Bose–Einstein condensation in magnetic insulators

The Bose–Einstein condensate (BEC) is a fascinating state of matter predicted to occur for particles obeying Bose statistics. Although the BEC has been observed with bosonic atoms in liquid helium and cold gases, the concept is much more general. We here review analogous states, where excitations in magnetic insulators form the BEC. In antiferromagnets, elementary excitations are magnons, quasiparticles with integer spin and Bose statistics. In certain experiments their density can be controlled by an applied magnetic field leading to the formation of a BEC. Furthermore, interactions between the excitations and the interplay with the crystalline lattice produce very rich physics compared with the canonical BEC. Studies of magnon condensation in a growing number of magnetic materials thus provide a unique window into an exciting world of quantum phase transitions and exotic quantum states, with striking parallels to phenomena studied in ultracold atomic gases in optical lattices.

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Not long after Bose and Einstein described quantum statistics of photons[1] and atoms[2], Bloch applied the same concept to excitations in solids. He treated misaligned spins in a ferromagnet as magnons, particles with integer spin and bosonic statistics, to describe the reduction of spontaneous magnetization by thermal fluctuations[3]. Matsubara and Matsuda pointed out an exact correspondence between a quantum antiferromagnet and a lattice Bose gas[4]. It is thus natural to ask whether these bosons can undergo Bose–Einstein condensation and become superfluid. The answer is yes. The question of order in spin systems and the possibility of Bose–Einstein condensation of magnons has been investigated theoretically for a number of quantum antiferromagnets[5–13]. Although the concept of a BEC can be applied in simple spin systems, in practice factors such as the large values of exchange constants and the presence of anisotropies violating rotational symmetry may restrict their usefulness. However, the analogy between spins and bosons has proven to be very fruitful in those antiferromagnets where closely spaced pairs of spins $S=1/2$ form dimers with a spin-singlet ($S=0$) ground state and triplet ($S=1$) bosonic excitations called triplons. The triplons are similar to magnetic excitations of an ordered antiferromagnet, the magnons; in particular, they have the same quantum numbers, so the two terms are sometimes used interchangeably. In the dimer systems, Bose–Einstein condensation has been predicted to occur[7] and was experimentally observed[8,14] in the magnetic insulator $TlCuCl_3$. This started a flurry of activity on the subject and the hunt for such transitions in other magnetic materials.

Other examples of condensates from elementary excitations were recently found in magnetic thin films[15], semiconductor microcavities[16], ^3He (refs 17,18) and models for ferromagnets[19,20]. Lattice bosons and several transitions occurring in these systems have been studied extensively in ultracold atomic gases in optical lattices[21–24]. Here we present an overview of recent developments in the field and discuss how quantum antiferromagnets offer several advantages that distinguish them from other model systems in which BEC occurs. We also discuss ways to go beyond the physics of simple Bose–Einstein condensation and to look at the fascinating new quantum phases of interacting bosons on a lattice.

BOSONS IN MAGNETS

Let us illustrate the basics of a magnon BEC in real dimerized antiferromagnets, such as the extensively studied $TlCuCl_3$ (refs 8,11,12,14,25–33) and $BaCuSi_2O_6$ (refs 34–40). The lattice of magnetic ions in such materials (Fig. 1a) can be visualized as a set of dimers, pairs of copper ions Cu^{2+} carrying $S=1/2$ each and interacting via Heisenberg exchange:

$$\mathcal{H}=\sum_i J_0 \mathbf{S}_{1,i}\cdot\mathbf{S}_{2,i}+\sum_{\langle mnij\rangle} J_{mnij}\mathbf{S}_{m,i}\cdot\mathbf{S}_{n,j}-g\mu_B H\sum_{\langle ni\rangle} S^z_{m,i}, \quad (1)$$

where μ_B is the Bohr magneton, H denotes an external magnetic field in the z-direction and i, j number dimers and $m,n=1,2$ their magnetic sites. The intradimer exchange is the strongest

