Superfluidity of ^{4}He : two-fluid model

$$\underbrace{\vec{j}(\vec{r})}_{\text{mass current}} = \underbrace{\rho_s(T)\,\vec{v}_s(\vec{r})}_{\text{superfluid}} + \underbrace{\rho_n(T)\,\vec{v}_n(\vec{r})}_{\text{normal}}$$

irrotational velocity field

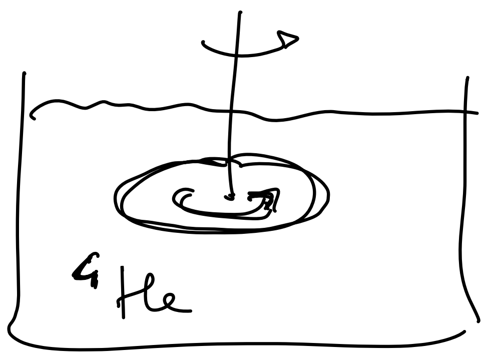
$$\vec{\nabla}\times\vec{v}_s = 0$$

in a stationary state: $\vec{v}_n(\vec{r})$ follows the motion of the boundaries

ρ, ρ_s, ρ_n mass densities

$$\rho = \rho_s + \rho_n$$

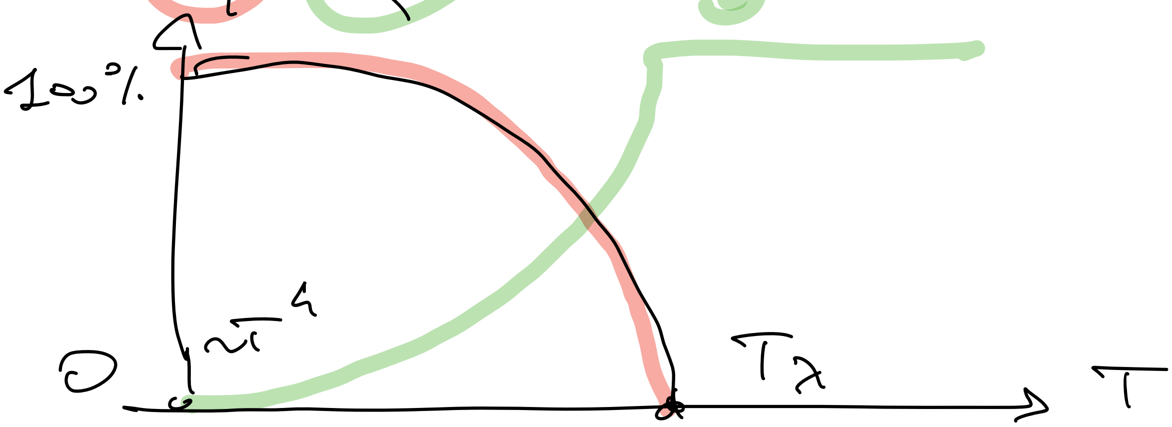
Andronikashvili experiment



$$I(T) = I_{cl}\left(1 - \frac{\rho_s(T)}{\rho}\right) + I_0$$

ρ_s/ρ, ρ_n/ρ

superfluid fraction



BEC / condensate depletion

$\chi_0(\vec{r})$, N_0 $\neq$ $\{ \chi_{\alpha \neq 0}(\vec{r}) \}$

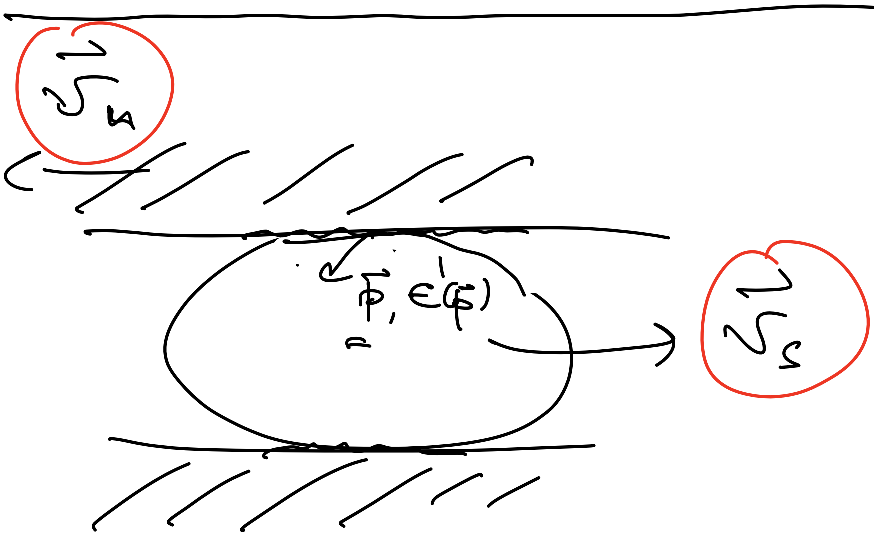
SF / normal

$\frac{N_0}{N}$ = condensate fraction $\simeq$ 7 % @ $T \geq 0$

$\frac{\rho_s}{\rho} \underset{T \to 0}{\sim}$ 100 %

$$\frac{\rho_s}{\rho} \neq \frac{N_0}{N}$$

Landau theory of the two-fluid model



$\epsilon'(\vec p) =$ energy cost of a quasi-particle excitation in the container frame ($\vec v_n$)

the only reference frame where there are no time-dependent forces → only frame in which I can define thermal equilibrium

$$\boxed{\epsilon'(\vec p) = \epsilon(\vec p) + \vec p \cdot (\vec v_s - \vec v_n)}$$

