of the transition

We learned from the previous section that, in metals, disorder can induce a transition from a metallic phase, governed by the Drude picture, to an insulating phase, in which electronic wave functions suffer a spatially exponential localization characterized by the localization length $\xi$. Going back to equation 3 for the dimensionless one-dimensional conductance:

$$G_0 = \frac{1}{G(L)} = \begin{cases} \frac{\xi}{e^{\frac{\xi}{L}} - 1} & \text{if } \xi \ll L \\ e^{-\frac{\xi}{L}} & \text{if } \xi \gg L \end{cases} \quad (4)$$

in which $\xi = \frac{1}{n|\ln(T)|}$. Notice that the conductance is a function of $\xi L$ only. Inside the localization length (for $\xi \gg L$), the system is Ohmic, i.e. described by the Drude picture and classically diffusive. Outside the localization length (for $\xi \gg L$), the conductivity is suppressed and the system is insulating. Generalizing to any dimension and different microscopic models, the transition between these two types of behaviour is called Anderson transition. From the expression of $\xi$ obtained in the one-dimensional model, there is manifestly two ways of bringing about an Anderson transition from a metallic phase. Either $|\ln(T_1)|$ is kept constant and $n$ is increased, which consists in adding more defects until their density $n$ reaches some critical value $n_c$. Or $n$ is kept constant and $|\ln(T_1)|$ is increased, which consists in lowering the Fermi energy $E_F$ past the mobility edge $E_c$.

Either way, it is worth reiterating that no broken symmetry is readily available. Therefore, writing down a Ginzburg-Landau free energy, in analogy with the thermal magnetic phase transition, is in itself a challenging task. It is still, however, possible to draw some analogies, following a number of author such as V. Dobrosavljevic in [1] and P. Wolffle and D. Vollhardt in [5]. What would be the order parameter? At the thermodynamic limit $L \to \infty$, such a quantity would vanish in one phase and not in the other. A quick glance at equation 4 tells us that the conductance does not fit the bill. The conductivity $\sigma$, however, is finite in an Ohmic system and, since it is related to the conductance of a $L$ sided cube in $d$ dimension by $\sigma = G L^{2-d}$, it vanishes in the localized phase at the thermodynamic limit. Therefore, $\sigma$ is a good order parameter. Next, what would play the role of the temperature $T - T_c$ in the magnetic transition, as a controlled parameter used to drive the system through the transition? Clearly, both $E_F - E_c$ and $n - n_c$ will do. As for the temperature in the Anderson transition, it is like an external field in the magnetic transition: for a finite temperature, $\sigma$ does not vanish in the insulating phase due to thermal excitations and there is no real transition. Though they are all-important for experiments, finite temperature corrections will not be discussed in this essay. And lastly, what would the correlation length be? It must be infinite in the 'ordered' phase. Since the metallic phase is characterized by $\xi > L$, the localization length is required to be infinite in this phase at the thermodynamic limit. Therefore, the localization length $\xi$ plays the role of the correlation length.

Another crucial question is the order of the transition. Is the Anderson transition a first or a second order transition? Historically, it was proposed by Mott that the metallic phase exhibits a minimal conductivity $\sigma_{\text{min}}$. As discussed by M. V. Sadovskii in [3], that would make the Anderson transition a first order one, since a jump in the order parameter would be expected as in Fig 5. An argument for the existence of $\sigma_{\text{min}}$...
Figure 5: From [3]. Curve 1 describes a first order transition under the hypothesis of a minimal metallic conductivity. Curve 2 describes a second order transition in which the conductivity vanishes continuously at the transition.

is that, in the Drude formula,

\[ \sigma = \frac{ne^2\tau}{m} = \frac{ne^2l}{mv_F} \]

the mean free path \( l \) cannot be shorter than the lattice spacing \( a \), and thus there is a minimal conductivity

\[ \sigma_{\text{min}} = \frac{ne^2a}{mv_F} \]

But subsequent experiments in the 1980’s showed that metallic phases exhibit in fact conductivities that are much smaller than \( \sigma_{\text{min}} \). Therefore, Anderson localization is a second order phase transition.