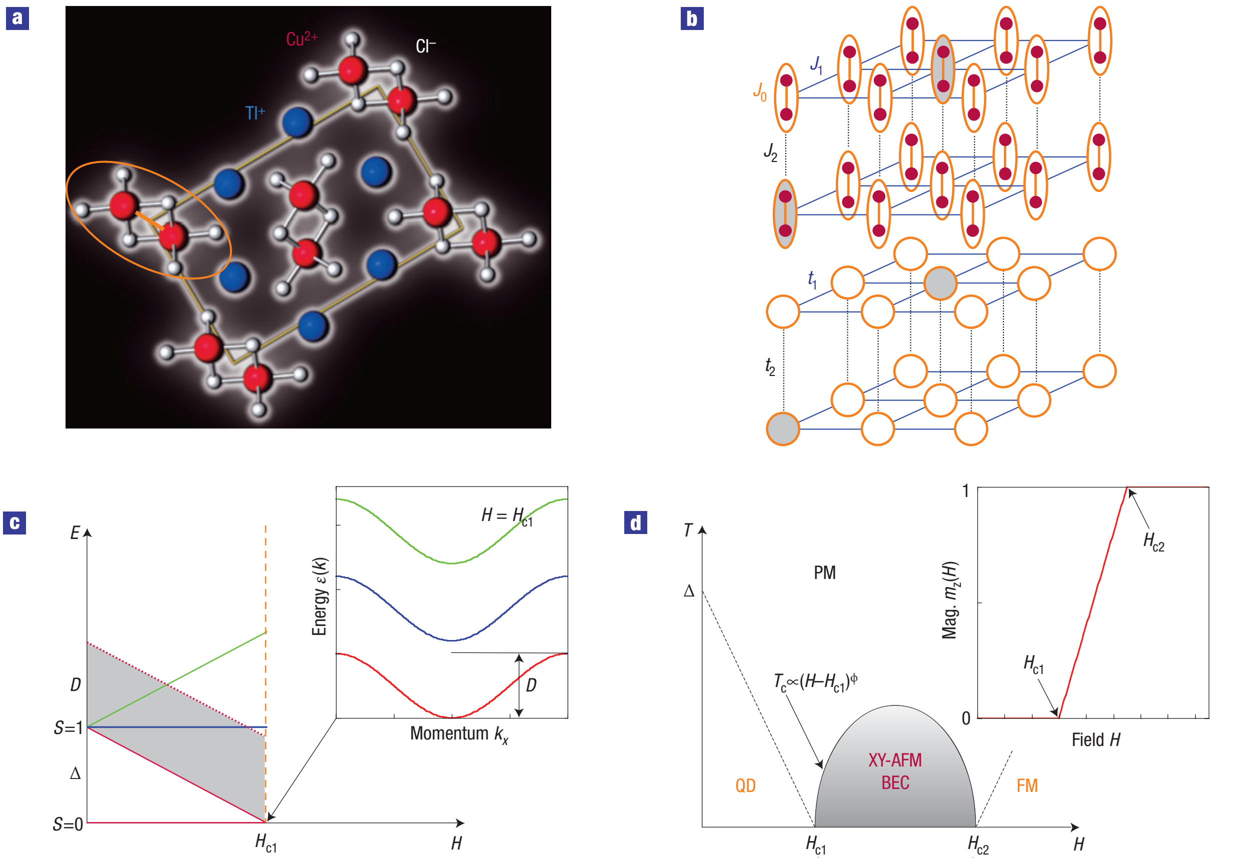

Figure 1 BEC of magnons in dimerized quantum antiferromagnets. **a**, Dimers in the real material $TlCuCl_3$ with $S = 1/2$ from Cu^{2+} ions and superexchange via Cl^- (refs 8,11,12,14,25,26,99). **b**, Dimers on a square lattice with dominant antiferromagnetic intradimer interaction J_0 and interdimer interactions J_i. Triplet states (grey, top) are mapped onto quasiparticle bosons (triplons, bottom). **c**, Zeeman splitting of the triplet modes with gap Δ and bandwidth D at $\mathbf{k}_0 = (\pi/a, \pi/a)$. Inset, Dispersion of triplons at the critical field H_{c1} (refs 11,12,14,25,26). **d**, Resulting phase diagram with paramagnetic (PM), quantum disordered (QD) and field-aligned ferromagnetic (FM) phases and canted-antiferromagnetic (XY-AFM) phase, where a magnon BEC occurs. Close to H_{c1} and H_{c2} the phase boundary follows a power law $T_c \propto (H - H_{c1})^{\phi}$ with a universal exponent $\phi = 2/3$ for a magnon BEC[7,8]. Inset, Magnetization curve $m_z(H)$ for a three-dimensional dimer spin system with a plateau at $m_z = 0, 1$ (gapped)[8,36,37].