normal component : gas of quasi-particles excited in the fluid

$$\vec J_n = \rho_n(T)\,(\vec v_n - \vec v_s)$$

calculated in the reference frame of the fluid

$= [\text{momentum density}]$

$\to \dfrac{(m v)}{V}$

$$= \frac{1}{V} \sum_{\vec{p}} \vec{p} \; n_T(\vec{p})$$

→ @ thermal equilibrium at temp. T

quasi-particles are bosonic

$$\int \frac{d^3p}{(2\pi\hbar)^3}$$

$$= \frac{1}{V}\sum_{\vec{p}} \frac{\vec{p}}{e^{\beta[\epsilon(\vec{p}) + \vec{p}\cdot(\vec{v}_s - \vec{v}_n)]} - 1}$$

$$\beta \vec{p}\cdot(\vec{v}_s - \vec{v}_n) \ll \beta\,\epsilon(\vec{p})$$

$$\frac{1}{e^{\beta\epsilon + \gamma} - 1} \simeq \frac{1}{e^{\beta\epsilon} - 1} - \gamma \frac{e^{\beta\epsilon}}{(e^{\beta\epsilon} - 1)^2}$$

$$+ O(\gamma^2)$$

$$\gamma = \beta \vec{p}\cdot(\vec{v}_s - \vec{v}_n)$$

$$\vec{j}_n = \int \frac{d^3p}{(2\pi\hbar)^3} \vec{p} \left[\cancel{\frac{1}{e^{\beta\epsilon(\vec{p})} - 1}} + \right.$$

$$+\beta\, \vec{p}\cdot(\vec{v}_n-\vec{v}_s)\, \frac{e^{\beta\epsilon(\vec{p})}}{\left(e^{\beta\epsilon(\vec{p})}-1\right)^2} + \cdots \Big]$$

$$\lfloor\ \epsilon(p)\ \rfloor$$

$$\epsilon(\vec{p}) = \epsilon(|\vec{p}|)$$

$$\vec{J}_n = \frac{\beta}{(2\pi\hbar)^3}\int_0^\infty dp\, p^2\, \frac{e^{\beta\epsilon(p)}}{\left(e^{\beta\epsilon(p)}-1\right)^2}\int d\Omega\, \underbrace{[\vec{p}\cdot(\vec{v}_n-\vec{v}_s)]\vec{p}}_{p_x\,\vec{p}}$$

