Superconductivity, Superfluidity & Magnetism

Systems of N identical quantum particles

$N \gg 1$

indistinguishable

electrons (fermions)

atoms

bosonic (integer spin)

ex: ^4He spin-0

fermionic (half-integer spin)

ex. ^3He spin-$\frac{1}{2}$

Distinguishable particles

$$|\Psi_N\rangle = \underset{1}{|\phi_1\rangle} \otimes \underset{2}{|\phi_2\rangle} \otimes \dots \otimes \underset{N}{|\phi_N\rangle}$$

$|\phi_\alpha\rangle \in H^{(1)}$ single particle Hilbert space

$|\Psi_N\rangle \in H^{(N)}$ N-particle Hilbert space

$$H^{(N)} = H^{(1)} \otimes H^{(1)} \otimes \dots \otimes H^{(1)}$$

most general

$$|\Psi_N\rangle = \sum_{\alpha_1 \alpha_2 \dots \alpha_N} c_{\alpha_1 \alpha_2 \dots \alpha_N} |\phi_{\alpha_1}\rangle \otimes |\phi_{\alpha_2}\rangle \otimes$$

$\ldots | \phi_{\alpha_N} \rangle$

Indistinguishable particles

Two particles

$$|\phi_1\rangle \otimes |\phi_2\rangle$$

$$\hookrightarrow \sum_P e^{i\vartheta_P} |\phi_{P(1)}\rangle \otimes |\phi_{P(2)}\rangle$$

$$P(1) = 1, 2$$
$$P(2) = 2, 1$$

$$e^{i\vartheta_P} = \eta^P$$ = parity of the permutation

$$\eta = \begin{cases} +1 & \text{bosons} \\ -1 & \text{fermions} \end{cases}$$

$\eta^P = \begin{cases} +1 & P \text{ contains an even number of pairwise permutations} \\ \eta & \text{" " " odd "} \end{cases}$

$$\eta^P = \begin{cases} 1 \\ \eta \end{cases}$$

$P(1) = 1 \quad P(2) = 2$

$P(1) = 2 \quad P(2) = 1$

$$|\phi_1\rangle \otimes |\phi_2\rangle \rightarrow \begin{cases} |\phi_1\rangle \otimes |\phi_2\rangle + |\phi_2\rangle \otimes |\phi_1\rangle \\ \text{symmetrized : bosons} \\ \text{NOT NORMALIZED} \\ |\phi_1\rangle \otimes |\phi_2\rangle - |\phi_2\rangle \otimes |\phi_1\rangle \end{cases}$$

anti-symmetrized : fermions

N particles

$$|\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots |\phi_N\rangle$$

$$\boxed{\rightarrow |\phi_1 \phi_2 \cdots \phi_N\rangle_\eta = \frac{1}{\sqrt{N!}} \sum_P \eta^P |\phi_{P(1)}\rangle \otimes |\phi_{P(2)}\rangle \otimes \cdots \otimes |\phi_{P(N)}\rangle}$$

physical state for indistinguishable particles

NOT PROPERLY NORMALIZED (for bosons)

$$|\phi_1 \phi_2 \cdots \phi_N\rangle_\eta \in \begin{cases} \mathcal{B}^{(N)} \subset \mathcal{H}^{(N)} & (\eta=1) \\ \mathcal{F}^{(N)} \subset \mathcal{H}^{(N)} & (\eta=-1) \end{cases}$$

(First quantization $\to$ second quantization)

<u>Theorem</u>

$${}_\eta\langle \phi_1 \phi_2 \cdots \phi_N | \psi_1 \psi_2 \cdots \psi_N\rangle_\eta = \begin{vmatrix} \langle\phi_1|\psi_1\rangle & \langle\phi_2|\psi_1\rangle & \cdots & \langle\phi_N|\psi_1\rangle \\ \vdots & & & \\ \langle\phi_N|\psi_1\rangle & & \cdots & \langle\phi_N|\psi_N\rangle \end{vmatrix}_\eta$$

