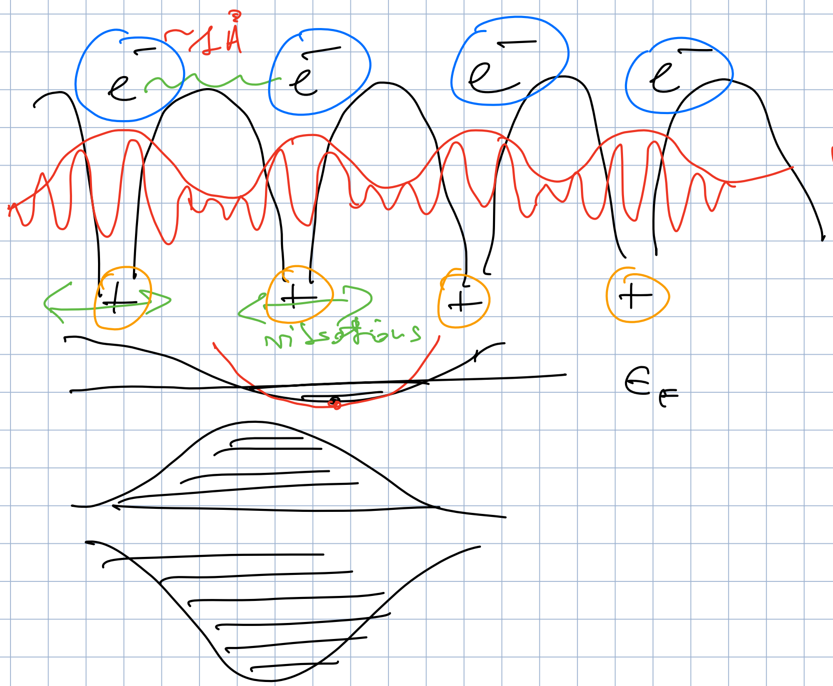
Normal state of electrons in a metal

($\hookrightarrow$ conventional superconductivity of simple metals)

$\simeq$ free electron gas ?



$$\epsilon_{\vec{k}} \simeq \frac{\hbar^2 k^2}{2m}$$

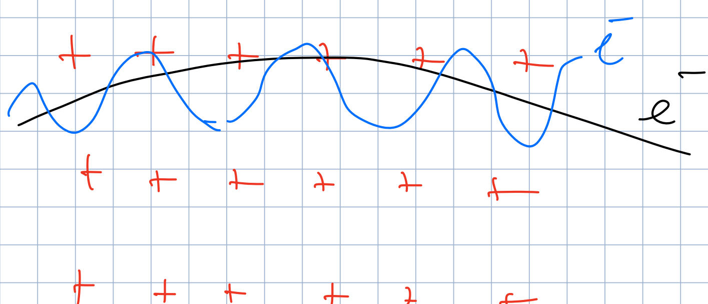
$$m = m_{eff}$$

1) $e^- - e^-$ interactions $\leftarrow$

2) lattice vibrations $\leftarrow$

Continuum approximation of a crystal

$\lambda \gg 1$ Å

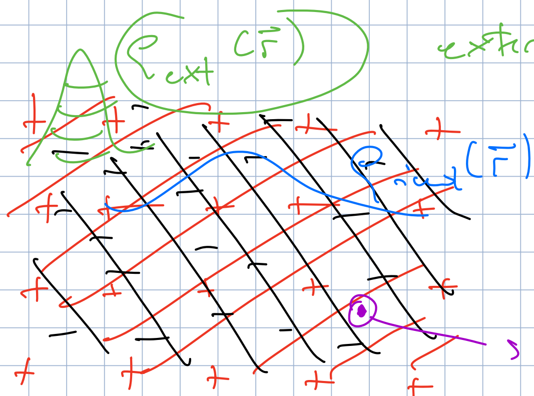


homogeneous distribution of positive charges mean-field picture → homogeneously spread electrons do not affect each other

↓

electrons interact with the average distribution of the electrons and with a positive, neutralizing background created by the ions.