interaction, which is antiferromagnetic, $J_0 > 0$, so that an isolated dimer has a ground state with total spin $S = 0$ and a triply degenerate excited state of energy J_0 and spin $S = 1$ (Fig. 1c). It is convenient to identify the triplet state with the presence of a triplon, a bosonic particle with $S = 1$, and the singlet state with the absence of a triplon; see Fig. 1b. As long as interdimer interactions are weak, the ground state consists of non-magnetic singlets. It remains disordered down to absolute zero temperature without long-range magnetic order. The triplon excitations are made mobile by weak interdimer couplings $J_{1,2,\ldots}$, from a sum over single-ion interactions J_{mnij}; see Fig. 1b. For dimers forming a square lattice (for simplicity), the energy of a triplon with spin projection $S^z = 0, \pm 1$ to first order in J_1 is

$$\varepsilon(\mathbf{k}) = J_0 + J_1[\cos(k_x a) + \cos(k_y a)] - g\mu_B H S^z, \quad (2)$$

where $\mathbf{k} = (k_x, k_y)$ is the quasiparticle wavevector, a is the lattice constant and $D = 4J_1$ is the bandwidth; see Fig. 1c. The energy–momentum dependence (dispersion) of the triplons and singlet–triplet correlations have been measured directly by inelastic neutron scattering[25,34,35,41].

The Zeeman term $-g\mu_B H S^z$ controls the density of triplons. As the magnetic field increases, the excitation energy of triplons with $S^z = +1$ is lowered and eventually crosses zero, as shown in Figs 1c and 2a. This defines two critical magnetic fields H_{c1} and H_{c2} in the phase diagram; see Fig. 1d. At zero temperature, below H_{c1} the magnetization $m_z(H)$ is zero and only singlets exist. Between H_{c1} and H_{c2} the magnetization increases with increasing field as more triplons appear in the ground state owing to the increased gain in Zeeman energy; see Fig. 1c. Above H_{c2} each site is occupied by a triplon and the magnetization saturates at one per dimer.

The bosonic nature of triplons is guaranteed by the simple fact that spin operators of two different dimers commute. However, because a dimer can hold at most one triplon, the bosonic picture requires the introduction of a hard-core constraint to exclude states with more than one quasiparticle per dimer. The constraint, which can be interpreted as a strong short-range repulsion between the bosons, poses a difficult theoretical problem. However, close to H_{c1} their density is small and collisions between bosons are rare; in this limit interactions can be fully taken into account. By particle–hole symmetry, a similar simplification occurs near H_{c2}.

BEC OF TRIPLONS

The nature of the ground state above H_{c1}, its interpretation as a BEC of magnetic quasiparticles and the tuning of the particle density become more accessible if the spin hamiltonian in equation (1) is rewritten in the second-quantized form[7,8]

$$\mathcal{H}=\sum_{i}(J_0-h)a_i^\dagger a_i+\sum_{i,j}t_{ij}a_i^\dagger a_j+\frac{1}{2}\sum_{i,j}U_{ij}a_i^\dagger a_j^\dagger a_j a_i \quad (3)$$

where $a_i^\dagger$ (a_i) creates (annihilates) a boson on dimer i and $h=g\mu_B H$ is the field in energy units. The term t_{ij}, generated by the coupling of the transverse spin components $S^x_{m,i}S^x_{n,j}+S^y_{m,i}S^y_{n,j}$, describes hopping between sites i and j, thus endowing the triplons with kinetic energy; see Fig. 1b. U_{ij} is the repulsion energy arising from the longitudinal term $S^z_{m,i}S^z_{n,j}$ when two triplons occupy neighbouring dimers i and j. The $O(2)$ symmetry of the spin hamiltonian (1) translates into a $U(1)$ symmetry of the boson hamiltonian (3) and the density of triplons is directly controlled by the applied magnetic field, which acts as a chemical potential. (We refer those purists who object to the interpretation of the magnetic field as a chemical potential to Shakespeare's *Romeo and Juliet* II.2.43: "What's in a name? That which we call a rose By any other word would smell as sweet".)

Below H_{c1} the ground state is a quantum-disordered paramagnet formed by the singlet sea (triplon vacuum) and can be approximated by the direct product of singlet states on each dimer, $|\psi\rangle_i=|S,S^z\rangle_i$ with $S=S^z=0$. Once the spin gap is closed at H_{c1}, a Bose condensate is formed. Because the bottom of the triplon band is located at a non-zero wavevector $\mathbf{k}_0=(\pi/a,\pi/a)$ (Fig. 1c), the wavefunction of the condensate varies in space as $\exp(i\mathbf{k}_0\cdot\mathbf{r})$. In this phase the state of an individual dimer at position $\mathbf{r}$ is well approximated by a coherent superposition of the singlet and the $S^z=+1$ triplet: $|\psi\rangle_i=\alpha_i(H)|0,0\rangle_i+\beta_i(H)|1,1\rangle_i$, where the amplitudes α_i and β_i depend on the magnetic field H (refs 7,11,12).