$\eta = -1$ determinant

$\eta = +1$ permanent

$$\begin{vmatrix} \langle\phi_1|\psi_1\rangle & \langle\phi_1|\psi_2\rangle \\ \langle\phi_2|\psi_1\rangle & \langle\phi_2|\psi_2\rangle \end{vmatrix}_\eta$$

$$= \langle\phi_1|\psi_1\rangle\langle\phi_2|\psi_2\rangle + \eta\langle\phi_1|\psi_2\rangle\langle\phi_2|\psi_1\rangle$$

$=$

$|\phi_\alpha\rangle \to |\vec{r}\rangle$ eigenstates of the position operator

$$\vec{r} = (x, y, z)$$

$${}_\eta\langle\vec{r}_1\, \vec{r}_2 \ldots \vec{r}_N|\psi_1\, \psi_2 \ldots \psi_N\rangle_\eta$$

$$= \begin{vmatrix} \psi_1(\vec{r}_1) & \psi_1(\vec{r}_2) & \ldots & \psi_1(\vec{r}_N) \\ \vdots & & & \\ \psi_N(\vec{r}_1) & - & - \; - \; - & \psi_N(\vec{r}_N) \end{vmatrix}_\eta$$

$\eta = -1$ Slater determinant

$N \times N$ matrix $\to$ determinant

costs $\mathcal{O}(N^3)$
operations

permanent
$\to$ costs $\mathcal{O}(2^N N^{(2)})$
operations

"Boson sampling problem"

Proper normalization of $|\phi_1 \phi_2 \dots \phi_N\rangle_\eta$

Fermions $\{|\phi_\alpha\rangle\}$ = orthonormal basis of $\mathcal{H}^{(1)}$

$$|\phi_1 \phi_2 - \phi_N\rangle_\eta = \frac{1}{\sqrt{N!}} \sum_P \eta^P |\phi_{P(1)}\rangle \dots |\phi_{P(N)}\rangle$$

Fermions : $|\phi_1\rangle \neq |\phi_2\rangle \neq |\phi_3\rangle \neq \dots$

$N!$ independent permutations

Bosons

independent permutations

$$\frac{N!}{\prod_\alpha (n_\alpha !)}$$

(ϕ_α)

$$|\phi_1 \phi_2 \ldots \phi_N\rangle_\eta$$

$$= \frac{1}{\sqrt{N!}} \sum_P \eta^P |\phi_{P(1)}\rangle \ldots |\phi_{P(N)}\rangle$$

$$= \frac{1}{\sqrt{N!}} \left(\prod_\alpha n_\alpha !\right) \sum_P{}' \eta^P |\phi_{P(1)}\rangle \ldots |\phi_{P(N)}\rangle$$

Normalized state

$$\sqrt{\frac{\prod_\alpha n_\alpha !}{N!}} \sum_P{}' \eta^P |\phi_{P(1)}\rangle \ldots |\phi_{P(N)}\rangle$$

$$= \sqrt{\prod_\alpha n_\alpha !} \; \frac{1}{\prod_\alpha n_\alpha !} \; \underbrace{\frac{1}{\sqrt{N!}} \sum_P \eta^P |\phi_{P(1)}\rangle \cdots |\phi_{P(N)}\rangle}_{|\phi_1 \phi_2 \ldots \phi_N\rangle_\eta}$$

$$= \frac{|\phi_1 \phi_2 \cdots \phi_N\rangle_\eta}{\sqrt{\prod_\alpha n_\alpha !}}$$

normalized

<u>Creation and destruction operators</u>

$$|\phi_1 \phi_2 \cdots \phi_N\rangle_\eta \in \mathcal{B}^{(N)}(\mathcal{F}^{(N)}) \qquad |\phi\rangle \in \mathcal{H}^{(1)}$$

