Superfluid ^{4}He

- can flow without dissipation at least partially

"Two-fluid" scenario

mass density
$$\rho^{(\vec r)} = \rho_s(\vec r, T) + \rho_n(\vec r, T)$$

superfluid density

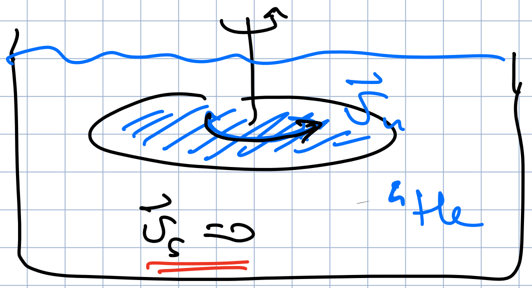
normal

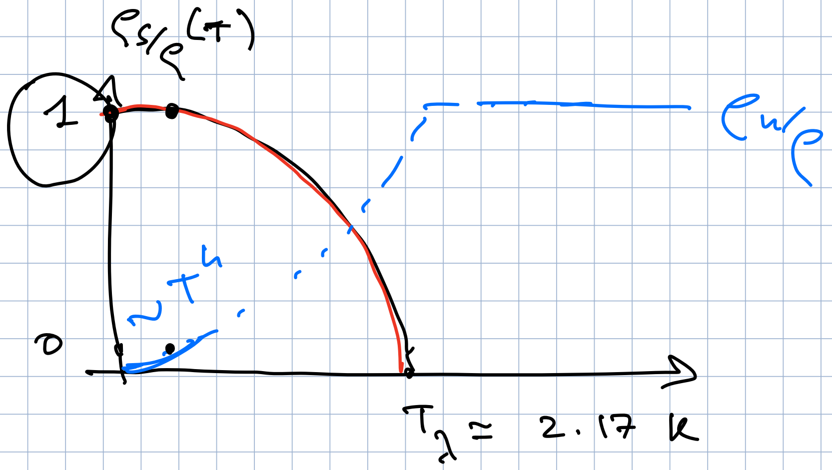
$$\frac{\rho_s}{\rho} = \text{superfluid fraction}$$

mass current
$$\vec j(\vec r)^{\,t} = \underbrace{\rho_s(\vec r, T)\,\vec v_s(\vec r)}_{\vec\nabla \times \vec v_s = 0} + \underbrace{\rho_n(\vec r, T)\,\vec v_n(\vec r)^{\,t}}_{\text{normal current}}$$

normal current : comes to equilibrium with / follows the boundaries of the container

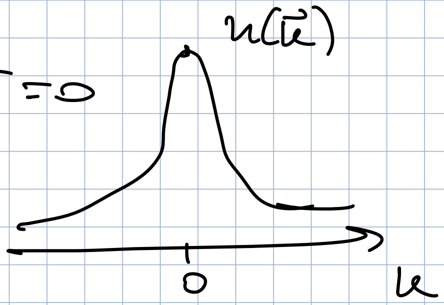
Andronikashvili experiment





Bose-Einstein condensation ?

Condensate fraction $\boxed{\frac{n_0}{n} \lesssim 7\%}$ @ $T=0$



Landau criterion for superfluidity

$$\underbrace{H - \overset{\mu=0}{\mu N} = \sum_{\vec p} \epsilon(\vec p)\, a^\dagger_{\vec p} a_{\vec p}} + \dots$$

free bosonic quasi particles

@ $T=0$

$\underline{\underline{\vec v}} = \vec v_s$

$$v \le v_c = \min_p \frac{\epsilon(p)}{p}$$

@ $T=0$ all the atoms participate in the superflow = dissipationless flow

$$\rho_s/\rho = 100\,\%$$

What if $T>0$?

@ $T=0$ (Bogolyubov theory)

vacuum of quasi-particles

[**Note**: ideal Bose gas @ $T=0$ is 100% superfluid but with vanishing $\underline{v_c}$.]