$$\vec{v}_n-\vec{v}_s \parallel x \qquad \lfloor\ \tfrac{1}{3}p^2(\vec{v}_n-\vec{v}_s)$$

$$\int d\Omega\, p_\mu p_\nu = \int d\Omega\, p_\mu^2\ \delta_{\mu\nu}$$

$$\mu,\nu = x,y,z \qquad = \frac{4\pi}{3}p^2\,\delta_{\mu\nu}$$

ρ_n

$$\vec{J}_n = \boxed{\frac{\beta\, 4\pi}{3(2\pi\hbar)^3}\int_0^\infty dp\, p^4\, \frac{e^{\beta\overset{cp}{\epsilon(p)}}}{\left(e^{\beta\overset{cp}{\epsilon(p)}}-1\right)^2}}\,(\vec{v}_n-\vec{v}_s)$$

$\epsilon(p) \simeq c_s p$ $\qquad x = \beta c p$

$=$

$T \ll$ roton energy

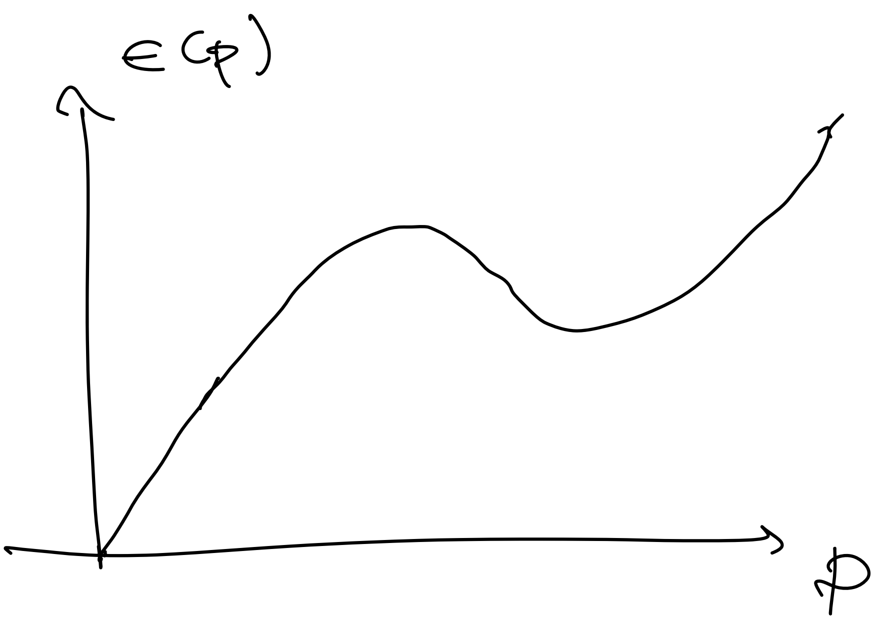
$p = \frac{x}{\beta c}$

$$= \frac{\beta}{3(2\pi\hbar)^3} \frac{4\pi}{(\beta c)^5} \int_0^\infty dx \, \frac{x^4 e^x}{(e^x - 1)^2} \, (\vec{v}_n - \vec{v}_s)$$