$\hat{a}^+_{\phi_0}$ creation of a particle in state $|\phi\rangle$

$$\underbrace{\hat{a}^+_\phi |\phi_1 \phi_2 \ldots \phi_N\rangle_\eta}_{} =: \underbrace{|\phi_0 \phi_1 \phi_2 \cdots \phi_N\rangle_\eta}_{}$$

$$= \frac{1}{\sqrt{(N+1)!}} \sum_P \eta^P |\phi_{P(0)}\rangle |\phi_{P(1)}\rangle \cdots |\phi_{P(N)}\rangle$$

$$\hat{a}^{\dagger}_{\phi} : \mathcal{B}^{(N)} \longrightarrow \mathcal{B}^{(N+1)}$$
$$\left(\mathcal{F}^{(N)} \longrightarrow \mathcal{F}^{(N+1)}\right)$$

destruction operator $\hat{a}_{\phi}$

$${}_{\eta}\langle \psi_1\, \psi_2 \cdots \psi_{N-1} | \underbrace{\hat{a}_{\phi} | \phi_1\, \phi_2 \cdots \phi_N \rangle_{\eta}}_{\mathcal{B}^{(N-1)}\ \left(\mathcal{F}^{(N-1)}\right)}$$

$$= {}_{\eta}\langle \phi_1\, \phi_2 \cdots \phi_N | \underbrace{a^{\dagger}_{\phi} | \psi_1\, \psi_2 \cdots \psi_{N-1} \rangle_{\eta}}^{*}$$

$$= {}_{\eta}\langle \phi_1\, \phi_2 \cdots \phi_N | \phi\, \psi_1\, \psi_2 \cdots \psi_{N-1} \rangle_{\eta}^{*}$$

$$= \left| \begin{matrix} \langle \phi_1 | \phi \rangle & \langle \phi_1 | \psi_1 \rangle & \cdots & \langle \phi_1 | \psi_{N-1} \rangle \\ \langle \phi_2 | \phi \rangle & & \cdots & \langle \phi_2 | \psi_{N-1} \rangle \\ \vdots & & & \\ \langle \phi_N | \phi \rangle & & \cdots & \langle \phi_N | \psi_{N-1} \rangle \end{matrix} \right|_{\eta}^{*}$$

$$= \left| \begin{matrix} \langle \phi_1 | \phi \rangle & \langle \phi_2 | \phi \rangle & \cdots & \langle \phi_N | \phi \rangle \\ \vdots & \langle \phi_2 | \psi_1 \rangle & \cdots & \langle \phi_N | \psi_1 \rangle \\ & \vdots & & \vdots \end{matrix} \right|_{\eta}^{*}$$

$$= \sum_{u=1}^{N} \eta^{u-1} \langle \phi_u | \phi \rangle^* {}_\eta\langle \phi_1 \phi_2 \cdots (\text{no } \phi_u) \cdots \phi_N | \psi_1 \psi_2 \cdots \psi_{N-1} \rangle_\eta$$

$$= \cancel{\eta^{1-1}}\, 1 \, \langle \phi_1 | \phi \rangle^* {}_\eta\langle \phi_2 \cdots \phi_N | \psi_1 \cdots \psi_{N-1} \rangle_\eta^*$$

$$+ \cdots$$

arbitrary

$${}_\eta\langle \psi_1 \psi_2 \cdots \psi_{N-1} | \hat{a}_\phi | \phi_1 \phi_2 \cdots \phi_N \rangle_\eta$$

$$= {}_\eta\langle \psi_1 \psi_2 \cdots \psi_{N-1} | \sum_{u=1}^{N} \eta^{u} \langle \phi | \phi_u \rangle | \phi_1 \phi_2 \cdots (\text{no } \phi_u) \cdots \phi_N \rangle_\eta$$

$$\hat{a}_\phi | \phi_1 \phi_2 \cdots \phi_N \rangle_\eta$$

$$= \sum_{u=1}^{N} \eta^{u-1} \langle \phi | \phi_u \rangle | \phi_1 \phi_2 \cdots (\text{no } \phi_u) \cdots \phi_N \rangle_\eta$$

