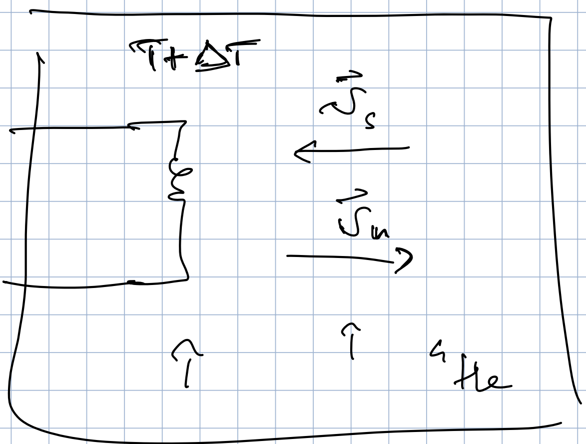
$$\min_p \frac{\epsilon(p)}{p} = \min_p \frac{\frac{p^2}{2m}}{p} = \frac{1}{2m} \underbrace{\min_{|\vec p|} p}_{=0}$$

$$\frac{\epsilon_p}{p} \simeq \frac{cp}{p} = c$$

$$\boxed{\epsilon(p) + \vec v \cdot \vec p < 0} \qquad |\vec v \cdot \vec p| > \epsilon(p) \quad (< 0)$$

$$vp > \vec v \cdot \vec p > \epsilon(p)$$

$$v > \frac{\epsilon(p)}{p} > \min_p \frac{\epsilon(p)}{p} = v_c$$



Theory of superconductivity : electron pairing

Normal metals : electrons are almost free even with

→ interaction with periodic potential

→ el - el interactions

↳ Landau theory of the Fermi liquid

unless you go too low in temperature ?

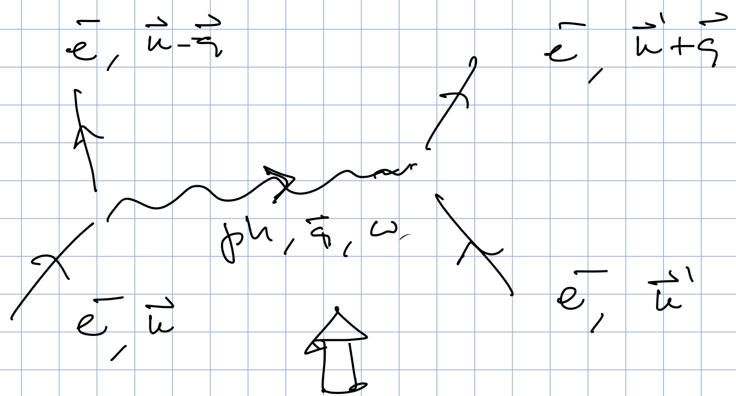
→ el - el interactions mediated by phonons

$$V_{el\text{-}el}(q,\omega) = \frac{e^2}{\epsilon_0(q^2+q_{TF}^2)}\left(1+\frac{\omega_q^2}{\omega^2-\omega_q^2}\right)$$

$$V_{bare}(\vec q) = \frac{e^2}{\epsilon_0 q^2}$$

ω_q ~ phonon frequency

picture :



$\omega < \omega_q$ attractive interaction

Leon Cooper (1956)

model attractive interactions

$\hat V_{eff}(\vec r_1 - \vec r_2)$ effective potential

matrix element

V = volume

$$\langle \vec k, -\vec k | \hat V_{eff} | \vec q, -\vec q\rangle = \frac{1}{V^2}\int d^3r_1\, d^3r_2\; e^{-i(\vec k\cdot\vec r_1 - \vec k\cdot\vec r_2)}\, V(\vec r_1 - \vec r_2)\, e^{i(\vec q\cdot\vec r_1 - \vec q\cdot\vec r_2)}$$