A consequence of this fact is that the conductivity and the localization length are expected to respectively vanish and diverge algebraically near the critical point:

\[ \sigma \propto (n - n_c)^\mu \]

\[ \xi \propto (n - n_c)^{-\nu} \]

in which \( \mu \) and \( \nu \) are the critical exponents, which are expected to be universal as discussed by B. Kramer, A. MacKinnony, T. Ohtsukiz and K. Slevin in [5]. In the absence of spatial symmetry due to disorder, the universality classes to which the Anderson transition can belong are defined by the dimensionality \( d \), the time-reversal symmetry and the spin-rotation symmetry. In the orthogonal class, the two symmetries are present. In the symplectic class, there is time-reversal invariance but spin-rotation symmetry is broken. And in the unitary class, time-reversal symmetry is broken. To give some examples, models featuring spin-orbit interactions belong to the symplectic class, while the presence of an external magnetic field breaks time-reversal invariance and thus places the model in the unitary class.

### 3.2 Finite size scaling theory

In 1979, E. Abrahams, P. W. Anderson, D. C. Licciardello and T. V. Ramakrishnan made a breakthrough in the understanding of the Anderson transition by proposing in
[9] a finite size scaling theory of the transition, using the dimensionless conductance
\( g(\delta n, L) = \frac{\delta n}{\xi} \) of a L sided cube in d dimension, in which \( \delta n = n - n_c \). One reason
for this is that the conductance, by its definition, is invariant under spatial renormalization. If every length is rescaled by a factor \( b \), we expect near the transition:

\[
g \to g \quad L \to \frac{L}{b} \quad \xi \to \frac{\xi}{b}
\]

This is the scaling hypothesis. Using \( \sigma = G_Q g L^{2-d} \) and \( \xi \propto (\delta n)^{-\nu} \), we further deduce

\[
\sigma \to \sigma b^{d-2} \quad \delta n \to \delta n b^{\frac{1}{\nu}}
\]

But using now \( \sigma \propto (n - n_c)^\mu \) yields

\[
\delta n \to \delta n b^{\frac{d-2}{\mu}}
\]

and so we find a scaling relation between \( \nu \) and \( \mu \), called the Wegner scaling relation:

\[
\mu = \nu(d - 2)
\]

We can already see that the dimension \( d = 2 \) plays a particular role, as will be confirmed shortly. Another consequence of the scaling appears if we write down the invariance of \( g \):

\[
g(\delta n, L) = g(\delta n b^{\frac{1}{\nu}}, \frac{L}{b})
\]

choosing \( b = (\delta n)^{-\nu} = \xi \) yields:

\[
g(\delta n, L) = F(\frac{L}{\xi})
\]

g is thus a function of \( \frac{L}{\xi} \) only, which is consistent with equation 4.

Having established the Wegner scaling law, we must now find the critical exponents around a transition. This also implies finding out in which dimensions (which universality classes, in fact) there is a transition. To achieve all this, E. Abrahams, P. W. Anderson, D. C. Licciardello and T. V. Ramakrishnan used in [9] a different approach which is essentially equivalent to the scaling hypothesis but allows the knowledge of \( g \) in the metallic and insulating limits to be used very easily. This approach, taken by many authors including C. A. Müller and D. Delande in [6] as well as V. Dobrosavljevic in [1], makes use of the beta function defined as:

\[
\beta(g) = \frac{d \ln(g)}{d \ln(L)}
\]

As the notation suggests, the key hypothesis is that \( \beta \) depends only on \( g \). Another is that \( \beta \) is smooth. Since the behaviour of \( g \) is known in the insulating and metallic limits, the behaviour of \( \beta \) is known in these limits as well. A simple monotonic interpolation between the two will then give us an idea of what \( \beta \) looks like.

In the metallic limit, \( \sigma \) is independent of \( L \) and so \( g \propto L^{d-2} \), therefore

\[
\beta(g) \approx d - 2
\]

In the insulating limit, the dimensionless conductance decreases exponentially with \( L \):

\[
g = e^{-\frac{L}{\xi}}, \text{so}
\]

\[
\beta(g) \approx \ln(g)
\]
A smooth and monotonic interpolation thus gives us something that looks like Fig 6. Given the definition of $\beta$, as $L$ is increased toward the thermodynamic limit, the system flows to the right above the abscisse line and to the left below the abscisse line.

For $d = 1$, $\beta$ is always negative and the system flows toward localization no matter how weak the disorder. This is consistent with our study of the one-dimensional model: as $\xi$ only depends on $n$ and $T_1$, the thermodynamic limit always implies $\xi << L$. A macroscopic one-dimensional system is always localized and insulating, there is no Anderson transition.

For $d = 2$, the same conclusions apply if the beta function really looks like Fig 6. This question will be further discussed in the next section.