$$a^\dagger_\phi \, a^\dagger_\psi \, |\phi_1 \phi_2 \cdots \phi_N\rangle_\eta$$

$$= a^\dagger_\phi \, |\psi \phi_1 \phi_2 \cdots \phi_N\rangle_\eta$$

$$= |\phi \psi \phi_1 \phi_2 \cdots \phi_N\rangle_\eta$$

$$a^\dagger_\psi \, a^\dagger_\phi \, |\phi_1 \phi_2 \cdots \phi_N\rangle_\eta$$

$$= |\psi \phi \phi_1 \phi_2 \cdots \phi_N\rangle_\eta$$

$$= \eta \, |\phi \psi \phi_1 \cdots \phi_N\rangle_\eta$$

$$= \eta \, a^\dagger_\phi \, a^\dagger_\psi \, |\phi_1 \cdots \phi_N\rangle_\eta$$

$$\eta a^\dagger_\psi \, a^\dagger_\phi = a^\dagger_\phi \, a^\dagger_\psi$$

$$a^\dagger_\phi \, a^\dagger_\psi - \eta \, a^\dagger_\psi \, a^\dagger_\phi = 0$$

$$[a^\dagger_\phi, a^\dagger_\psi]_\eta = 0$$

$$= \begin{cases} [a^{\dagger}_{\phi_1}, a^{\dagger}_{\psi}] = 0 & \text{bosons} \\ \{a^{\dagger}_{\phi_1}, a^{\dagger}_{\psi}\} = 0 & \text{fermion} \end{cases}$$

$$a^{\dagger}_{\phi} a^{\dagger}_{\psi} + a^{\dagger}_{\psi} a^{\dagger}_{\phi}$$

$$\boxed{[a_{\phi_1}, a_{\psi}]_{\eta} = 0}$$

Consequence : fermions

$$(a^{\dagger}_{\phi})^2 = -(a^{\dagger}_{\phi})^2 = 0$$

$$a_{\psi} a^{\dagger}_{\phi} | \phi_1 \phi_2 \ldots \phi_N \rangle_{\eta}$$

$$= a_{\psi} | \phi_0 \phi_1 \phi_2 \ldots \phi_N \rangle_{\eta}$$

$$= \sum_{k=0}^{N} \eta^k \langle \psi | \phi_k \rangle | \phi_0 \phi_1 \ldots (\text{no } \phi_k) \ldots \phi_N \rangle_{\eta}$$

$$a^{\dagger}_{\phi} a_{\psi} | \phi_1 \phi_2 \ldots \phi_N \rangle_{\eta}$$

$$= a^\dagger_\phi \sum_{k=1}^{N} \eta^{k-1} \langle\psi|\phi_k\rangle \, |\phi_1 \cdots (\text{no } \phi_k) \cdots \phi_N\rangle_\eta$$

$$= \eta \sum_{k=1}^{N} \eta^{k} \langle\psi|\phi_k\rangle \, |\phi\,\phi_1 \cdots (\text{no } \phi_k) \cdots \phi_N\rangle_\eta$$

$$= \eta \sum_{k=0}^{N} \eta^{k} \langle\psi|\phi_k\rangle \, |\phi\,\phi_1 \cdots (\text{no } \phi_k) \cdots \phi_N\rangle_\eta$$

$$- \eta \langle\psi|\phi\rangle \, |\phi_1\,\phi_2 \cdots \phi_N\rangle_\eta$$

$$= \left(\eta\, a_\psi a^\dagger_\phi - \eta \langle\psi|\phi\rangle\right) |\phi_1\,\phi_2 \cdots \phi_N\rangle_\eta$$

$$\eta\, a^\dagger_\phi a_\psi = \cancel{\eta}\, a_\psi a^\dagger_\phi - \cancel{\eta} \langle\psi|\phi\rangle$$

$$a_\psi a^\dagger_\phi - \eta\, a^\dagger_\phi a_\psi = \langle\psi|\phi\rangle$$