Perturb the system with an external charge



$$\nabla^2 \phi_{ext}(\vec{r}, t) = -\frac{\rho_{ext}(\vec{r}, t)}{\epsilon_0} \leftarrow$$

ϕ_{ext} ↓ scalar potential created by the ext. charges

$$\nabla^2 \phi(\vec{r}) = -\frac{\rho(\vec{r})}{\epsilon} \leftarrow \qquad \rho(\vec{r}) = \rho_{ext}(\vec{r}) + \rho_{ind}(\vec{r})$$

↑ potential felt by the test charge

↑ induced in the electron gas by the perturbation

$\rho_{ind}(\vec{r}) = 0$ if $\rho_{ext}(\vec{r}) = 0$

$\downarrow$

Consistent with mean-field picture

Relate ϕ and ϕ_{ext} : theory of dielectric screening

$\downarrow$

$$\phi_{ext}(\vec{r}, t) = \underbrace{\int d^3r' \int dt' \, \epsilon(\vec{r}-\vec{r}', t-t') \, \phi(\vec{r}', t')}_{\text{linear response}}$$

$\uparrow$

dielectric function

FT : $\phi_{ext}(\vec{q}, \omega) = \epsilon(\vec{q}, \omega) \, \phi(\vec{q}, \omega)$

$$\phi(\vec{q}, \omega) = \frac{\phi_{ext}(\vec{q}, \omega)}{\epsilon(\vec{q}, \omega)}$$

$\uparrow$

Consider only $\underbrace{\text{static fields}}$: drop the ω dependence

(instantaneous response of $\phi(\vec{r})$ to $\phi_{ext}(\vec{r})$)

FT of the Laplace equations

$$-q^2 \phi(\vec{q}, \omega) = -\frac{\rho(\vec{q}, \omega)}{\epsilon_0} \qquad -q^2 \phi_{ext}(\vec{q}, \omega) = -\frac{\rho_{ext}(\vec{q}, \omega)}{\epsilon_0}$$

$$\rho(\vec{q}, \omega) = \rho_{ext}(\vec{q}, \omega) + \rho_{int}(\vec{q}, \omega)$$

$$\phi(\vec{q},\omega) = \frac{\rho_{ext}(\vec{q},\omega) + \rho_{ind}(\vec{q},\omega)}{q^2\epsilon_0}$$

$$= \phi_{ext}(\vec{q},\omega) + \frac{\rho_{ind}}{q^2\epsilon_0}$$

$$1 = \epsilon(\vec{q},\omega) + \frac{\rho_{ind}}{q^2\epsilon_0\,\phi(\vec{q},\omega)}$$

$$\boxed{\epsilon(\vec{q},\omega) = 1 - \frac{\rho_{ind}(\vec{q},\omega)}{q^2\epsilon_0\,\phi(\vec{q},\omega)}}$$

$$\boxed{\epsilon(\vec{q},\omega) = \frac{\phi_{ext}(\vec{q},\omega)}{\phi(\vec{q},\omega)} = \frac{\rho_{ext}(\vec{q},\omega)}{\rho_{ext}(\vec{q},\omega)+\rho_{ind}(\vec{q},\omega)}}$$

Calculate $\epsilon(\vec{q},\omega)$: Thomas – Fermi approximation

mean-field picture + "local density approximation"

system behaves locally as if it were homogeneous at a local chemical potential

electron in $\phi(\vec{r})$

$$\left[-\frac{\hbar^2}{2m}\nabla^2 - e\phi(\vec{r})\right]\psi(\vec{r}) = E\psi(\vec{r})$$

$$\phi(\vec{r}) \sim \phi_{ext}$$

. if $\phi(\vec{r})$ varies slowly with respect the wavelength of the electrons

$$-\frac{\hbar^2}{2m}\nabla^2\psi = \left[E + e\phi(\vec{r})\right]\psi(\vec{r})$$

$$\psi \sim e^{i\vec{k}\cdot\vec{r}}$$

$\perp$

nearly constant

$$E + e\phi(\vec r) \simeq \epsilon_{\vec k} = \frac{\hbar^2 k^2}{2m} \quad \leftarrow$$