In spin language, the condensate corresponds to magnetic order, formed by the transverse components $\langle S_i^x\rangle$ and $\langle S_i^y\rangle$, which spontaneously breaks the rotational $O(2)$ symmetry of the spin hamiltonian (1). To make the analogy with the traditional BEC manifest, we can form a $U(1)$ order parameter $\langle S_i^x+iS_i^y\rangle$. The phase corresponding to the angle of the spin in the XY plane is thus the phase of the wavefunction in boson language. At H_{c1} the paramagnetic phase at low fields makes a transition into a canted antiferromagnet with long-range magnetic order in the plane perpendicular to the field[7] (Figs 1d, 3a). The staggering of the transverse components of magnetization reflects a non-zero wavevector $\mathbf{k}_0$ of the condensate. The critical properties of the magnet in the vicinity of this phase transition are governed by the quantum critical point (QCP) of the BEC universality class located at $T=0$ and $H=H_{c1}$. Indeed, close to H_{c1} the bosons are extremely diluted, weakening the effects of the hard-core interactions. Close to the QCP, the phase boundary $T_c(H)$ follows[7] a power law $T_c\propto(H-H_{c1})^\phi$ with a universal critical exponent $\phi=z/d$, which depends only on the dimensionality d and dynamical critical exponent $z=2$ for a quadratic triplon energy band. The upper critical dimension for the QCP is $d_c=2$.

The BEC quantum phase transition has been observed in a growing number of dimer-based magnetic insulators, such as $ACuCl_3$ (A = Tl, K, NH_4), $BaCuSi_2O_6$, $Cu(NO_3)_2\cdot2.5D_2O$, $Cs_3Cr_2Br_9$, $(CH_3)_2CHNH_3CuCl_3$ and $(C_4H_{12}N_2)Cu_2Cl_6$ (refs 8,14, 36,37,40,42–49). The physics of triplon condensation, presented above for a weakly coupled dimer system, remains essentially unchanged, because these QCPs are in the same universality class. Quasi-one-dimensional arrangements of $S=1$ moments in spin chains, for example Haldane chains, and even field-saturated frustrated antiferromagnets can be described within the same framework, as has been successfully done for nickel-based materials and Cs_2CuCl_4, respectively[50–54].

Table 1 Correspondence between a Bose gas and a quantum antiferromagnet.

Bose gas	Antiferromagnet
Particles	Spin excitations ($S=1$ quasiparticles)
Boson number N	Spin component S^z
Charge conservation $U(1)$	Rotational invariance $O(2)$
Condensate wavefunction $\langle\psi(\mathbf{r})\rangle$	Transverse magnetic order $\langle S_i^x+iS_i^y\rangle$
Chemical potential μ	Magnetic field H
Superfluid density ρ_s	Transverse spin stiffness
Mott insulating state	Magnetization plateau

Experimentally, the static and dynamic magnetic properties have been studied at and around the BEC QCPs in such materials by many experimental techniques. The evolution of magnetic excitations across the QCP at H_{c1} in a three-dimensional network of dimers was studied in $TlCuCl_3$; see Figs 1c, 2a,c[11,12,14,26]. The softening of the triplet mode $S^z=+1$ in the quantum disordered phase, Fig. 2a, is followed by a dramatic change in the nature of the excitation spectrum above the critical field H_{c1}. In particular, as can be expected in a system with a spontaneously broken XY symmetry in the plane perpendicular to the applied magnetic field, a Goldstone mode appears with a soundlike dispersion $\hbar\omega(\mathbf{k})\sim\hbar s|\mathbf{k}-\mathbf{k}_0|$, where s is a velocity (refs 11,12).

Historically, the temperature dependence of the magnetization $m_z(T)$ at fixed field $H>H_{c1}$ in $TlCuCl_3$ (Fig. 2d) and the scaling of the critical temperature $T_c(H)$ were the key experimental observations that confirmed the theoretical model[7,8] of the magnon BEC in dimerized quantum antiferromagnets. The minimum and cusp in $m_z(T)$ at the finite-temperature transition cannot be explained within simple mean-field theory[55], but are a consequence of magnon condensation[7,8] (and as such also occur in lower-dimensional magnets). Even in the one-dimensional case, where the condensate is disordered by thermal and quantum fluctuations, similar minima in the magnetization can occur, but now as a simple crossover[56]. Comprehensive data from bulk measurements establish the phase diagram and critical exponents for $BaCuSi_2O_6$ (Fig. 2b). The temperature crossover exponent is found to be $\phi=2/3$, as expected for a BEC QCP in three dimensions. However, below 1 K and down to 35 mK the phase boundary unexpectedly becomes linear, indicating[37–40] that $\phi=1$. We discuss this anomaly below.