$$\boxed{[a_\psi, a^\dagger_\phi]_\eta = \langle\psi|\phi\rangle}$$

orthonormal states

$$[a_{\psi}, a^{+}_{\phi}]_{\eta} = \delta_{\phi\psi}$$

Examples

$[a_{\psi}, a^{+}_{\psi}] = 1$ bosons

$\{a_{\psi}, a^{+}_{\psi}\} = 1$ fermions

a^{+}_{ϕ} creates a particle in $|\phi\rangle$

$\{|\psi_\alpha\rangle\}$ basis of $H^{(1)}$

$$|\phi\rangle = \sum_\alpha \langle\psi_\alpha|\phi\rangle \, |\psi_\alpha\rangle$$

a^{+}_{ϕ} in terms of $a^{+}_{\psi_\alpha}$

$$a^{+}_{\phi} |\phi_1 \phi_2 \ldots \phi_N\rangle_\eta = |\phi\, \phi_1 \ldots \phi_N\rangle_\eta = \sum_\alpha \langle\psi_\alpha|\phi\rangle \underbrace{|\psi_\alpha\, \phi_1\, \phi_2 \ldots \phi_N\rangle_\eta}_{a^{+}_{\psi_\alpha}|\phi_1 \phi_2 \ldots \phi_N\rangle_\eta}$$

$$\begin{cases} a^{+}_{\phi} = \sum_\alpha \langle\psi_\alpha|\phi\rangle \, a^{+}_{\psi_\alpha} \\ a_{\phi} = \sum_\alpha \langle\phi|\psi_\alpha\rangle \, a_{\psi_\alpha} \end{cases}$$

basis change formulas

Special basis of $H^{(1)}$: $|\vec{r}\rangle$

$$\hat{a}_\phi, \hat{a}^\dagger_\phi \to \hat{a}_{\vec{r}}, \hat{a}^\dagger_{\vec{r}}$$

$$\hat{\psi}(\vec{r}), \hat{\psi}^\dagger(\vec{r})$$ field operators

$$[\hat{\psi}] = \frac{1}{\sqrt{V}}$$

$$\{|\psi_\alpha\rangle\} \to \{|\vec{r}\rangle\}$$

$$a^\dagger_\phi = \text{"} \sum_{\vec{r}} \underbrace{\langle \vec{r}|\phi\rangle} a^\dagger_{\vec{r}} \text{"}$$

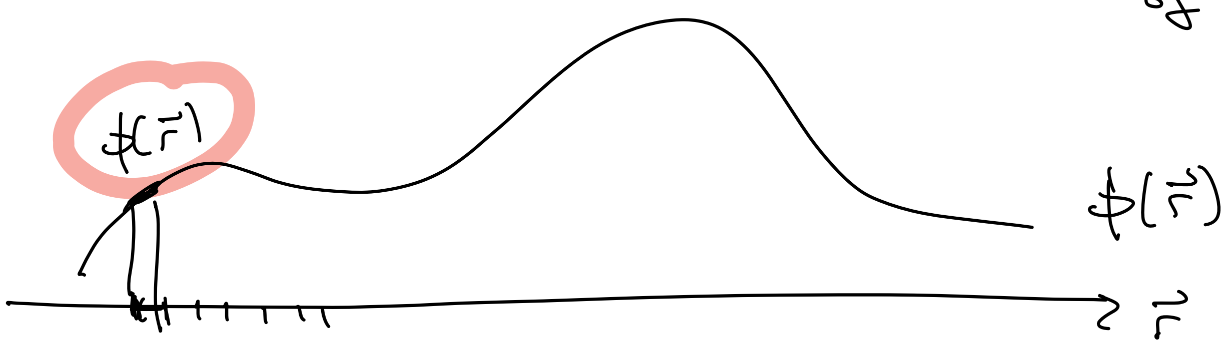
$[a^\dagger] = 1$

$$= \int d^d r \; \phi(\vec{r}) \; \underbrace{\hat{\psi}^\dagger(\vec{r})}$$

$[\phi(\vec{r})] = \frac{1}{\sqrt{V}}$ $\quad [\hat{\psi}] = \frac{1}{\sqrt{V}}$

$$\boxed{\hat{a}^\dagger_\phi = \int d^d r \; \phi(\vec{r}) \; \hat{\psi}^\dagger(\vec{r})}$$

d = # of dimensions of space



single particle in state $|\phi\rangle$

$$\hat{\psi}(\vec{r})|\phi\rangle = \langle \vec{r}|\phi\rangle \; |(\text{no } \phi)\rangle$$