All BEC are superfluid

BEC $\Rightarrow$ superfluidity

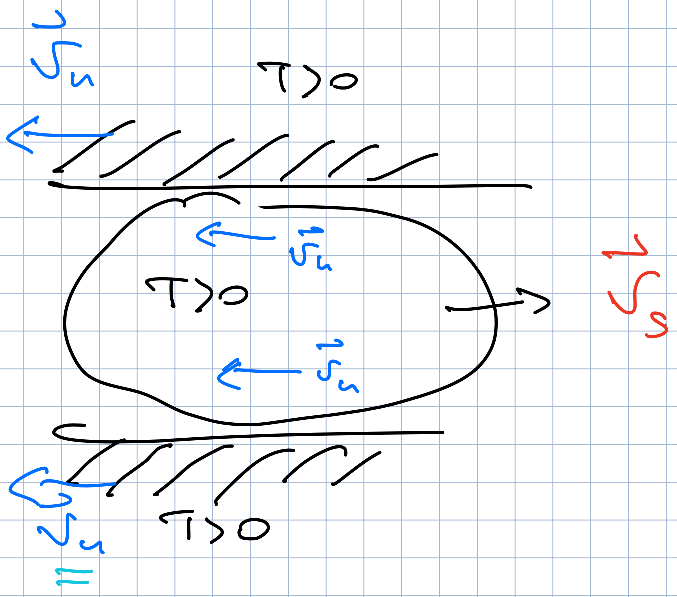
BEC $\not\Leftarrow$ superfluid

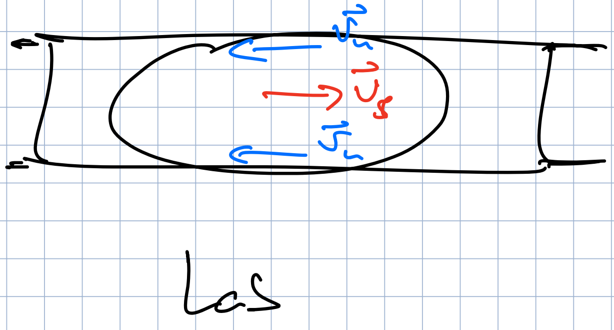
quasi-BEC $\Leftarrow$ superfluid

$N_0 \sim O(N^{\alpha})$

$\alpha \leq 1$

Idea: since @ $T=0$ there are NO quasiparticle excitations and NO normal fraction





Thermal equilibrium: can only be defined in the reference frame of the boundary walls because it is the only ref. frame without time-dependent forces

In the superfluid frame (moving @ $\vec{v}_s$)

$\epsilon(\vec{p})$ disp. relation , $\vec{p}$ momentum

In the boundary frame (moving @ $\vec{v}_n$)

$$\epsilon'(\vec{p}) = \epsilon(\vec{p}) + \vec{p}\cdot(\vec{v}_s - \vec{v}_n) \quad \leftarrow$$

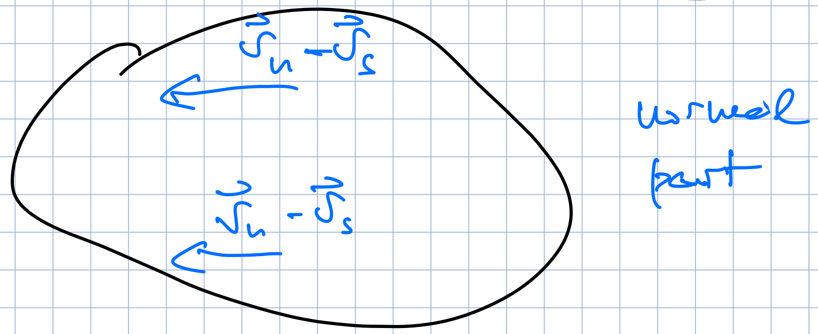
where thermal equilibrium can be defined

calculate the current density of quasiparticles

$$\underset{\substack{\uparrow \\ \text{mass current density}}}{[\vec{j}]} = \left[\frac{m t^{-1}}{\ell^2}\right] = \underset{\text{density of momentum}}{\left[\frac{p}{V}\right]} = \left[\frac{m \ell t^{-1}}{\ell^3}\right]$$

normal component @ $T > 0$: mass current at velocity $\vec{v}_n - \vec{v}_s$ in the frame of the superfluid

in the superfluid ref. frame



if there is a current moving with velocity $\vec{v}_n - \vec{v}_s$ in the superfluid frame $\Rightarrow$ it comes from the normal part

Current density in the $\boxed{\text{superfluid frame}}$

$$\underline{\underline{\vec{J}}} = \frac{1}{V} \underbrace{\sum_{\vec{p}} \vec{p}\; n(\vec{p})}_{\text{total momentum of quasi-particles}}$$

$$\vec{J} = \rho_n(\vec{v}_n - \vec{v}_s)$$

$$\rho_s\,\cancel{(\vec{v}_s - \vec{v}_s)}$$

$$= \frac{1}{V}\sum_{\vec{p}}\; \vec{p}\; \frac{1}{e^{\beta\, \epsilon'(\vec{p})} - 1}$$