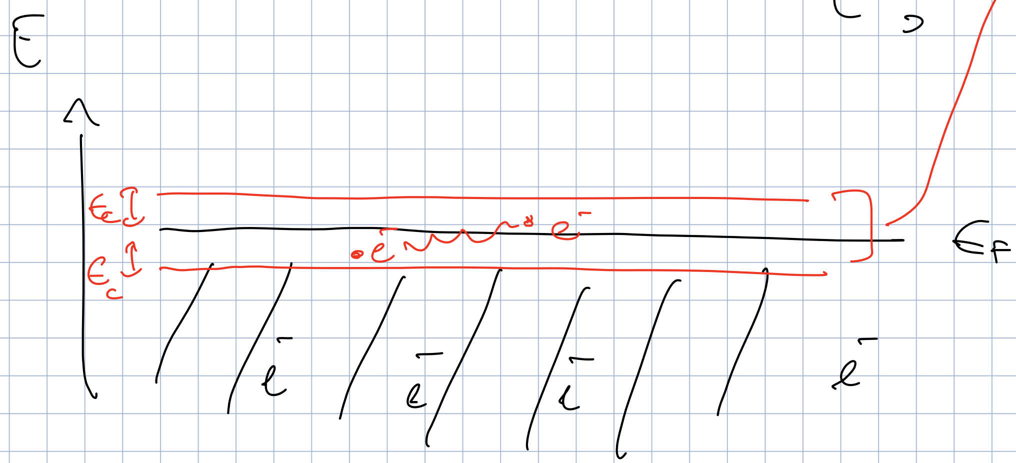
$$= \frac{1}{V^2}\int d^3r_1\, d^3r_2\; e^{i(\vec k-\vec q)(\vec r_1-\vec r_2)}\, V(\vec r_1-\vec r_2)$$

$$= \underbrace{\frac{1}{V}\int d^3r\; e^{i(\vec k-\vec q)\cdot\vec r}\, V(\vec r)}$$

$$= V_{\vec k-\vec q}$$

$V_0 > 0$

$$V_{\vec{k}\,\vec{q}} = \nu \langle \vec{k}\uparrow, -\vec{k}\uparrow | \hat{U}_{eff} | \vec{q}\uparrow, -\vec{q}\uparrow \rangle = \begin{cases} -V_0 & \text{if } |\epsilon_{\vec{k}} - \epsilon_F|, \ |\epsilon_{\vec{q}} - \epsilon_F| < \epsilon_c \\ 0 & \text{otherwise} \end{cases}$$



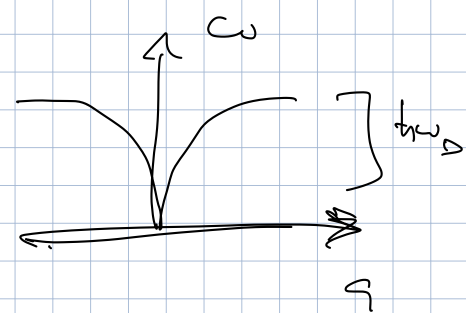
$\epsilon_c \ll \epsilon_F$

$\epsilon_c \simeq$ energy of phonons

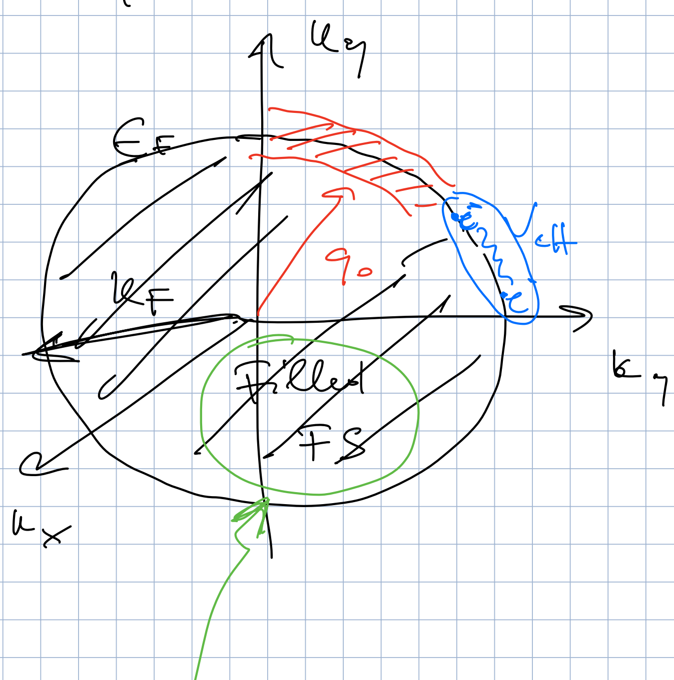
$\simeq \hbar\omega_D$: Debye energy

$= k_B T_D$

$T_D \sim 100\,K$



- Cooper problem : two-body problem



$$\left[-\frac{\hbar^2}{2m}\nabla_1^2 - \frac{\hbar^2}{2m}\nabla_2^2 + V_{eff}(\vec r_1 - \vec r_2)\right]\psi(\vec r_1, \vec r_2) = E\,\psi(\vec r_1, \vec r_2)$$

$$\psi(\vec r_1, \vec r_2) = \psi(\vec r_1 - \vec r_2)\ \cancel{\Phi(\vec R)}\quad 1$$