$|\emptyset\rangle$ Vacuum

$$= \phi(\vec{r}) \, |\phi\rangle$$

"second quantization"

Fock space

so far $\mathcal{B}^{(N)}$, $\mathcal{F}^{(N)}$

$$F_{\text{bosons}} = \mathcal{B}^{(0)} \oplus \mathcal{B}^{(1)} \oplus \mathcal{B}^{(2)} \oplus \cdots \qquad (\mathcal{B}^{(0)} \to |0\rangle)$$

$$F_{\text{fermions}} = \mathcal{F}^{(0)} \oplus \mathcal{F}^{(1)} \oplus \mathcal{F}^{(2)} \oplus \cdots$$

harmonic oscillator $\hat{H} = \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)$

$[a, a^+] = 1$

$|0\rangle, |1\rangle, |2\rangle, |3\rangle, \ldots$

$\hat{a}^+\hat{a}\,|n\rangle = n\,|n\rangle$

$$\{|0\rangle\} \oplus \{|1\rangle\} \oplus \{|2\rangle\} \oplus \cdots$$

Normalized states in Fock space

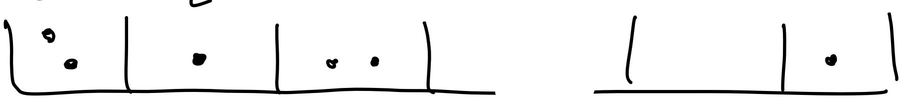
occupation numbers

$$\frac{|\phi_1 \phi_2 \cdots \phi_N\rangle_{(M)}}{\sqrt{\prod_\alpha n_\alpha!}} =: |n_{\alpha_1}, n_{\alpha_2}, \ldots, n_{\alpha_M}\rangle_{(M)}$$

$\{|\phi_\alpha\rangle\}$ orthonormal basis

$N = \sum_\alpha n_\alpha$

M dimensional Hilbert space $H^{(1)}$



$\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad --- \quad \alpha_M$

states of the

$|\phi_\beta\rangle$ state in the $\{|\phi_\alpha\rangle\}$ basis

You need to order sequentially the single-particle states

$$|n_{\alpha_1} n_{\alpha_2} -- n_{\alpha_N}\rangle = \frac{1}{\sqrt{\prod_\alpha n_\alpha!}} |\phi_1 \phi_2 -- \phi_N\rangle_\eta$$

ordered in the same sequence

ex. $|\phi_{\alpha_2} \phi_{\alpha_4} \phi_{\alpha_5} -- \phi_{\alpha_{100}}\rangle$

$a^\dagger_{\phi_\beta} |n_{\alpha_1} n_{\alpha_2} --- n_{\alpha_M}\rangle$ → second quantization

$$= a^\dagger_{\phi_\beta} \frac{|\phi_1 \phi_2 --- \phi_N\rangle_\eta}{\sqrt{\prod_\alpha n_\alpha!}}$$ → first quantization

$$= \frac{|\phi_\beta \, \phi_1 \, \phi_2 \, \cdots \, \phi_N\rangle_\eta}{\sqrt{\prod_\alpha n_\alpha!}}$$

$$= \eta^{\sum_{\alpha<\beta} n_\alpha} \frac{|\phi_1 \, \phi_2 \cdots \phi_\beta \cdots \phi_N\rangle}{\sqrt{\prod_{\alpha\neq\beta} n_\alpha!} \; \sqrt{n_\beta!}} \; \frac{\sqrt{(n_\beta+1)!}}{\sqrt{(n_\beta+1)!}}$$

$$= \eta^{\sum_{\alpha<\beta} n_\alpha} \; \sqrt{n_\beta+1} \; |n_{\alpha_1} \, n_{\alpha_2} \cdots, n_\beta+1, \cdots n_{\alpha_M}\rangle$$

