Consequences of Bose-Einstein condensation
for weakly interacting Bose gas

So far: working assumption that
weak interactions do not destroy BEC.

nearly
100% condensate: spontaneous-symmetry breaking (SSB)

$$\hat{\psi}(\vec{r}) = \Psi_0(\vec{r}) + \delta\hat{\psi}(\vec{r})$$

nearly 100% condensate $\langle \delta\hat{\psi}^\dagger \delta\hat{\psi} \rangle \ll |\Psi_0(\vec{r})|^2$

density of condensate depletion

condensate density

$$\int d^dr \, |\Psi_0|^2 = N_0 \sim O(N)$$

100% BEC $\langle \delta\psi^\dagger \delta\psi \rangle = 0$

Ψ_0 has it all

Gross-Pitaevskii equation (GPE)

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + U_{ext}(\vec{r}) + \int d^d r' V(\vec{r}-\vec{r}')\left|\Psi_0(\vec{r}')\right|^2\right]\Psi_0(\vec{r}) = \mu\, \Psi_0(\vec{r})$$

$\Rightarrow$ solution minimizes the energy in the absence of condensate depletion

time-dependent GPE

$$i\hbar \frac{\partial}{\partial t}\Psi_0(\vec{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + U_{ext}(\vec{r},t) + \cancel{\tfrac{1}{2}}\int d^d r' V(\vec{r}-\vec{r}')\left|\Psi_0(\vec{r}',t)\right|^2\right]\Psi_0(\vec{r},t)$$

how to obtain ?

$$i\hbar \frac{\partial}{\partial t}\hat{\psi}(\vec{r},t) = [\hat{\psi}(\vec{r}), \mathcal{H}] = \qquad \hat{\psi} \to \Psi_0$$

$$= \left[-\frac{\hbar^2}{2m}\nabla^2 + U_{ext} + \cancel{\tfrac{1}{2}}\int d^d r' V(\vec{r}-\vec{r}')\,\hat{\psi}^\dagger(\vec{r}',t)\,\hat{\psi}(\vec{r}',t)\right]\hat{\psi}(\vec{r},t)$$

$$\left[\hat{\mathcal{H}} = \hat{\mathcal{H}}_{one\text{-}body} + \frac{1}{2}\int d^d r' d^d r'' V(\vec{r}'-\vec{r}'')\,\psi^\dagger(\vec{r}')\psi^\dagger(\vec{r}'')\psi(\vec{r}'')\psi(\vec{r}')\right]$$

$$i\hbar \frac{\partial \hat{\psi}^\dagger}{\partial t} = [\psi^\dagger, \mathcal{H}] = -\left[\qquad\qquad\right]^\dagger$$

$$\left[\left(\frac{\partial}{\partial t}\psi^\dagger\right)\psi + \psi^\dagger\frac{\partial \psi}{\partial t}\right] = \frac{\partial}{\partial t}(\psi^\dagger\psi)$$

$$= \frac{1}{i\hbar}\left[\qquad\right]$$

$$= -\vec{\nabla}\cdot\vec{j}$$

$$\frac{\partial}{\partial t}\left(\hat{\psi}^{+}\hat{\psi}\right) + \vec{\nabla}\cdot\hat{\vec{j}} = 0$$

$$\langle \hat{\vec{j}}(\vec{r},t)\rangle = \frac{\hbar}{2mi}\left[\langle \hat{\psi}^{+}\vec{\nabla}\hat{\psi}\rangle - \langle(\vec{\nabla}\hat{\psi}^{+})\hat{\psi}\rangle\right]$$

$$\langle \underset{=}{\psi^{+}(\vec{r})}\cdot\vec{\nabla}\underset{=}{\psi(\vec{r})}\rangle = \lim_{\vec{r}'\to\vec{r}} \vec{\nabla}_{\vec{r}'}\underbrace{\langle\psi^{+}(\vec{r})\,\psi(\vec{r}')\rangle}_{g^{(1)}(\vec{r},\vec{r}')}$$