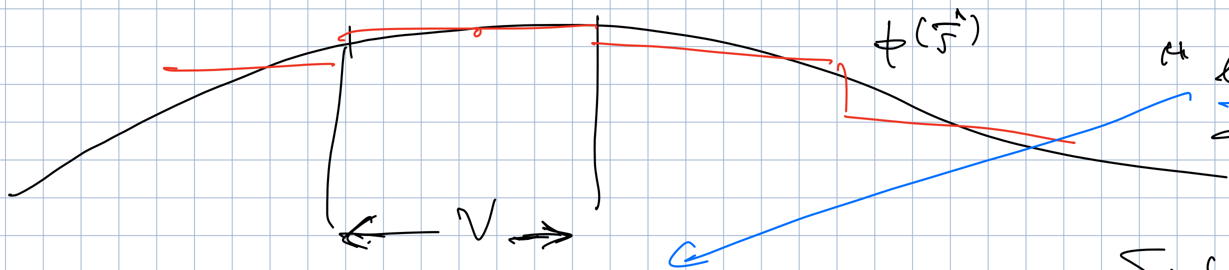
$$\Rightarrow \quad E_{\vec k} = \epsilon_{\vec k} - e\phi(\vec r) \quad \Leftarrow \qquad \text{II}$$

$$n(\vec r) \overset{@T}{=} \frac{1}{V}\sum_{\vec k} \frac{1}{e^{\beta(E_{\vec k}-\mu)}+1} = \int \frac{d^3k}{(2\pi)^3}\, \frac{1}{e^{\beta[\epsilon - e\phi(\vec r) - \mu]}+1}$$

density electrons

$$\mu \to \mu + e\phi(\vec r)$$

"local density approximation"



$$\delta n(\vec r) \rightarrow$$

$$n(\vec r) = n_0(\mu + e\phi(\vec r)) = n_0(\mu) + e\frac{\partial n_0}{\partial \mu}\phi(\vec r) + \dots$$

density of an homogeneous electron gas

linear response

induced density distribution

$$e\phi(\vec r) \ll \mu \simeq \epsilon_F$$

$$\rho_{ind}(\vec r) = -e\,\delta n(\vec r)$$

TF Theory

$$= -e^2\frac{\partial n_0}{\partial \mu}\phi(\vec r) \Rightarrow \frac{\rho_{ind}}{\phi} = -e^2\frac{\partial n_0}{\partial \mu}$$

Thomas-Fermi dielectric function

$$\epsilon_{TF}(\vec q) = 1 + \frac{e^2}{q^2\epsilon_0}\frac{\partial n_0}{\partial \mu} = 1 + \frac{q_{TF}^2}{q^2}$$

q_{TF}^2

$$\epsilon_{TF}^{-1}(\vec q) = \frac{q^2}{q^2 + q_{TF}^2}$$

$$\phi(\vec q) = \frac{q^2}{q^2+q_{TF}^2}\,\phi_{ext}(\vec q)$$

$$\rho_{ext}(\vec r) = Q\,\delta(\vec r) \qquad \text{point charge}$$

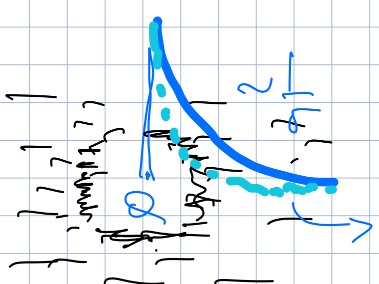
$$\boxed{\phi_{ext}(\vec r) = \frac{Q}{4\pi\epsilon_0 r}} \xrightarrow{FT} \phi_{ext}(\vec q) = \frac{Q}{\epsilon_0 q^2}$$

$$\phi(\vec q) = \frac{\cancel{q^2}}{q^2+q_{TF}^2}\,\frac{Q}{\epsilon_0 \cancel{q^2}} = \frac{Q}{\epsilon_0(q^2+q_{TF}^2)} \leftarrow \text{Lorentzian}$$