$$\boxed{a^+_{\phi_\beta} |n_{\alpha_1} \, n_{\alpha_2} \cdots n_\alpha\rangle = \eta^{\sum_{\alpha<\beta} n_\alpha} \sqrt{n_\beta+1} \; |n_{\alpha_1}, n_{\alpha_2}, \cdots, n_\beta+1, \cdots, n_{\alpha_M}\rangle}$$

special case : bosons in a single state

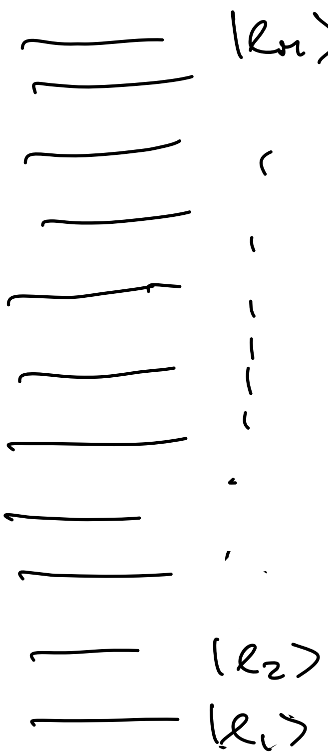
$$a^+|n\rangle = \sqrt{n+1} \; |n+1\rangle \qquad (\eta = 1)$$

Bette notation

$$|n_1 n_2 \dots n_M\rangle =: \frac{|\phi_{\alpha_1} \phi_{\alpha_2} \dots \phi_{\alpha_N}\rangle_\eta}{\sqrt{\prod_\alpha n_\alpha!}}$$

$$\alpha_1 < \alpha_2 < \dots < \alpha_M$$

$$|\phi_{\alpha_1}\rangle \otimes |\phi_{\alpha_2}\rangle \otimes \dots \otimes |\phi_{\alpha_N}\rangle$$



$$|n_1 n_2 \dots n_M\rangle =: |e_3\, e_5\, e_7\, e_{99}\rangle_\eta$$

Exercise

$$\boxed{a_{\phi_\beta}|n_1, n_2, \dots, n_M\rangle = \eta^{\sum_{\alpha<\beta} n_\alpha} \sqrt{n_\beta}\, |n_1, n_2 \dots n_\beta - 1, \dots n_M\rangle}$$

bosons $\quad \hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$

single single-particle state

$$a^\dagger|0\rangle = |1\rangle$$

$$(a^+)^2|0\rangle = \sqrt{2}\,|2\rangle$$

$$(a^+)^3|0\rangle = \sqrt{3\cdot 2}\,|2\rangle$$

$$\vdots$$

$$(a^+)^n|0\rangle = \sqrt{n!}\,|n\rangle$$

$$|n\rangle = \frac{(a^+)^n}{\sqrt{n!}}|0\rangle$$

many single-particle states

$|0\rangle$

$$|n_1, n_2, \dots, n_M\rangle = \underbrace{\frac{(a_1^+)^{n_1}}{\sqrt{n_1!}}\,\frac{(a_2^+)^{n_2}}{\sqrt{n_2!}}\cdots\frac{(a_M^+)^{n_M}}{\sqrt{n_M!}}}_{\text{ORDER IS IMPORTANT}}|0,0,\dots,0\rangle$$

same for fermions

$$\sqrt{n!} = 1$$

Second-quantized operators $\Longleftrightarrow$ observables

Distinguishable particles : Hilbert space $\mathcal{H}^{(N)}$

$$|\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_N\rangle$$

single-particle operator $\hat{O}_i^{(1)}$

$$O_i^{(1)}\left(|\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_i\rangle \otimes \cdots \otimes |\phi_N\rangle\right)$$

$$= |\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes \underbrace{\hat{O}_i^{(1)} |\phi_i\rangle} \otimes \cdots \otimes |\phi_N\rangle$$

Symmetrized observables

$$O^{(1)}_{symm} = \sum_{i=1}^{N} \hat{O}_i^{(1)}$$

$$\hat{O}^{(1)} |\lambda_\alpha\rangle = \lambda_\alpha |\lambda_\alpha\rangle \qquad \text{single-particle eigenstates}$$