$$\langle\vec{j}(\vec{r},t)\rangle = \frac{\hbar}{2mi}\lim_{\vec{r}'\to\vec{r}} \nabla_{\vec{r}'}\left[g^{(1)}(\vec{r},\vec{r}';t) - g^{(1)}(\vec{r}',\vec{r};t)\right]$$

$$g^{(1)}(\vec{r},\vec{r}';t) = \sum_{\alpha} N_\alpha\, \chi_\alpha^{*}(\vec{r},t)\,\chi_\alpha(\vec{r}',t)$$

$$= \sum_{\alpha}\vec{j}_\alpha(\vec{r},t)$$

$$N_\alpha\frac{\hbar}{2mi}\left(\chi_\alpha^{*}\vec{\nabla}\chi_\alpha - (\vec{\nabla}\chi_\alpha^{*})\chi_\alpha\right)$$

$$= \vec{j}_0(\vec{r},t) + \underline{\sum_{\alpha\neq 0}\vec{j}_\alpha(\vec{r},t)}$$

dominant term

$$\chi_0(\vec r,t) = |\chi_0(\vec r,t)|\, e^{i\phi_0(\vec r,t)}$$

$$\vec j_0(\vec r,t) = \frac{\hbar}{m}\, N_0\, |\chi_0(\vec r,t)|^2\, \vec\nabla \phi_0(\vec r,t)$$

$\rho(\vec r,t)$

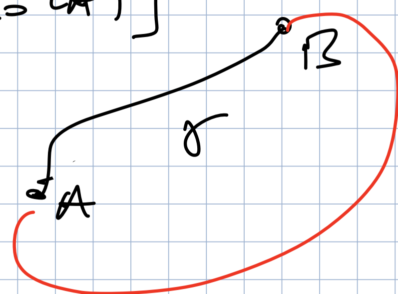
particle # density

$$= \rho(\vec r,t)\; \vec v_s(\vec r,t)$$

"superfluid velocity"

$$\vec v_s(\vec r,t) = \frac{\hbar}{m}\, \vec\nabla \phi_0(\vec r,t)$$

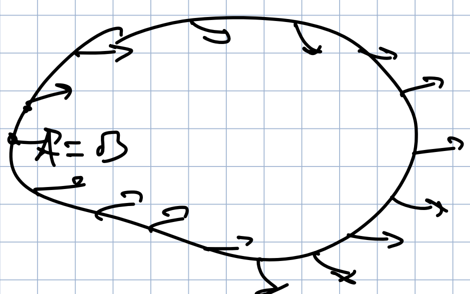
$$\kappa = \int_A^B d\vec l \cdot \vec v_s(\vec r,t) = \frac{\hbar}{m}\left[\phi_0(B) - \phi_0(A)\right]$$



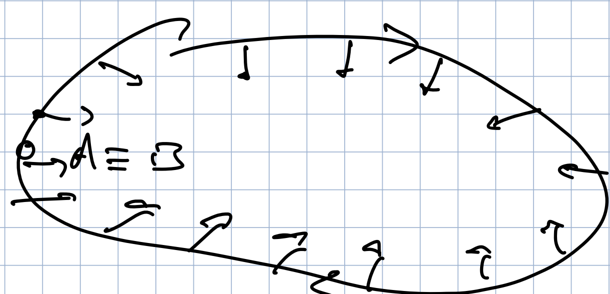
$$\underset{A\equiv B}{=} \frac{\hbar}{m}\, 2\pi p$$