$\pi^4/15$

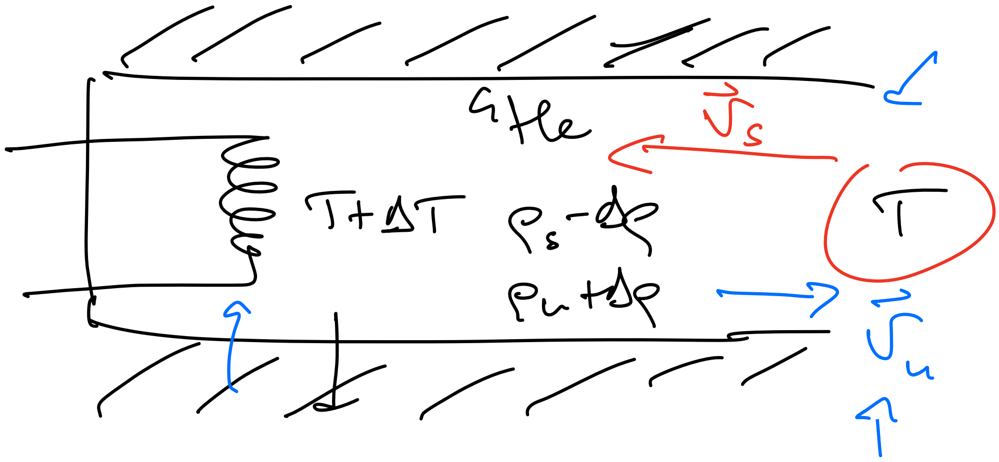
$$= \dots = \frac{2\pi^2 (k_B T)^4}{45 \hbar^3 c^5} (\vec{v}_n - \vec{v}_s)$$

$\rho_n(T)$

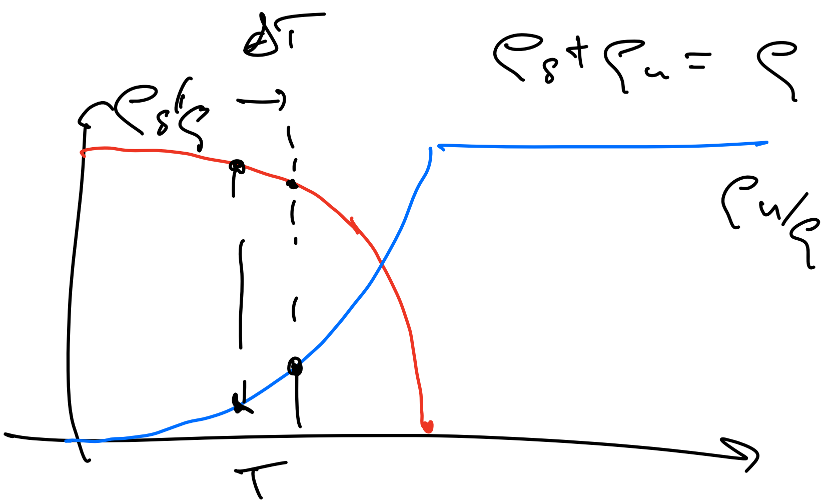


Thermomechanical effects

superfluid motion : carries no entropy



$$\rho_s + \rho_n = \rho$$

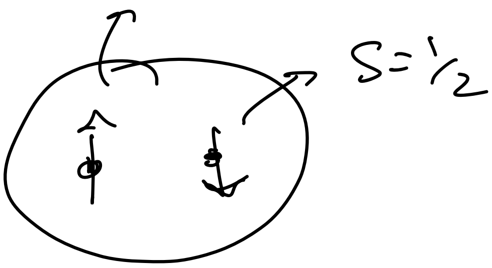


second sound : heat wave

Fountain effect

Fermions → super-conductivity
(when metals are cooled down to $T \sim 1\,K$)

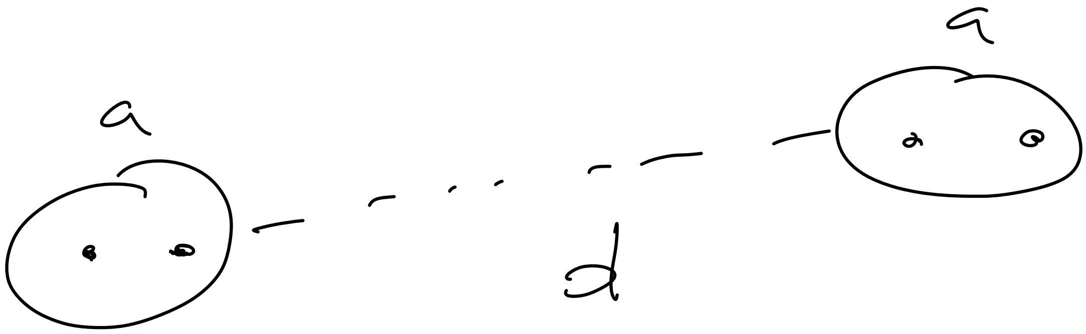
Fermions can condense if they form pairs (or even-numbered groups)