COMPARISON WITH OTHER BOSON SYSTEMS

The correspondence between a Bose gas and an antiferromagnet is summarized in Table 1. Despite many similarities, there are also a few important differences between triplons in a magnet and atoms in a Bose gas[21–24]. Most importantly, the number of atoms is usually controlled directly (microcanonical ensemble), whereas the number of triplons is typically set by the magnetic field acting like a chemical potential (canonical ensemble).

Considerable differences also exist at the practical level. Triplons are much lighter and have a much higher density than atomic gases. As a result, condensates survive to much higher temperatures: kelvins in magnets, even room temperature in some cases[15], as opposed to nanokelvins in atomic gases. As solid-state systems, they also allow for a variety of static and dynamic probes (magnetization, specific heat, NMR, neutrons and so on). On the other hand, the cold atomic systems enable us potentially to study some out-of-equilibrium situations that are hard to maintain in a solid-state device because the condensate is still strongly coupled to the dissipative environment.

From the point of view of realization, systems with cold atoms have a high degree of control and tunability in terms of the interactions and structure. Indeed, the structure of the lattice and

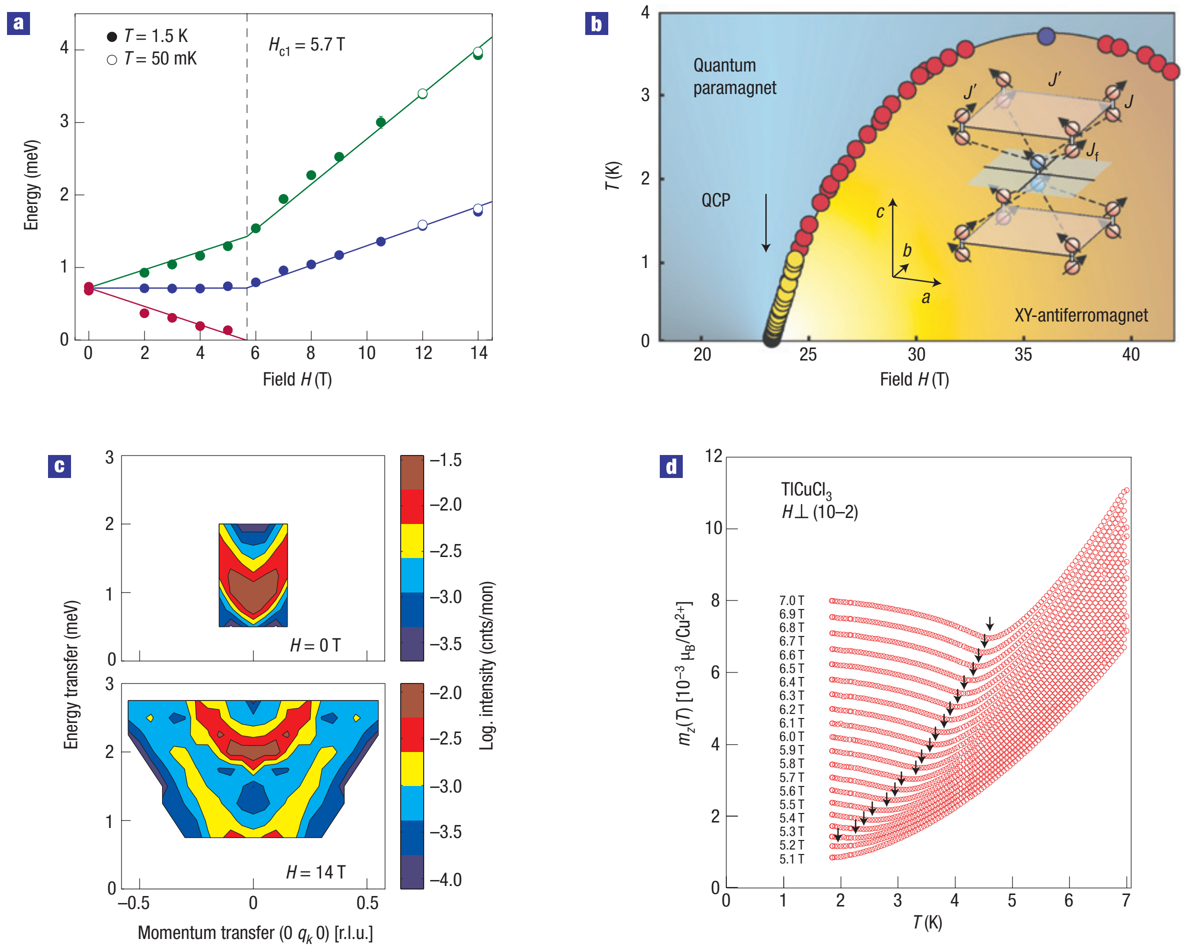

Figure 2 Experimental results on the magnon BEC. **a**, Zeeman splitting of the triplet modes in $TlCuCl_3$ up to $H > H_{c1}$ measured by inelastic neutron scattering[14,31,32]. **b**, Phase diagram of $BaCuSi_2O_6$ measured by torque magnetization, magneto-caloric effect and specific heat[36,37]. Dimensional reduction was reported in this material, with a crossover from the three-dimensional BEC critical exponent $\phi = 2/3$ to $\phi = 1$ for two dimensions at temperatures close to the QCP[37]. **c**, Excitations in the BEC of triplons realized in $TlCuCl_3$ (refs 14,31,32). Goldstone mode with linear dispersion around $\mathbf{k}_0$, corresponding to $q_k = 0$ in the notation of ref. 14. Spin anisotropy generally leads to a spin gap in real materials[31–33]. **d**, Temperature dependence of the magnetization $m_z(T)$ in $TlCuCl_3$ for fixed magnetic field H, as indicated[8]. Minima at the finite-temperature phase transition (vertical arrows), as expected for a BEC of triplons.