$$\downarrow FT^{-1}$$

$$\frac{Q}{4\pi\epsilon_0 r}\, e^{-q_{TF} r}$$

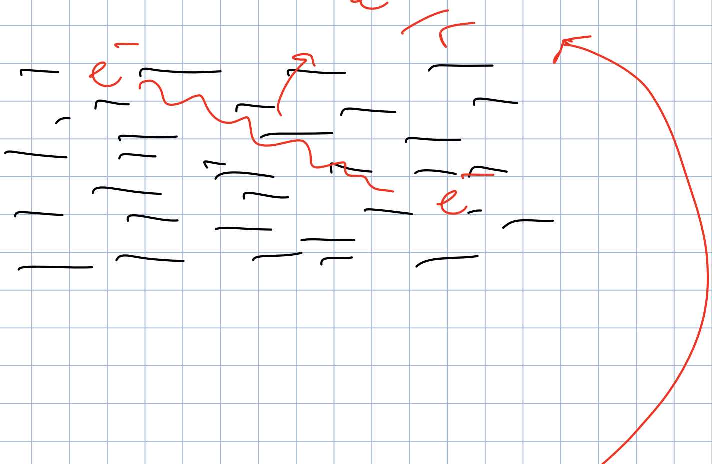
exponential screening

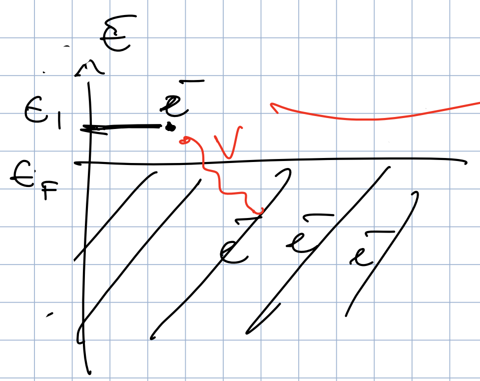


$$\frac{e^{-q_{TF} r}}{r} \sim \phi(\vec r)$$

estimate: $q_{TF} \sim k_F \sim \frac{1}{1\,\text{Å}}$

Dielectric screening from the other electrons





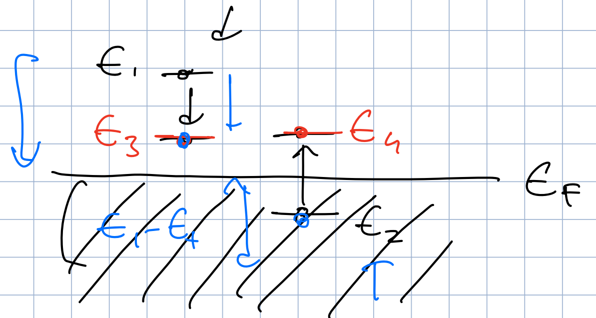
$$V(\vec r) \simeq \frac{e^2}{4\pi\epsilon_0 r}\, e^{-q_{TF} r}$$

Fermi sea

Perturbative treatment of the el-el interaction starting from a Fermi sea @ $T=0$ + 1 electron above the Fermi sea

Time-dependent perturbation theory: how long is the extra ele. living at energy ϵ_1?

<u>Fermi's golden rule</u>



$$\epsilon_1 + \epsilon_2 = \epsilon_3 + \epsilon_4$$

$$\Gamma \sim \sum_{2,3,4} \underbrace{|\langle \vec k_1, \vec k_2 | V | \vec k_3, \vec k_4 \rangle|^2}\ \underbrace{\delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4)}$$

ϵ_2 can be chosen from $2\epsilon_F - \epsilon_1$ to ϵ_F

ϵ_3 " " " in a interval from ϵ_F to ϵ_1

ϵ_4 is fixed by energy cons.