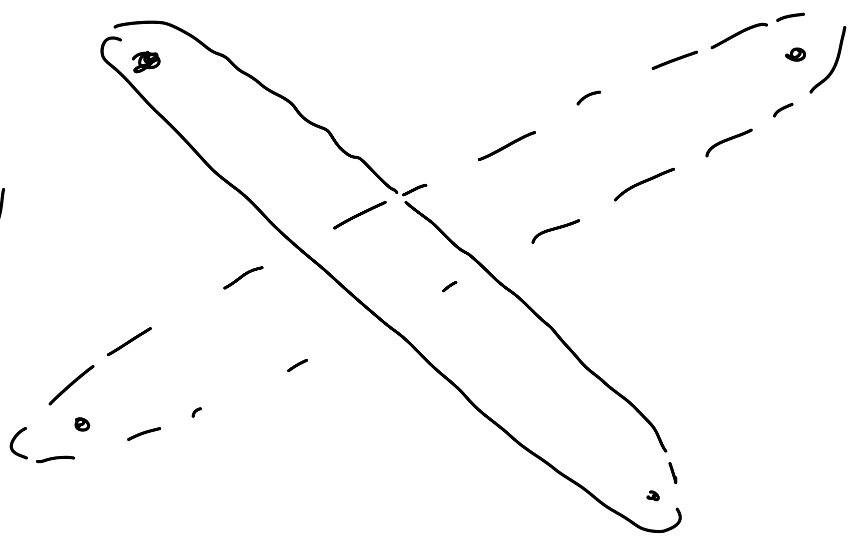
$S_{tot} = 1$

^{87}Rb , ^{23}Na , ^{4}He

Composite particle with even total spin → composite boson tightly bound fermions → composite bosons



$d \gg a$



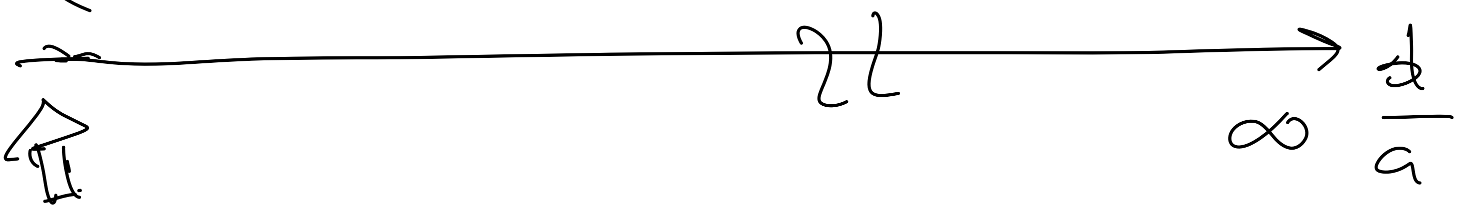
$d \lesssim a$

$d \ll a$

BEC - BCS crossover

composite bosons BEC

superconductivity (BCS limit)



(Ideal)

Normal Fermi gas

$$\mathcal{H} = \sum_\alpha \epsilon_\alpha \, a_\alpha^\dagger a_\alpha$$

$$\{a_\alpha, a_\beta^\dagger\} = \delta_{\alpha\beta}$$

$$\{a_\alpha, a_\beta\} = \{a_\alpha^\dagger, a_\beta^\dagger\} = 0$$

$$a_\alpha^\dagger a_\alpha = 0, 1$$

$$\langle n_\alpha \rangle = \frac{\sum_{n_\alpha = 0,1} n_\alpha e^{-\beta(\epsilon_\alpha - \mu) n_\alpha}}{\sum_{n_\alpha = 0,1} e^{-\beta(\epsilon_\alpha - \mu) n_\alpha}}$$

$$\mathcal{H} - \underline{\mu} N = \sum_\alpha (\epsilon_\alpha - \underline{\mu}) \, a_\alpha^\dagger a_\alpha$$