vorticity

$p \in \mathbb{Z}$



$p=0$

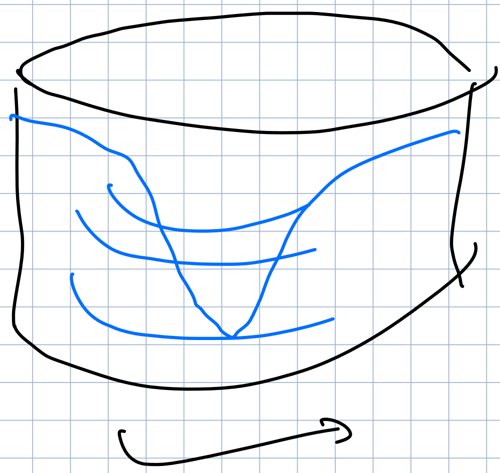


$p=-1$

$$\kappa = \frac{h}{m}\,p$$

vortex quantum

vortex quantization



regular fluid

BEC

Bogolyubov theory of the condensate depletion

$$\hat{\psi}(\vec{r}) = \Psi_0(\vec{r}) + \delta\hat{\psi}(\vec{r})$$

$$\hat{H} - \mu\hat{N} = E_{GP}[\Psi_0, \Psi_0^*] + \underset{\text{linear in } \delta\psi,\, \delta\psi^\dagger}{\hat{H}_1} + \underset{\text{quadratic}}{\hat{H}_2} + \cancel{\hat{H}_3 + \hat{H}_4}$$

$$-\mu$$

$$\hat{H}_1 = \int d^d r \; \delta\hat{\psi}^{\dagger} \underbrace{\left[-\frac{\hbar^2}{2m}\nabla^2 + U_{ext} + \int d^d r' \, V(\vec{r}-\vec{r}\,') \, |\Phi_0(\vec{r}\,')|^2 \right] \Phi_0(\vec{r})}_{\text{GPE}} $$

$$+ \; h.c. \qquad = 0$$

$$\psi^{\dagger}(\vec{r})\, \psi^{\dagger}(\vec{r}\,')\, \psi(\vec{r}\,')\, \psi(\vec{r})$$

$$\hat{H}_2 = \int d^d r \; \delta\psi^{\dagger} \left[-\frac{\hbar^2}{2m}\nabla^2 + U_{ext} - \mu \right] \delta\psi$$

$$+ \frac{1}{2} \int d^d r \int d^d r' \; V(\vec{r}-\vec{r}\,') \Big[\delta\psi^{\dagger}(\vec{r})\, \delta\psi^{\dagger}(\vec{r}\,') \, \Phi_0 \Phi_0 + h.c.$$

$$+ 2\, \delta\psi^{\dagger}(\vec{r})\, \delta\psi(\vec{r}) \, |\Phi_0(\vec{r}\,')|^2$$

$$+ 2\, \delta\psi^{\dagger}(\vec{r})\, \delta\psi(\vec{r}\,') \, \Phi_0^{*}(\vec{r}\,') \, \Phi(\vec{r}) \Big]$$

Weakly interacting Bose gas in free space

$$U_{ext} = 0$$

$$|\vec{r}-\vec{r}\,'|$$

$$V(\vec{r}-\vec{r}\,') = g\, \delta(\vec{r}-\vec{r}\,')$$

$$g = \frac{4\pi\hbar^2 a_s}{m} \qquad a_s = \text{s-wave scattering length}$$

GPE:

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + g\,|\Phi_0|^2 \right] \Phi_0 = \mu\, \Phi_0$$

solution $\qquad \Phi_0(\vec r) = \frac{\sqrt{N_0}}{\sqrt{V}}\cancel{e^{i\vartheta}} = \sqrt{n_0}\,\cancel{e^{i\vartheta}}$

$$\boxed{g n_0 = \mu} \qquad\qquad \Phi_0 = \sqrt{n_0}$$

$$\hat{H}_2 = \int d^d r\; \delta\psi^\dagger \left(-\frac{\hbar^2}{2m}\nabla^2 - \mu\right)\delta\psi$$

$$+ \frac{g n_0}{2}\int d^d r \left[\delta\psi^\dagger\delta\psi^\dagger + h.c. + 4\,\delta\psi^\dagger\delta\psi\right]$$

$$\delta\hat{\psi} = \sum_{\vec k\neq 0}\frac{e^{i\vec k\cdot\vec r}}{\sqrt{V}}\, a_{\vec k}$$

$$\hat{H}_2 = \sum_{\vec k\neq 0}\left(\frac{\hbar^2 k^2}{2m} - g n_0\right) a^\dagger_{\vec k} a_{\vec k}$$

$$+ 2 g n_0 \sum_{\vec k\neq 0} a^\dagger_k a_k$$

$$+ \frac{g n_0}{2}\sum_{\vec k\neq 0}\left(a^\dagger_k a^\dagger_{-k} + h.c.\right)$$