For $d = 3$, there is an unstable fixed point at $g = g_c$. If $g > g_c$, the system will be metallic at the thermodynamic limit, and if $g < g_c$ it will be insulating. Therefore, there is an Anderson transition and $g = g_c$ is the critical point. In the spirit of renormalization groups methodology, studying $\beta$ around the critical point should give us the critical exponents. Let us then linearize $\beta$ around $g = g_c$:

$$
\beta(ln(g)) = s \ln\left(\frac{g}{g_c}\right)
$$

in which $s$ is the slope at the critical point. We will now integrate $\beta$ between a generic $L$ and $L_0 = \xi$, for which we know that $g_0 \approx 1$:

$$
\ln \left(\frac{L}{\xi}\right) = \int_{\ln(\frac{1}{\xi})}^{\ln(\frac{1}{\xi})} d\ln\left(\frac{g}{g_c}\right) = \frac{1}{s} \ln \left(\frac{\ln\left(\frac{g}{g_c}\right)}{\ln\left(\frac{1}{g_c}\right)}\right)
$$
so

\[
\left( \frac{L}{\xi} \right)^s = \frac{\ln\left( \frac{g}{g_c} \right)}{\ln\left( \frac{1}{g_c} \right)}
\]

and therefore

\[
\xi \propto L \left( \ln\left( \frac{g}{g_c} \right) \right)^{-\frac{1}{s}}
\]

Now, we have \( \ln\left( \frac{g}{g_c} \right) = \ln(1 + \frac{g-g_c}{g_c}) \approx \frac{g-g_c}{g_c} \) and, linearizing the dependancy of \( g \) to \( n \): \( \frac{g-g_c}{g_c} \propto \frac{n-n_c}{n_c} \). Thus, we find

\[
\xi \propto (\delta n)^{-\frac{1}{s}}
\]

which implies \( \nu = \frac{1}{s} \). Using a simple interpolation function such as

\[
\beta(g) = d - 1 - (1 + g) \ln(1 + \frac{1}{g})
\]

gives \( \nu = 1.68 \) which is not terribly off compared to the numerically estimated value \( \nu = 1.68 \).

### 3.3 Weak localization corrections and lower critical dimension

Coming back to the case \( d = 2 \), we expect that \( \beta \) for high \( g \) behaves as

\[
\beta(g) = \frac{\alpha}{g} + O\left( \frac{1}{g^2} \right)
\]

If \( \alpha < 0 \), there is no transition like for \( d = 1 \). If \( \alpha > 0 \), however, there are transitions. To determine the sign of \( \alpha \), we need a microscopic model. The standard Hamiltonian for this task is called the Anderson Hamiltonian and reads:

\[
H = -t \sum_{<ij>} c_i^\dagger c_j + \sum_i W_i c_i^\dagger c_i
\]

in which the first term is the usual first-neighbor hopping term and the second term, featuring random variables \( W_i \), stands for the disorder. A perturbative treatment of the second term, explained by C. A. Müller and D. Delande in [6], leads to weak localization corrections which allow to find the sign of \( \alpha \). For the orthogonal and unitary classes, \( \alpha \) is found to be negative, therefore the lower critical dimension for these classes is \( d = 2 \). For the symplectic class, however, \( \alpha \) is positive and the lower critical dimension is \( d = 1 \).

Numerical calculations using the Anderson Hamiltonian have found the critical exponents, regrouped in Fig 7.

### 4 Conclusion

I have gained a basic understanding of the Anderson localization phenomenon and of the Anderson transition. Yet it still feels like I have barely scratched the surface here, the integration of correlation effects in particular are crucial when it comes to experiments. The field is clearly extremely rich and I have much to explore in the future.
Figure 7: From [1]. Numerical estimates of the critical exponent, with 95% confidence intervals.

<table>
<thead>
<tr>
<th>Critical exponent</th>
<th>Universality class</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nu = 1.57 \pm 0.02 )</td>
<td>3D orthogonal symmetry</td>
<td>Slevin and Ohtsuki (1999)</td>
</tr>
<tr>
<td>( \nu = 1.43 \pm 0.04 )</td>
<td>3D unitary symmetry</td>
<td>Slevin and Ohtsuki (1997)</td>
</tr>
<tr>
<td>( \nu = 1.375 \pm 0.016 )</td>
<td>3D symplectic symmetry</td>
<td>Asada et al., (2005)</td>
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<tr>
<td>( \nu = 2.73 \pm 0.02 )</td>
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<td>Asada et al., (2002)</td>
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<tr>
<td>( \nu = 2.59 \pm 0.006 )</td>
<td>Integer quantum Hall effect</td>
<td>Slevin and Ohtsuki (2009)</td>
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References