the hopping of the bosons can directly be controlled by varying the strength of an optical lattice. The interaction can also be controlled via a Feshbach resonance, enabling us in principle to design our pet hamiltonian. In spin systems the parameters can be changed only in a limited way by the application of pressure or by changing the chemical composition. Efforts in quantum chemistry have produced good realizations of three-, two- and one-dimensional dimer materials, for example $TlCuCl_3$, $BaCuSi_2O_6$ and $(C_5H_{12}N)_2CuBr_4$, respectively. In addition, by the very nature of the mapping from spins to bosons, spin systems offer a definite advantage in reaching the limit of strong on-site repulsion as well as the effects of interactions between nearest neighbours (see the next section). Spin systems are therefore an optimal starting point to study situations for which these ingredients are important.

The cold atomic systems have the important advantage that the phase $U(1)$ symmetry is exact. In the magnets, the corresponding $O(2)$ symmetry in the plane perpendicular to the magnetic field can be broken by weak anisotropic interactions (crystalline anisotropies, spin–orbit coupling and dipolar interactions). As a matter of fact, they may always be present in real magnetic materials at some, preferably low, energy scale (compared with interactions J_{mnij})[31,57,58], or it may be challenging experimentally to apply the magnetic field exactly along a symmetry direction. Even when symmetry-breaking terms are weak, they become important at low temperatures and modify the physics in the vicinity of the QCP. Although such additional terms in the spin hamiltonian should be part of any detailed experimental characterization, many of the currently studied magnetic materials provide a clean experimental window where critical exponents, effects of dimensionality and so on can be investigated in the anticipated physical limits[8,37,38,57]. The physics of BEC may also be altered in the presence of coupling between spins and lattice distortions. Fortunately, a thorough understanding of BEC in $O(2)$-symmetric magnets enables us to treat theoretically small deviations from the idealized case[30,32,33,59–62].

Finally, these spin systems provide an excellent opportunity to study the critical behaviour in the immediate vicinity of a QCP because the high degree of homogeneity in boson density offered by crystalline solids is difficult to attain in the presence of a trap potential confining an atomic gas.