$$\Gamma \sim \overbrace{(\epsilon_1 - \epsilon_F)\, g(\epsilon_F)}^{\text{choice of } \epsilon_2}\ \overbrace{(\epsilon_1 - \epsilon_F)\, g(\epsilon_F)}^{\text{choice of } \epsilon_3}$$

$$\sim \quad (\epsilon_1 - \epsilon_F)^2 \, g^2(\epsilon_F) \longrightarrow 0$$

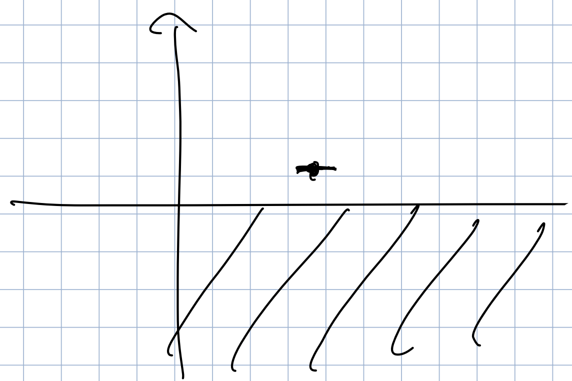
$$\epsilon_1 \to \epsilon_F$$

Pauli blocking

Landau theory of the electron liquid

$\Rightarrow$ electrons close to the Fermi energy behave essentially as free particles

states of the electron liquid : $|\{n_{\vec{k},\sigma}\}\rangle$



quasi-free electrons
(quasi-particles)

$m \xrightarrow{\text{eff}} m^*$

renormalized effective mass

$\Rightarrow$ interaction of electrons with lattice vibrations change things!

Include lattice vibrations and their effect on dielectric response

No ext. charge $\Rightarrow$ lattice $\equiv$ homogeneous positive charge distribution

+ ext. charge

$$\rho(\vec q,\omega) = \rho_{ext}(\vec q,\omega) + \rho^{(e)}_{ind}(\vec q,\omega) + \overset{\text{ions}}{\rho^{(i)}_{ind}}(\vec q,\omega)$$

for the electrons

$$\rho^{(e)}_{ext.} = \rho_{ext} + \rho^{(i)}_{ind}$$

$$\frac{\rho^{(e)}_{ind}}{\rho_{ext}} = -1 + \frac{1}{\epsilon_{TF}} = -1$$

$$\epsilon(\vec q,\omega) = \frac{\rho^{(e)}_{ext}}{\rho^{(e)}_{ext} + \rho^{(e)}_{ind}} \qquad \frac{1}{\epsilon} = 1 + \frac{\rho^{(e)}_{ind}}{\rho^{(e)}_{ext}}$$

created by $\rho_{ext} + \rho^{(i)}_{ind}$

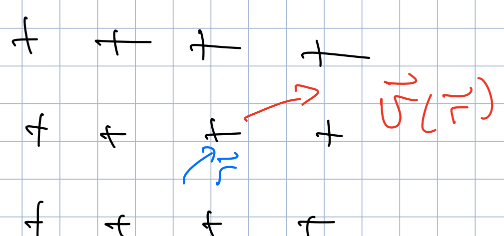
$$\frac{\phi^{(e)}_{ext}}{\epsilon} = \phi \qquad \frac{\phi^{(e)}_{ext}}{\phi} = \frac{\rho^{(e)}_{ext}}{\rho} = \epsilon_{TF} = 1 + \left(\frac{q_{TF}}{q}\right)^2$$

$$= \frac{\rho - \rho^{(e)}_{ind}}{\rho} = 1 - \frac{\rho^{(e)}_{ind}}{\rho}$$

$$\boxed{\rho^{(e)}_{ind} = -\left(\frac{q_{TF}}{q}\right)^2 \rho}$$

$\rho^{(i)}_{ind}$?   microscopic treatment of the ions

$\Rightarrow$ classical treatment



continuous medium

$$M \dot{\vec v} = (Ze)\vec E = -Ze \vec\nabla \phi(\vec r, t)$$

$$\frac{\partial \rho^{(i)}_{ind}}{\partial t} = -\vec{\nabla}\cdot\vec{j}$$

ionic current (charge)

$$\vec{j} = (Ze)\, n\, \vec{v}$$

density

$$= -\vec{\nabla}\cdot\left((Ze)\, n\, \vec{v}\right)$$

$$\frac{\partial^2 \rho^{(i)}_{ind}(\vec{r},t)}{\partial t^2} = -(Ze)\, n\, \vec{\nabla}\cdot\dot{\vec{v}}$$

$$= \frac{(Ze)^2 n}{M}\, \underbrace{\nabla^2 \phi}_{-\rho/\epsilon_0} = -\frac{(Ze)^2 n}{M}\, \frac{\rho(\vec{r},t)}{\epsilon_0}$$