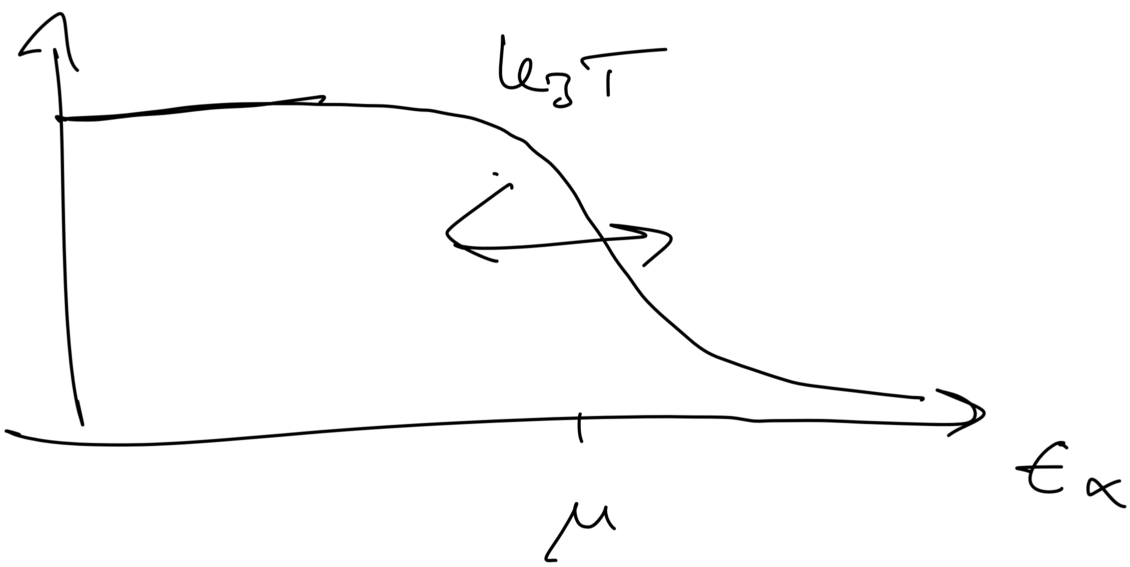
$$Z_G = \mathrm{Tr}\left[e^{-\beta(\mathcal{H} - \mu N)} \right]$$

$$= \sum_{\{n_\alpha\}} e^{-\beta \sum_\alpha (\epsilon_\alpha - \mu) n_\alpha}$$

$$\Omega_G = - k_B T \log Z_G$$

$$N = -\frac{\partial \Omega_G}{\partial \mu} = \sum_\alpha \langle n_\alpha \rangle$$

$$\langle n_\alpha \rangle = \frac{e^{-\beta(\epsilon_\alpha - \mu)}}{e^{-\beta(\epsilon_\alpha - \mu)} + 1} = \frac{1}{e^{\beta(\epsilon_\alpha - \mu)} + 1}$$