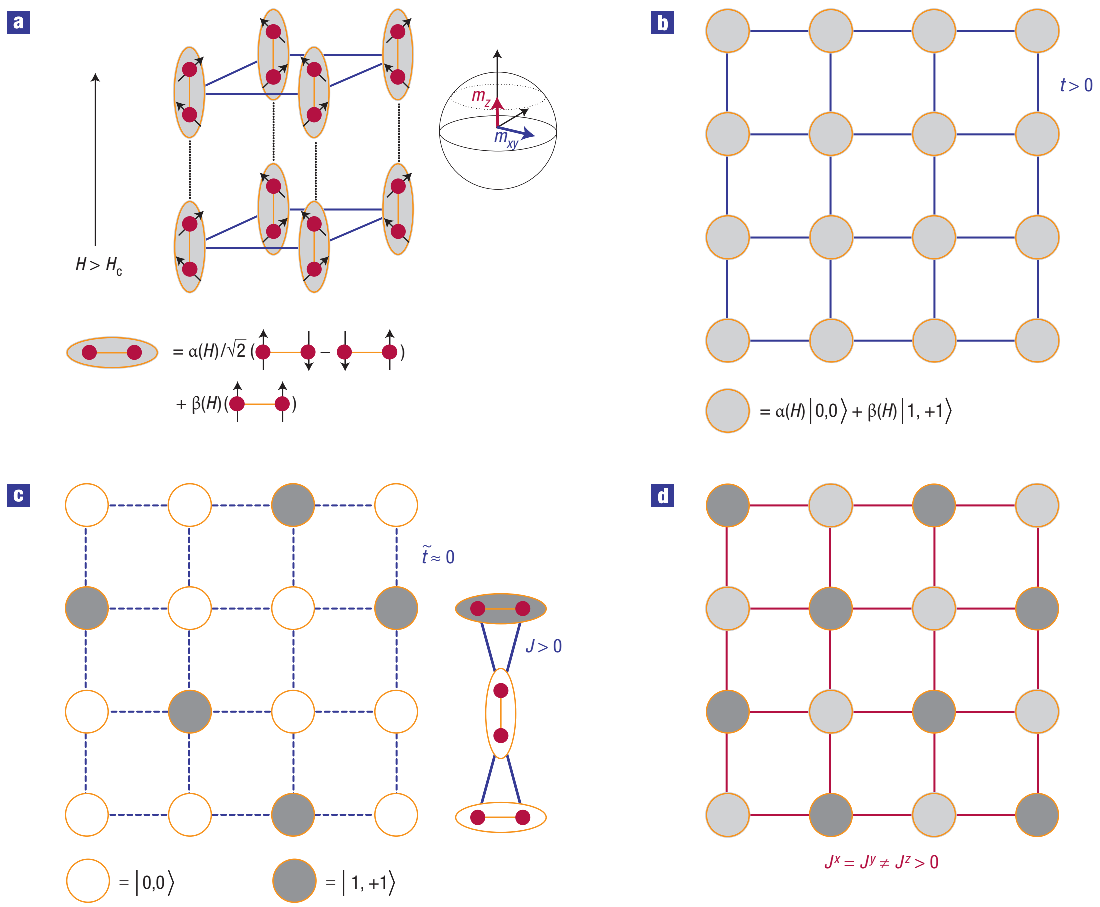

Figure 3 Lattice boson phases. **a**, Canted antiferromagnetic state at $H_{c1} < H < H_{c2}$ in a BEC of magnons. Longitudinal m_z (see Fig. 1d) and transverse (XY) magnetization m_{xy}. **b**, Triplon condensate in the superfluid phase. The hopping amplitude t of the quasiparticles leads to a uniform mixture of singlet and triplet states. **c**, Analogue of the Mott insulating phase for triplons with a magnetization plateau at $m_z = 1/3$: triplons 'crystallize' to form a superstructure. This phase is observed in $SrCu_2(BO_3)_2$, where quasiparticle hopping is strongly suppressed by geometrical frustration[10,67,68,70]. **d**, Supersolid on a square lattice of dimers with exchange anisotropy as proposed theoretically[89,91], and for other lattice geometries[86–88,90]. Both translation and $U(1)$ symmetry are spontaneously broken, that is, the superfluid and 'crystal' of lattice bosons coexist.

BEYOND SIMPLE BEC

The interplay of an atomic lattice and strong boson interactions leads to new phenomena beyond the simple BEC paradigm. When the boson density approaches a value commensurate with the lattice periodicity, we often find magnetization plateaux where the triplon density stays constant in a finite range of magnetic fields at $T = 0$. We have already discussed two simple examples of such plateaux for $H < H_{c1}$ with zero magnetization and for $H > H_{c2}$ where each site is occupied by one triplon (Fig. 1d). These simple $m_z = 0$ or 1 plateaux correspond to states with an integer number of bosons per unit cell of the lattice. More generally, because in spin systems the $S^z_{m,i}S^z_{n,j}$ terms in the hamiltonian (1) translate into triplon repulsion, magnetization plateaux at fractional fillings can also occur[63,64]. Such intermediate plateaux arise for strong enough repulsion between spin excitations on adjacent lattice sites and beyond, and correspond in boson language to Mott-insulator phases[10,65,66]. In the simplest case (square lattice, repulsive nearest-neighbour interactions) triplons may form an incompressible state by 'crystallizing' in a superstructure pattern (Fig. 3c). Unlike the ground state of an integer magnetization plateau (Fig. 3b), such a state violates the translational symmetry of the lattice, and the spontaneous symmetry breaking is characterized by a discrete order parameter. Fractional plateaux require strong magnon interactions (in comparison to the kinetic energy) and are therefore less robust then integer ones. Nonetheless, plateaux at magnon fillings 1/3, 1/4 and 1/8, and recently for other fractions have been observed in $SrCu_2(BO_3)_2$ (refs 67–69). The kinetic energy in this material is suppressed by geometrical frustration and magnon repulsion is therefore more pronounced[70].

Outside the plateaux, magnons can be easily added to the condensate, ensuring a continuously varying magnetization $m_z(H)$. This state has transverse magnetic order violating the continuous $O(2)$ symmetry, is therefore gapless and corresponds to the superfluid phase of bosons. Its excitations are Goldstone modes, characteristics of this superfluid phase[11,14]. The incompressible plateau state and the gapless state with a continuously varying magnetization are again separated by a QCP.