→ thermal equilibrium in the boundary frame

$$= \rho_n (\vec{v}_n - \vec{v}_s) \neq 0$$

$$\approx \int \frac{d^3p}{(2\pi\hbar)^3}\; \frac{\vec{p}}{e^{\beta(\epsilon(\vec{p}) + \underbrace{\vec{p}\cdot(\vec{v}_s - \vec{v}_n)}_{\text{perturbation to } \epsilon(\vec{p})})} - 1}$$

$$\frac{1}{e^{\beta(\epsilon+\delta)} - 1} \simeq \frac{1}{e^{\beta\epsilon} - 1} - \frac{\beta\, \delta\, e^{\beta\epsilon}}{(e^{\beta\epsilon} - 1)^2} + \cdots$$

$$\simeq \int \frac{d^3p}{(2\pi\hbar)^3}\; \vec{p} \left[\overset{0}{\cancel{\frac{1}{e^{\beta\epsilon(\vec{p})} - 1}}} + \beta\, \vec{p}\cdot \overset{\vec{v}}{(\vec{v}_n - \vec{v}_s)}\; \underbrace{\frac{e^{\beta\epsilon(\vec{p})}}{(e^{\beta\epsilon(\vec{p})} - 1)^2}}_{f(p)} + \cdots \right]$$

$$\epsilon(\vec{p}) = \epsilon(p)$$

$$\int d^3p\ \vec{p}_x(\vec{p}\cdot\vec{v})f(p) = \int d^3p\ \vec{p}_x\left(\underbrace{p_x v_x + p_y}\cancel{v_y} + \cancel{p_z v_z}\right) f(p)$$

$$= \vec{v}_x \int d^3p\ \boxed{p_x^2}\ f(p)$$

$$= \frac{\vec{v}_x}{3} \int d^3p\ \underline{p^2}\ f(p)$$

$$\vec{J}_n \simeq (\vec{v}_n - \vec{v}_s)\cdot \underbrace{\frac{1}{3}\int \frac{d^3p}{(2\pi\hbar)^3}\ \beta p^2\ \frac{e^{\beta\epsilon(p)}}{\left(e^{\beta\epsilon(p)}-2\right)^2}}_{\rho_n(T)}$$

$\epsilon(p)$?

$$\epsilon(p) \underset{p\to 0}{\simeq} cp$$

$$\rho_n(T) = \frac{1}{3}\ \frac{4\pi}{(2\pi\hbar)^3}\ \beta \int_0^\infty dp\ \frac{p^4 e^{\beta cp}}{\left(e^{\beta cp}-1\right)^2}$$

$$x = \beta cp$$

$$= \frac{1}{3}\ \frac{4\pi}{(2\pi\hbar)^3}\ \beta\ \frac{1}{(\beta c)^5} \underbrace{\int_0^\infty dx\ \frac{x^4 e^x}{\left(e^x-1\right)^2}}_{\frac{4\pi^2}{15}}$$

$$\cdot \sim \; = \frac{2\pi^2 (k_B T)^4}{45\, \hbar^3 \underset{\uparrow}{\boxed{c}}^{\,3}} \sim T^4$$

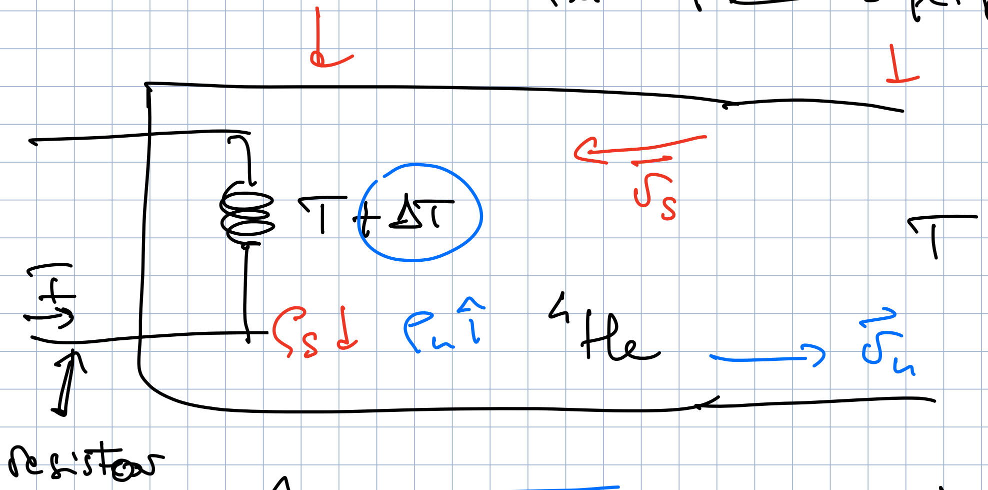
Two-fluid Hydrodynamics (^{4}He)