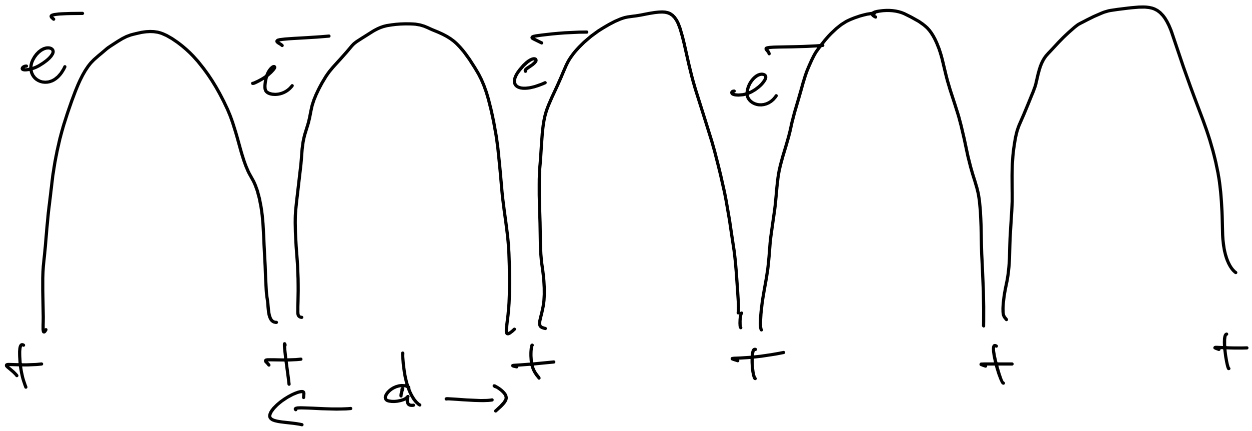


$$k_B T \lesssim \mu$$

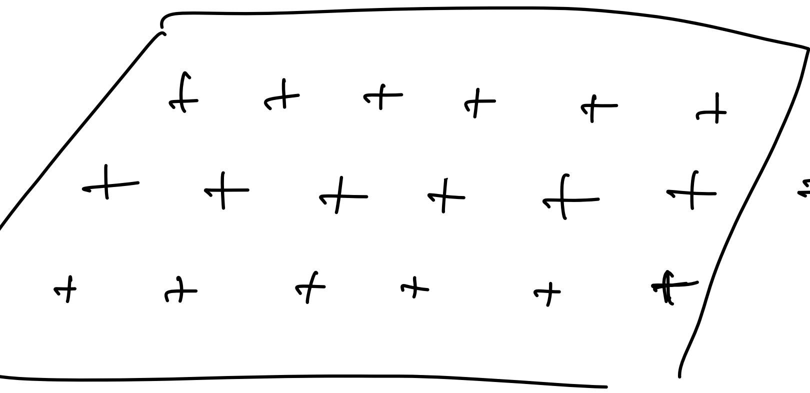
Electrons in metals

1) lattice ions

e^- → <u>Bloch waves</u> $\simeq$ plane waves $e^{i\vec{k}\cdot\vec{r}}$

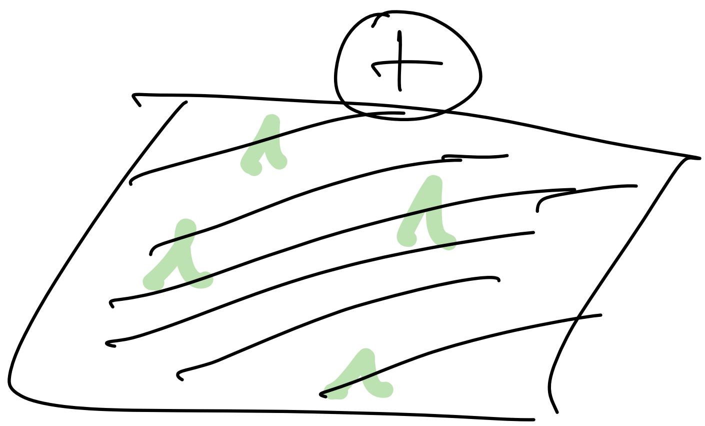


$kd \ll 1$



lattice

$\Rightarrow$



uniform distribution of positive charges ("jellium" model)

2) imperfections / dislocations / impurities
(phonons)

τ : typical scattering time between two scattering events

→ density of electrons

$$\sigma = \frac{n e^2 \tau}{m}$$

3) electrons interact with each other

$$e^- \overset{\sim d}{\cdot - - - - - \cdot} e^- \simeq \frac{e^2}{4\pi\varepsilon_0 d}\, e^{-d/\lambda}$$

Screening