Although the dimensionality of an antiferromagnet is determined mostly by quantum chemistry, it may also change on the fly as the magnet is cooled down. The 'Han purple' $BaCuSi_2O_6$ contains layers of dimers with finite exchange couplings between the layers. Yet the critical behaviour shows two-dimensional exponents and an interesting crossover close to the QCP[37,40]. Attempts to interpret these surprising facts were made on the basis of ideas that the interplane triplon hopping is suppressed by geometrical frustration and vanishes exactly at the wavevector of the condensate[37,38]. This would lead to a dimensional reduction at low temperature. An alternative explanation points to the existence of inequivalent layers with different values of the spin gap as the cause of the two-dimensional behaviour[35,39,40].

Another spectacular example of reduced dimensionality is provided by $(C_5H_{12}N)_2CuBr_4$ (refs 71,72) and related organometallic compounds. These materials can be thought of as a collection of chains of dimers nearly decoupled from one another. As a result, a large part of the H–T phase diagram can be understood by starting in the one-dimensional limit[7,73–77]. At low energies the triplons on a single chain form a Luttinger liquid[78]. Such a compound is thus particularly well suited for studying the dimensional crossover between one and three dimensions. In the former case, spin excitations behave as fermions, as described by the Jordan–Wigner transformation; in the latter, the excitations act like 'regular' bosons[79]. The excellent degree of control over the density by the magnetic field as well as the various probes providing access to the dynamical spin–spin correlations make quasi-one-dimensional magnets ideal for studying the dimensional crossover, a phenomenon of general importance for several major areas in physics[78].

We would like to mention the tantalizing possibility of finding an exotic phase known as the supersolid[80]. A hybrid of a solid and a superfluid, this phase violates both the translational symmetry by forming a density wave and the $U(1)$ phase symmetry by showing transverse magnetic order (Fig. 3d). It was conjectured long ago[81–83] and its existence is under intense debate at the moment following experiments in helium[84,85]. Such a phase could occur at the end of a fractional plateau because the transition must accomplish two tasks: create a superfluid and destroy a solid (that is, restore the broken translational symmetry). According to Landau's theory of phase transitions, such transitions are either discontinuous or happen in two stages with an intervening, possibly supersolid, phase. Several theoretical calculations show that supersolids exist in realistic magnetic models[86–91]. The observation of such a phase would thus be an interesting realization of the physics of interacting bosons on a lattice.

Last but not least, advances in growth techniques may enable us in the future to study in detail the effects of disorder on bosons[66,92]. Here Bose glasses, which can be made by adding bond disorder or site dilution, are of current interest[66,93–96], as is the influence of defects on the quantum critical states[97,98]. Other directions include thermodynamics of strongly interacting bosons[41,99] and quantum coherence at the mesoscopic scale[100].

CONCLUSIONS AND OUTLOOK

Quantum spins and quantum dimer systems offer remarkable opportunities to study phenomena related to the Bose–Einstein condensation of interacting quantum particles. The high density and low mass of spin excitations, magnons or triplons, leads to robust condensates, which survive to temperatures as high as 10 K. The availability of numerous model magnets and physical probes has enabled a detailed study of the critical phenomena and magnetic properties in the vicinity of the BEC quantum phase transition. Spin systems have thus proven to be nicely complementary to other systems, such as cold atomic gases, for the investigation of the BEC. The success of this approach and present state of knowledge in the field is the result of a collective effort across several areas of expertise, including materials science and chemistry, where high-quality samples are prepared.

In addition to providing a direct connection with the remarkable BEC phenomenon, the boson picture offers an intuitive understanding for complex quantum properties of matter that are much less physically transparent in the original spin language and provides access to a cornucopia of exciting new problems and questions. For example, a classical description of the dimerized magnetic materials in the high-field ordered phase as canted antiferromagnets misses entirely the relevant spin physics close to the quantum phase transition. The transverse order vanishes at the QCP and all localized magnetic moments are in a highly fluctuating, perfectly entangled quantum-mechanical state, which finds a clear-cut and intuitive physical description in boson language. The possibility to fine-tune the density of bosons by using the magnetic field provides a tool to determine the role of the interactions in such systems. A wide range of different lattice geometries and dimensionalities is available for further study. In combination with the large number of experimental probes for spins, this enables us to enter the world of exotic phases of strongly interacting quantum particles such as the BEC in various dimensions, Luttinger-liquid physics, commensurate solids with a fractional number of bosons per unit cell and supersolids combining superfluidity with a broken translational symmetry.

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