$\downarrow$ FT

$$-\omega^2 \rho^{(i)}_{ind}(\vec{q},\omega) = -\underbrace{\boxed{\frac{(Ze)^2 n}{M\epsilon_0}}}_{\omega_i^2}\, \rho(\vec{q},\omega)$$

$$\boxed{\rho^{(i)}_{ind}(\vec{q},\omega) = \frac{\omega_i^2}{\omega^2}\, \rho(\vec{q},\omega)}$$

$$\boxed{\underline{\rho^{(e)}_{ind}} = -\left(\frac{q_{TF}}{q}\right)^2 \underline{\rho}}$$

$$\rho^{(e)}_{ind} + \rho^{(i)}_{ind} = \rho - \rho_{ext} = \left(\frac{\omega_i^2}{\omega^2} - \frac{q_{TF}^2}{q^2}\right)\rho$$

$$\rho_{ext}(\vec{q},\omega) = \left(1 - \frac{\omega_i^2}{\omega^2} + \frac{q_{TF}^2}{q^2}\right)\rho(\vec{q},\omega)$$

$$\underbrace{\qquad\qquad}_{\epsilon(q,\omega)}$$

$$\frac{1}{\epsilon(\vec{q},\omega)} = \Big(\underbrace{1+\frac{q_{TF}^2}{q^2}}_{TF} - \frac{\omega_i^2}{\omega^2}\Big)^{-1}$$

when does $\epsilon^{-1}(\vec{q},\omega)$ diverge?

$$\phi(\vec{q},\omega) = \frac{1}{\epsilon(\vec{q},\omega)}\,\phi_{ext}(\vec{q},\omega) \to 0 \neq 0$$

$$\frac{1}{\epsilon(\vec{q},\omega)} \to \infty$$

hitting an elementary excitation / resonance

$$1+\frac{q_{TF}^2}{q^2} - \frac{\omega_i^2}{\omega_q^2} = 0 \qquad \omega = \omega_q$$

$$\omega_q^2 = \omega_i^2 \frac{q^2}{q^2+q_{TF}^2} \underset{q \ll q_{TF}}{\simeq} \frac{\omega_i^2}{q_{TF}^2} q^2$$

$$\omega_q \simeq \frac{\omega_i}{q_{TF}} q = c\, q$$

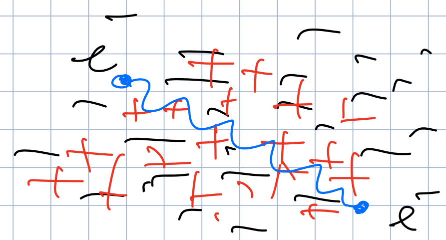
acoustic phonon

$$\frac{1}{\epsilon(\vec{q},\omega)} = \dots = \frac{1}{\epsilon_{TF}(\vec{q})} + \underbrace{\frac{q^2}{q^2+q_{TF}^2}\,\frac{\omega_q^2}{\omega^2 - \omega_q^2}}$$

$$\omega \to \omega_q \qquad \frac{1}{\epsilon} \to 0$$

$\frac{1}{\epsilon(\vec{q},\omega)}$ can be <u>negative</u> if $\omega \to \omega_q^-$

Q



$$\phi(\vec{q},\omega) = \underbrace{\frac{1}{\epsilon(\vec{q},\omega)}}_{\text{can be } <0} \overbrace{\phi_{ext}(\vec{q},\omega)}^{\text{bare Coulomb}}$$

two electrons <u>can attract each other!</u>

<u>retardation effect</u>