→ $\vec{v}_s$ is an independent thermodynamic property (T, p, ...) because the superfluid decouples mechanically from the boundaries

→ $\vec{v}_s$ is irrotational $\quad \vec{\nabla} \times \vec{v}_s = 0$

→ the superfluid component is a $T=0$ Bose fluid carrying no entropy

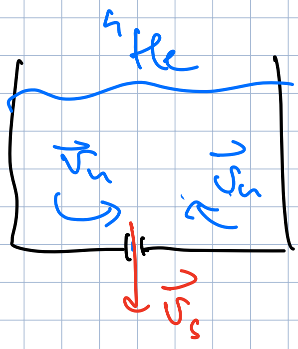
Second sound: wave-like propagation of heat / entropy in the superfluid



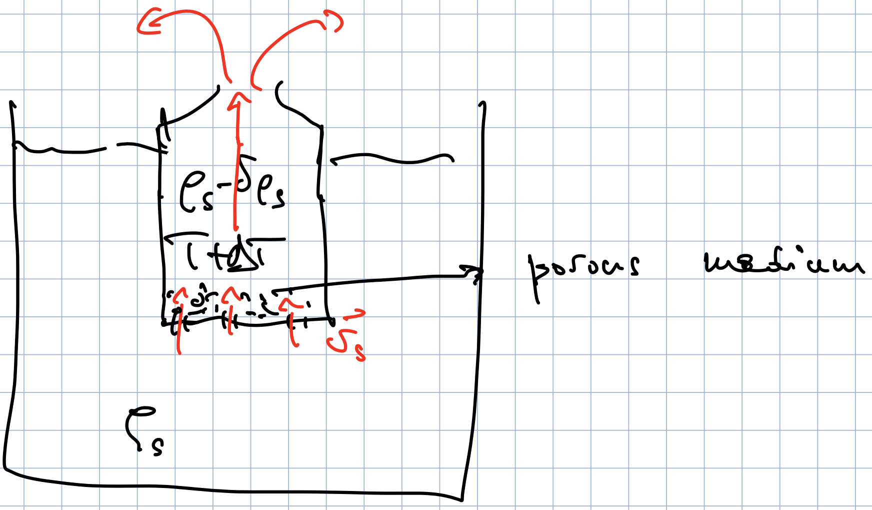
entropy
heat transport without mass transport

and ballistic

Super leaks :



Fountain effect

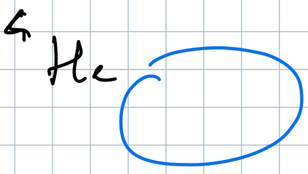


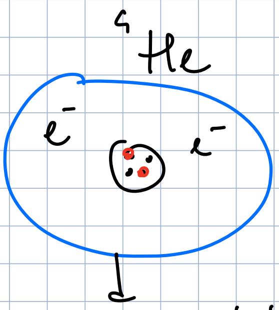
FERMIONS

-> normal state : Fermi liquid

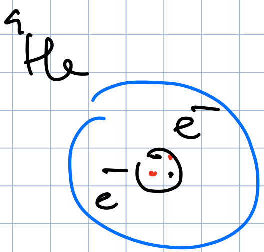
superconducting state

"fermionic condensation", "fermionic superfluidity"