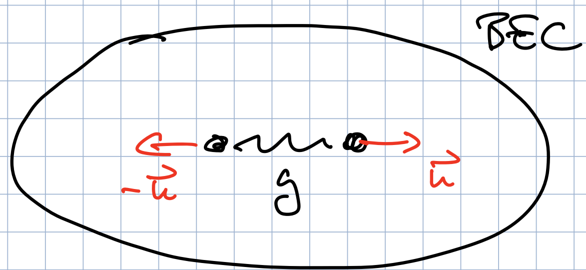
Bogoliubov Hamiltonian

$$\left[H, \sum_{\vec k}\hbar\vec k\, a^\dagger_{\vec k} a_{\vec k}\right] = 0$$

$$\hat{H}_2 = \sum_{\vec{k}\neq 0}\Big(\overbrace{\frac{\hbar^2 k^2}{2m}}^{e_{\vec{k}}} + g n_0\Big)\, a^+_{\vec{k}} a_{\vec{k}} + \frac{g n_0}{2}\sum_{\vec{k}\neq 0}\left(a^+_k a^+_{-k} + h.c.\right)$$

$\hat{a}_0 \hat{a}_0 \leftarrow |\Psi_0|^2$

quadratic in a, a^+



$$\hat{a}^+_k \hat{a}^+_{-k} \underbrace{\hat{a}_0 \hat{a}_0}_{n_0}$$

Bogolyubov diagonalization

$$\hat{H}_2 = \sum_{\vec{k}\neq 0} E_{\vec{k}}\, \hat{b}^+_k \hat{b}_k + \text{const.}$$

$$\hat{H}_2 = \frac{1}{2}\sum_{k\neq 0} \begin{pmatrix} \hat{a}^+_k \\ \hat{a}_{-k} \end{pmatrix}^T \begin{pmatrix} A_{\vec{k}} & B_{\vec{k}} \\ B_{\vec{k}} & A_{\vec{k}} \end{pmatrix} \begin{pmatrix} \hat{a}_k \\ \hat{a}^+_{-k} \end{pmatrix} + \text{const.}$$

$$A_{\vec{k}} = e_{\vec{k}} + g n_0 \qquad B_k = g n_0$$

$$\begin{cases} \hat{b}_{\vec{k}} = u_{\vec{k}}\, \hat{a}_{\vec{k}} + v_{\vec{k}}\, \hat{a}^+_{-\vec{k}} \\ \hat{a}_k = u_{\vec{k}}\, \hat{b}_{\vec{k}} - v_{\vec{k}}\, \hat{b}^+_{-k} \end{cases}$$

$u_{\vec{k}}, v_{\vec{k}}$ depend only on $|\vec{k}|$

Bogolyubov transformation

$$\begin{pmatrix} \hat{a}_k \\ \hat{a}^+_{-k} \end{pmatrix} = \begin{pmatrix} u & -v \\ -v & u \end{pmatrix} \begin{pmatrix} \hat{b}_k \\ \hat{b}^+_{-k} \end{pmatrix}$$

$$\hat{H}_2 = \frac{1}{2} \sum_{k \neq 0} \begin{pmatrix} b_k^\dagger & b_k \end{pmatrix} \boxed{\begin{pmatrix} u & -v \\ -v & u \end{pmatrix} \begin{pmatrix} A & B \\ B & A \end{pmatrix} \begin{pmatrix} u & -v \\ -v & u \end{pmatrix}} \begin{pmatrix} b_k \\ b_k^\dagger \end{pmatrix} + \text{const}$$

$$[\hat{b}_k, \hat{b}_k^\dagger] = \boxed{1 = u_k^2 - v_k^2}$$

$$\begin{pmatrix} A(u^2+v^2) - 2Buv & B(u^2+v^2) - 2Auv \\ B(u^2+v^2) - 2Auv & A(u^2+v^2) - 2Buv \end{pmatrix}$$

