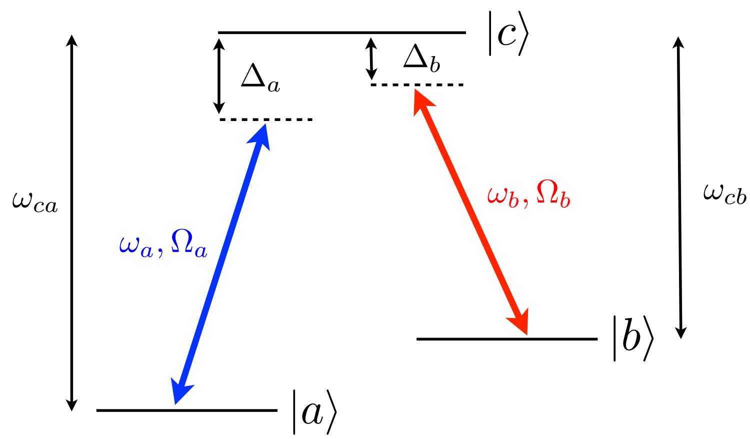
Light and Matter - part 2 (M1) - T. Roscilde, S. Ciliberto, F. Kulzer

TD2: Three-level atom

Adiabatic elimination, dark states, electromagnetically induced transparency

1 Λ-scheme and adiabatic elimination



In this exercise we explore a generalization of the two-level atom coupled to a single monochromatic light field, by moving to a typical setup in atomic physics, called the Λ-scheme. It consists of an atom which has two nearly degenerate low-energy states $|a\rangle$ and $|b\rangle$ (think of two hyperfine states, for instance) and a higher energy state $|c\rangle$ (with a higher principal quantum number). The atom interacts with *two* laser beams a and b, whose polarizations are such that laser a couples to the transition $|a\rangle \to |c\rangle$ and laser b couples to the transition $|b\rangle \to |c\rangle$. We need to use lasers here, because we assume that the two beams are mutually coherent in phase.
Laser a has frequency ω_a, and it is detuned by $\Delta_a = \omega_{ca} - \omega_a$ to the frequency of the $|a\rangle \to |c\rangle$ transition, $\omega_{ca} = (E_c - E_a)/\hbar$. Similarly laser b has frequency ω_b, and it is detuned by $\Delta_b = \omega_{cb} - \omega_b$ to the frequency of the $|b\rangle \to |c\rangle$ transition, $\omega_{cb} = (E_c - E_b)/\hbar$.
The 3-level Hamiltonian reads

$$\begin{aligned}
\mathcal{H} &= \mathcal{H}_0 + \mathcal{H}'(t) \\
\mathcal{H}_0 &= E_a\, |a\rangle\langle a| + E_b\, |b\rangle\langle b| + E_c\, |c\rangle\langle c| \\
\mathcal{H}' &= \frac{1}{2}\left(\hbar\Omega_a e^{-i\omega_a t} + \text{c.c.}\right)\left(|c\rangle\langle a| + |a\rangle\langle c|\right) + \frac{1}{2}\left(\hbar\Omega_b e^{-i\omega_b t} + \text{c.c.}\right)\left(|c\rangle\langle b| + |b\rangle\langle c|\right) \quad (1)
\end{aligned}$$

In the following we will set $E_a = 0$ for simplicity.

1.1

In the case of the 3-level system, we move to a "rotating" frame defined by the following transformation:

$$|\tilde{\psi}\rangle = e^{i\xi t}|\psi\rangle \quad (2)$$

where ξ is the following diagonal operator

$$\xi = \frac{\delta}{2}\ |a\rangle\langle a| + \left(\frac{E_b}{\hbar} - \frac{\delta}{2}\right)\ |b\rangle\langle b| + \left(\frac{E_c}{\hbar} - \Delta\right)\ |c\rangle\langle c|\ . \tag{3}$$

Here

$$\delta = \Delta_a - \Delta_b \qquad\qquad \Delta = \frac{\Delta_a + \Delta_b}{2}\ . \tag{4}$$

Show that $|\tilde{\psi}\rangle$ obeys the Schrödinger equation

$$i\hbar\frac{d}{dt}|\tilde{\psi}\rangle = \tilde{\mathcal{H}}|\tilde{\psi}\rangle \tag{5}$$

where

$$\tilde{\mathcal{H}} \approx -\frac{\hbar\delta}{2}\ |a\rangle\langle a| + \frac{\hbar\delta}{2}\ |b\rangle\langle b| + \hbar\Delta\ |c\rangle\langle c| + \frac{\hbar}{2}\left(\ \Omega_a|c\rangle\langle a|\ +\ \Omega_b|c\rangle\langle b|\ +\ \text{h.c.}\right) \tag{6}$$

The latter expression is obtained by using the rotating-wave approximation.

1.2

Using the decomposition $|\tilde{\psi}(t)\rangle = \tilde{\psi}_a(t)\ |a\rangle + \tilde{\psi}_b(t)\ |b\rangle + \tilde{\psi}_c(t)\ |c\rangle$, write the equation of motion for the coefficients $\tilde{\psi}_a$, $\tilde{\psi}_b$, and $\tilde{\psi}_c$.
Writing $\tilde{\psi}_c = e^{-i\Delta t}\phi_c$ show that

$$i\dot{\phi}_c = \left(\frac{\Omega_{ca}}{2}\ \tilde{\psi}_a + \frac{\Omega_{cb}}{2}\ \tilde{\psi}_b\right)e^{i\Delta t}\ . \tag{7}$$

1.3

Integrating formally the previous equation, show that

$$\tilde{\psi}_c(t) = -\frac{\Omega_{ca}\tilde{\psi}_a + \Omega_{cb}\tilde{\psi}_b}{2\Delta} + \mathcal{O}\left(\frac{\Omega}{\Delta}\right)^2 \tag{8}$$