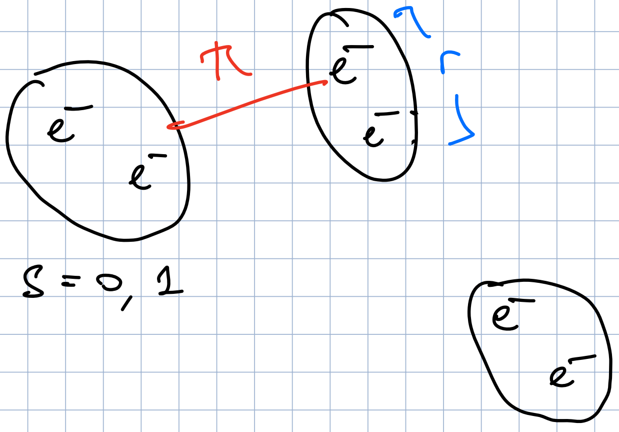




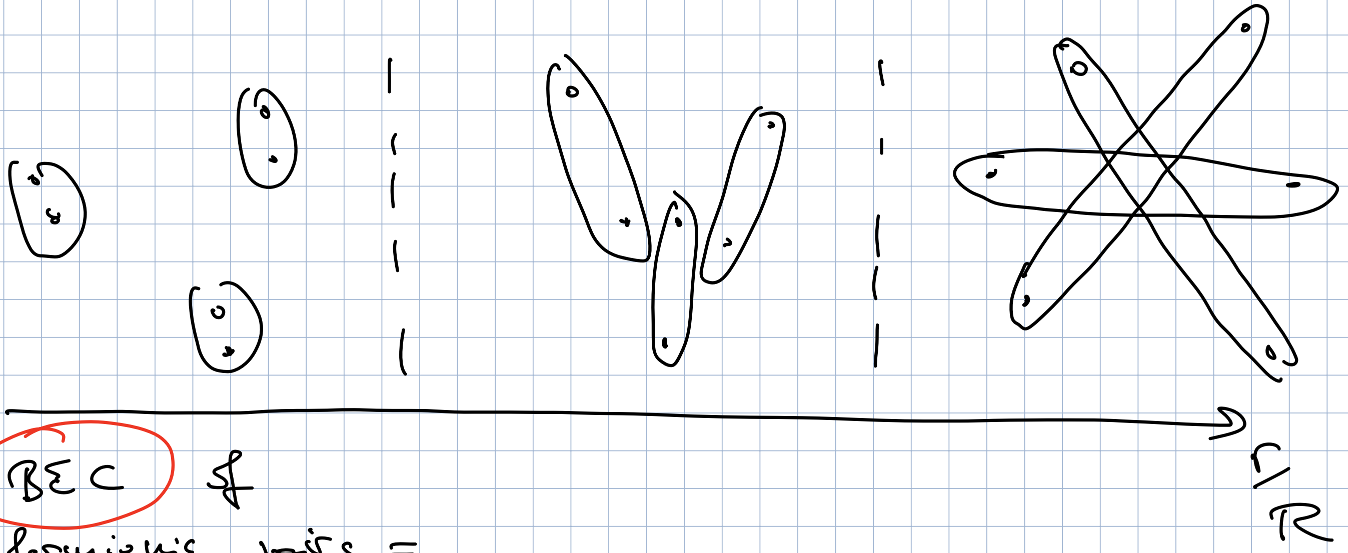
boson : total spin $S=0$



$R \gg r$



Condensate of pairs



BEC of fermionic pairs = composite bosons

Condensate ?

BCS (Bardeen-Cooper-Schrieffer) regime

Crossover

✓

conventional superconductors

? (✓)

Take into account the fermionic nature of the constituents

Fermi gas : ideal

$|\psi_\alpha\rangle$ single particle states, ϵ_α eigenenergies

$n_\alpha = 0, 1$ Pauli exclusion principle

μ = chemical potential

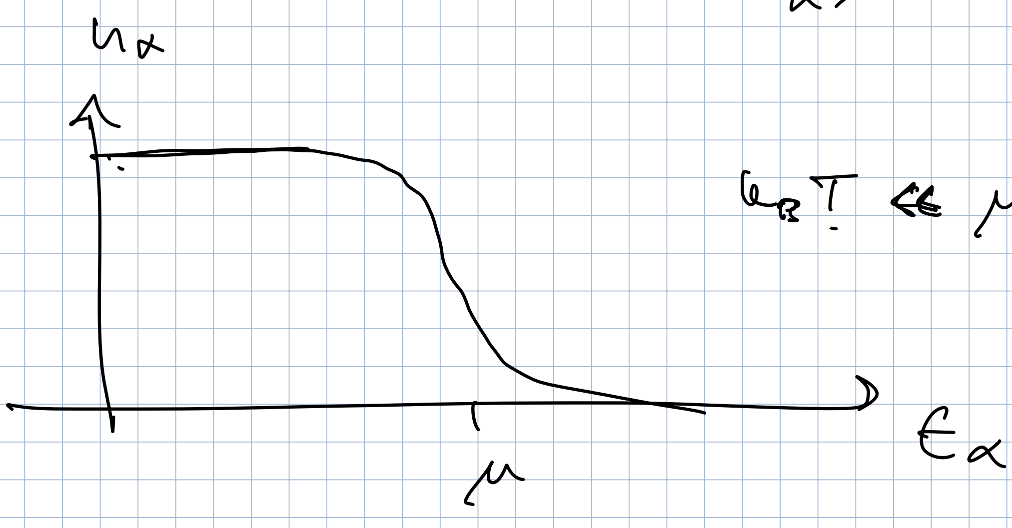
$$Z_G = \prod_\alpha \left(\sum_{n_\alpha = 0,1} e^{-\beta \overbrace{(\epsilon_\alpha - \mu) n_\alpha}} \right)$$