$$B(u^2+v^2) - 2Auv = 0$$

$$\boxed{\mathcal{E} = \mathcal{E}_k = \sqrt{e_k(e_k + 2g n_0)} = \sqrt{A_k^2 - B_k^2}}$$

$$A = A_{|\vec{k}|} \geq 0$$

$$\mathcal{E} = \mathcal{E}_{|\vec{k}|}$$

$$\begin{cases} u^2 = \frac{1}{2}\left(\frac{A}{\mathcal{E}} + 1\right) \\ v^2 = \frac{1}{2}\left(\frac{A}{\mathcal{E}} - 1\right) \end{cases}$$

$$uv = \frac{B}{2A}(u^2+v^2)$$

$$\boxed{\begin{aligned} u_{\vec{k}} &= \sqrt{\frac{1}{2}\left(\frac{A_{\vec{k}}}{\mathcal{E}_{\vec{k}}} + 1\right)} \\ v_{\vec{k}} &= \operatorname{sign}(B_{\vec{k}}) \sqrt{\frac{1}{2}\left(\frac{A_{\vec{k}}}{\mathcal{E}_{\vec{k}}} - 1\right)} \end{aligned}}$$

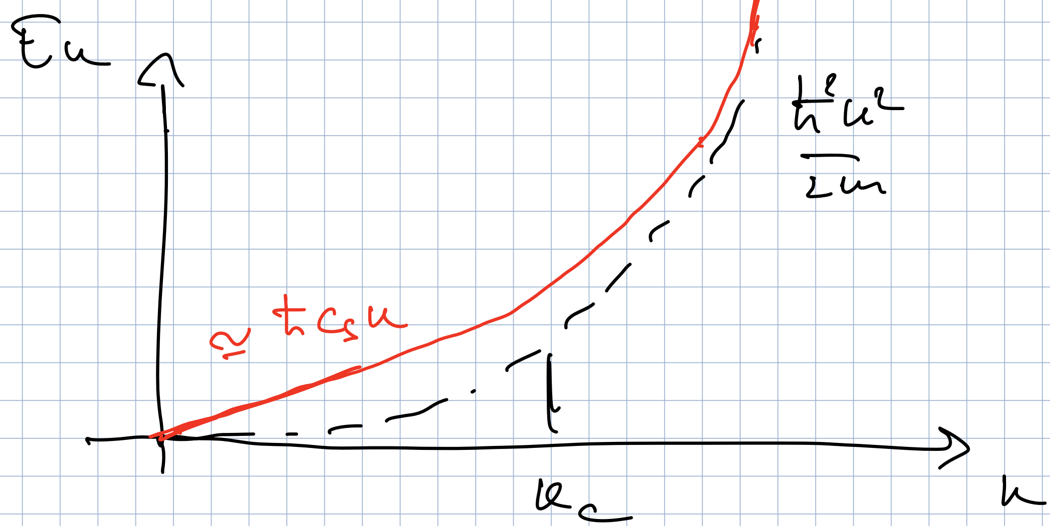
$$\hat{H}_2 = \sum_{\vec{k} \neq 0} \varepsilon_{\vec{k}} \, b^+_{\vec{k}} b_{\vec{k}} + \text{const.}$$

b_k, b_k^+ : destroy / create Bogolyubov quasi-particles

$$\varepsilon_{\vec{k}} = \sqrt{\frac{\hbar^2 k^2}{2m}\left(\frac{\hbar^2 k^2}{2m} + 2 g n_0\right)}$$

c_s B. speed of sound

$$\simeq \begin{cases} \hbar \sqrt{\dfrac{g n_0}{m}}\, k & \dfrac{\hbar^2 k^2}{2m} \ll g n_0 \\[2ex] \dfrac{\hbar^2 k^2}{2m} & \dfrac{\hbar^2 k^2}{2m} \gg g n_0 \end{cases}$$



$$\frac{\hbar^2 k_c^2}{2m} \simeq g n_0 \qquad \xi = \frac{1}{k_c} = \frac{\hbar}{\sqrt{2 m g n_0}}$$

healing length