In the limit of large detuning $\Omega \ll \Delta$, the second order term can be safely neglected, leading to the so-called *adiabatic elimination*.
Show that this condition allows to obtain the restrict the dynamics of the system to states $|a\rangle$ and $|b\rangle$ only, with the effective Hamiltonian matrix

$$\frac{\mathcal{H}_{\text{eff}}}{\hbar} = \begin{pmatrix} -\frac{\delta}{2} - \frac{|\Omega_{ca}|^2}{4\Delta} & -\frac{\Omega_{ac}\Omega_{cb}}{4\Delta} \\ -\frac{\Omega_{bc}\Omega_{ca}}{4\Delta} & \frac{\delta}{2} - \frac{|\Omega_{ca}|^2}{4\Delta} \end{pmatrix}\ . \tag{9}$$

Can you interpret physically this result?

2 Dark states, electromagnetically induced transparency

We now move back to the full three-level system, and from the equation of motion for $|\tilde{\psi}(t)\rangle = \tilde{\psi}_a(t)\ |a\rangle + \tilde{\psi}_b(t)\ |b\rangle + \tilde{\psi}_c(t)\ |c\rangle$ as already derived at point 1.2.

2.1

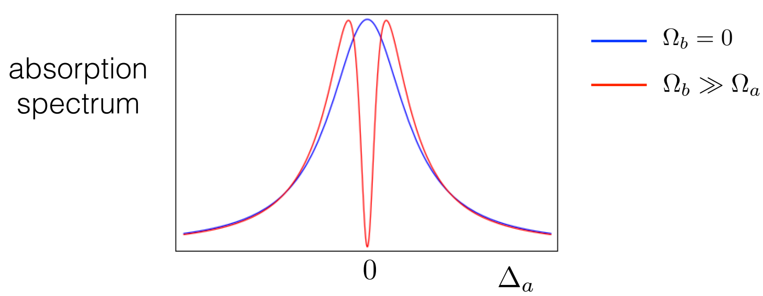
We consider the initial condition $\tilde{\psi}_c(0) = 0$. Show that the condition $d\tilde{\psi}_c/dt = 0$ at $t = 0$ is obtained if

$$\Omega_a \tilde{\psi}_a(0) + \Omega_b \tilde{\psi}_b(0) = 0 \ . \tag{10}$$

Prove that if $\delta = 0$ (Raman condition), a stationary solution can be found which verifies the condition Eq. 10 at all times.
Write down the normalized stationary state $|\psi_D\rangle$. This is called in the literature a *dark state*, as the lasers fail to excite it to the state $|c\rangle$ even if they are resonant with the respective transition frequencies. Under this situation the atom does not scatter light, and hence it looks transparent.
(An explicit time-dependent calculation shows that the transitions induced by the two lasers interfere destructively when starting from this state – see question 2.3).

2.2

Consider the situation in which laser b is much more intense than laser a (in which case laser b is called the *pump* laser, while laser a is called the *probe* laser).
Show that, under these conditions $|\psi_D\rangle \approx |a\rangle$.
If $\Delta_b \approx 0$, justify that the absorption spectrum for the a laser (proportional to the probability that light from the a laser is absorbed by the atom), takes the form of the figure below.
Why in your opinion is this phenomenon called *electromagnetically induced transparency* (EIT)?



2.3 (Extra questions)

We can further understand why $\psi_c(t) = 0$ is a dark state using perturbation theory. In the laboratory frame we write the state evolved by the Hamiltonian $\mathcal{H}$ as

$$|\psi(t)\rangle = \sum_{n=a,b,c} e^{-\frac{i}{\hbar}E_n t} \ \phi_n(t) \ |n\rangle \ . \tag{11}$$

As you may have seen in past courses, first-order perturbation theory provides the following equation for the ϕ_c coefficient

$$\frac{d\phi_c}{dt} = \frac{1}{i\hbar} \sum_n \ \phi_n(0) \ \langle c|\mathcal{H}'(t)|n\rangle \ e^{i\omega_{cn} t} \tag{12}$$

where $\omega_{cn} = (E_c - E_n)/\hbar$.

Integrating the above equation with the initial condition $\phi_c(0) = 0$, show that it gives the solution

$$\phi_c(t) \approx -\frac{1}{2} \sum_n \phi_n(0) \ \langle c|O(t)|n\rangle = -\frac{1}{2} \ \langle c|O(t)|\psi(0)\rangle \tag{13}$$

where

$$\begin{aligned} O(t) &= O_a(t) + O_b(t) \\ O_a(t) &= \Omega_a \ e^{i\Delta_a t} \ \mathrm{sinc}(\Delta_a t) \ |c\rangle\langle a| \\ O_b(t) &= \Omega_b \ e^{i\Delta_b t} \ \mathrm{sinc}(\Delta_b t) \ |c\rangle\langle b| \end{aligned} \tag{14}$$

Justify the neglected terms in Eq. 13.
Show that if the initial state $|\psi(0)\rangle$ is the dark state $|\psi_D\rangle$, and if the Raman condition is satisfied, then

$$O_a(t)|\psi_D\rangle = -O_b(t)|\psi_D\rangle \ . \tag{15}$$

Conclude that the effects of the two lasers on the dark state interfere destructively (that's why it is dark!).

Appendix

Rabi frequency

$$\Omega_{eg} = -\frac{E_0}{e\hbar} \ \langle e|(\hat{\epsilon} \cdot \boldsymbol{D})|g\rangle \tag{16}$$

Intensity

$$I = \frac{c\epsilon_0}{2}|E_0|^2 \tag{17}$$

$$\mathrm{sinc}(x) = \frac{\sin(x)}{x} \tag{18}$$