$$= \prod_\alpha \left[1 + e^{-\beta(\epsilon_\alpha - \mu)} \right]$$

$$\Omega_G = - k_B T \log Z_G = - k_B T \sum_\alpha \log\left(1 + e^{-\beta(\epsilon_\alpha - \mu)}\right)$$

$$N = - \frac{\partial \Omega}{\partial \mu} = \ldots = \sum_\alpha \boxed{\frac{1}{e^{\beta(\epsilon_\alpha - \mu)} + 1}}$$

$\langle n_\alpha \rangle$ = Fermi-Dirac distribution

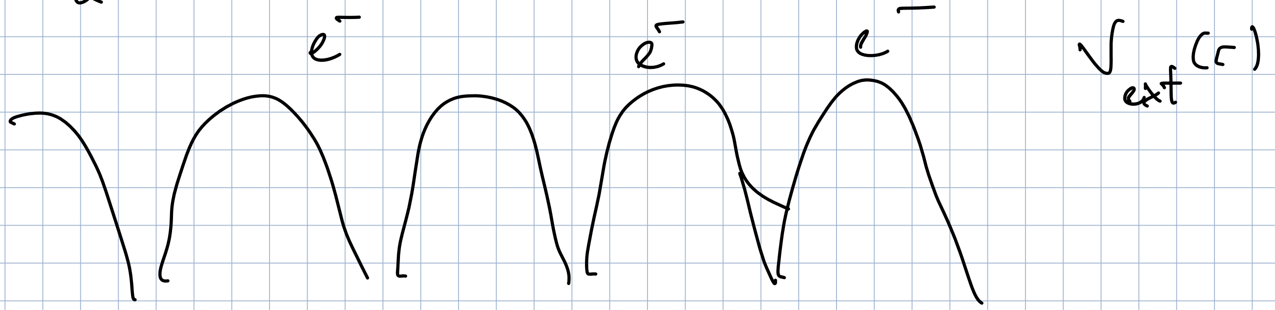


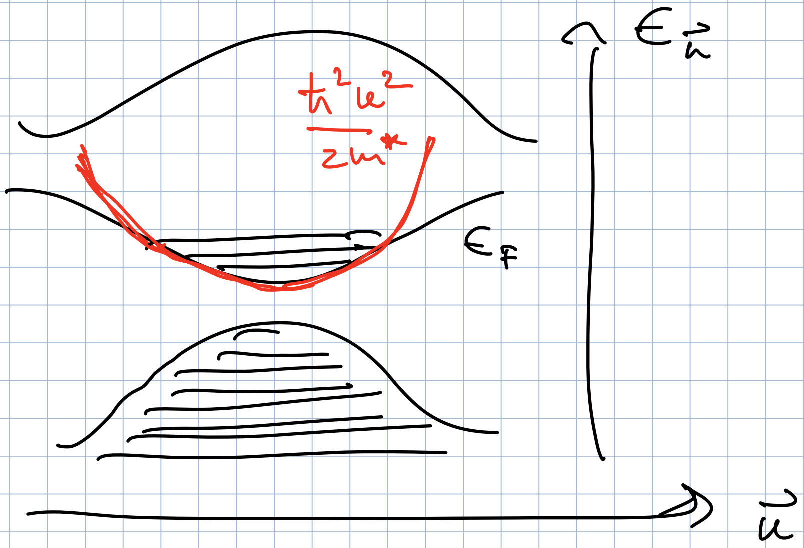
$k_B T \ll \mu$

Ideal Fermi gas → electron gas in a metal

1) lattice ✓

ϵ_α ?





$T = 0$

$\mu = \epsilon_F$

2) Interaction between electrons & lattice vibrations (phonons)

electrons & electrons