Fock state in the $\{|\lambda_\alpha\rangle\}$ basis

$$|n_1, n_2, \ldots n_M\rangle$$

n of particles in $|\lambda_2\rangle$

$$\hat{O}^{(1)}_{symm} \underbrace{|n_1, n_2, \ldots, n_M\rangle}$$

$$= \cdots = \left(\sum_\alpha \lambda_\alpha n_\alpha\right) |n_1, n_2, \ldots, n_M\rangle$$

$$\underbrace{\hat{a}^+_\alpha \hat{a}_\alpha}_{\hat{n}_\alpha \text{ number operator}} | n_1, n_2, \ldots, n_M \rangle = n_\alpha | n_1, n_2, \ldots, n_M \rangle$$

$$= \Big(\underbrace{\sum_\alpha \lambda_\alpha \hat{a}^+_\alpha \hat{a}_\alpha}_{O^{(1)}_{\text{system}}} \Big) | n_1 n_2 \ldots n_M \rangle$$

example $\quad \hat{N} = \sum_\alpha \hat{n}_\alpha = \sum_\alpha \hat{a}^+_\alpha \hat{a}_\alpha$

$$O^{(1)}_{\text{system}} = \sum_\alpha \lambda_\alpha a^+_\alpha a_\alpha$$

$$\{ |\lambda_\alpha\rangle \} \to \{ |\phi_\rho\rangle \}$$

$$\begin{cases} a^+_\alpha = \sum_\rho \langle \phi_\rho | \lambda_\alpha \rangle a^+_\rho \\ a_\alpha = \sum_\rho \langle \lambda_\alpha | \phi_\rho \rangle a_\rho \end{cases}$$

$$= \sum_\alpha \sum_{\rho\rho'} \lambda_\alpha \langle \phi_\rho | \lambda_\alpha \rangle \langle \lambda_\alpha | \phi_{\rho'} \rangle a^+_\rho a_{\rho'}$$

$$\hat{O}^{(1)} = \sum_\alpha \lambda_\alpha |\lambda_\alpha\rangle\langle\lambda_\alpha|$$

$$= \sum_{\rho\rho'} \langle \phi_\rho | \hat{O}^{(1)} | \phi_{\rho'} \rangle a^+_\rho a_{\rho'}$$

Ex. $\quad |\phi_p\rangle \to |\vec{r}\rangle$

$$\hat{O}^{(1)}_{\text{system}} = \int d^d r \, d^d r' \, \langle \vec{r}|\hat{o}^{(1)}|\vec{r}'\rangle \, \psi^\dagger(\vec{r})\,\psi(\vec{r}')$$

$$\hat{o}^{(1)} = \frac{\vec{p}^{\,2}}{2m}$$

$$\langle \vec{r}|\frac{\vec{p}^{\,2}}{2m}|\vec{r}'\rangle = -\frac{\hbar^2}{2m}\vec{\nabla}^2 \, \delta(\vec{r}-\vec{r}')$$

$$= -\frac{\hbar^2}{2m}\vec{\nabla}'^2 \, \delta(\vec{r}-\vec{r}')$$

$$\hat{K} = \sum_i \frac{p_i^2}{2m}$$

$$= \int d^d r \, d^d r' \left[-\frac{\hbar^2}{2m}\left(\vec{\nabla}'^2 \, \delta(\vec{r}-\vec{r}')\right)\right]\hat{\psi}^\dagger(\vec{r})\,\hat{\psi}(\vec{r}')$$

$$= \ldots = \int d^d r \, d^d r' \, \delta(\vec{r}-\vec{r}') \, \hat{\psi}^\dagger(\vec{r})\left(-\frac{\hbar^2}{2m}(\vec{\nabla}')^2\right)\hat{\psi}(\vec{r}')$$

$$= \int d^d r \, \hat{\psi}^\dagger(\vec{r})\left(-\frac{\hbar^2}{2m}\nabla^2\right)\hat{\psi}(\vec{r})$$