$\uparrow$ 1 $\qquad$ $\uparrow$ center of mass

$|\vec k| \geq k_F$

$$= \sum_{\vec k} e^{i\vec k\cdot(\vec r_1 - \vec r_2)}\,\psi_{\vec k} = \psi(|\vec r_1 - \vec r_2|)$$

$\psi_{\vec k} = \psi_{|\vec k|}$ $\qquad$ $\vec k$ $\qquad$ $-\vec k$ $\qquad$ s-wave symmetric

$$\psi(\vec r_1, \vec r_2)\ \underbrace{\frac{|\uparrow_1\downarrow_2\rangle - |\downarrow_1\uparrow_2\rangle}{\sqrt 2}}_{\text{singlet}} \longrightarrow \underline{\text{anti-symmetric !}}$$

s-wave

hope : $E - 2\epsilon_F < 0$ $\qquad$ bound state ?

$$V_{eff}(\vec r_1 - \vec r_2) = \frac{1}{V}\sum_{\vec q} V_{\vec q}\, e^{i\vec q\cdot(\vec r_1 - \vec r_2)}$$

1

$$\sum_{\vec k}\underbrace{\left[2\frac{\hbar^2 k^2}{2m} + \frac{1}{V}\sum_{\vec q} V_{\vec q}\, e^{i\vec q\cdot(\vec r_1 - \vec r_2)} - E\right]}\psi_{\vec k}\, e^{i\vec k\cdot(\vec r_1 - \vec r_2)} = 0$$

$$\sum_{\vec k}\left[\left(\overset{\epsilon_{\vec k}}{2\frac{\hbar^2 k^2}{2m}} - E\right)\psi_{\vec k} + \frac{1}{V}\sum_{\vec q} V_{\vec k - \vec q}\,\psi_{\vec q}\right] e^{i\vec k\cdot(\vec r_1 - \vec r_2)} = 0$$

$= 0$

$$\psi_{\vec{k}} = -\frac{1}{(2\epsilon_{\vec{k}} - E)} \frac{1}{V} \sum_{\vec{q}} V_{\vec{k}-\vec{q}} \psi_{\vec{q}}$$

$$= -\frac{1}{(2\epsilon_k - E)} \int \frac{d^3q}{(2\pi)^3} V_{\vec{k}-\vec{q}} \psi_{\vec{q}}$$

only depends on ϵ_q

only depends on ϵ_k, ϵ_q

$$\psi_{\vec{k}} = \psi_{|\vec{k}|} = \psi_{\epsilon_{\vec{k}}}$$

$$\cancel{\int_{\epsilon_F}^{\epsilon_F+\epsilon_c} d\epsilon\, \psi_\epsilon} = \int d\epsilon \, -\frac{1}{2\epsilon - E} \underbrace{\int_{\epsilon_F}^{\epsilon_F+\epsilon_c}}_{=} d\epsilon' \underbrace{g(\epsilon')}_{\simeq g(\epsilon_F)} \underbrace{(-V_0)}_{} \underbrace{\psi_{\epsilon'}}_{=}$$

$\epsilon_c \ll \epsilon_F$

$$= \int_{\epsilon_F}^{\epsilon_F+\epsilon_c} d\epsilon \frac{g(\epsilon_F) V_0}{2\epsilon - E} \cancel{\int_{\epsilon_F}^{\epsilon_F+\epsilon_c} d\epsilon'\, \psi_{\epsilon'}}$$

$$1 = g(\epsilon_F) V_0 \int_{\epsilon_F}^{\epsilon_F+\epsilon_c} d\epsilon \frac{1}{2\epsilon - E} = \frac{g(\epsilon_F) V_0}{2} \log |2\epsilon - E| \Big|_{\epsilon_F}^{\epsilon_F+\epsilon_c}$$

$$= \frac{g(\epsilon_F) V_0}{2} \log \left| \frac{2\epsilon_F - E + 2\epsilon_c}{2\epsilon_F - E} \right|$$

$$\tilde{E} = E - 2\epsilon_F$$

$$= \frac{g(\epsilon_F) V_0}{2} \log \left| \frac{-\tilde{E} + 2\epsilon_c}{-\tilde{E}} \right| = 1$$

$$\frac{\tilde{E} - 2\epsilon_c}{\tilde{E}} = e^{\frac{2}{V_0 g(\epsilon_F)}}$$

$$\left(1 - e^{\frac{2}{V_0 g(\epsilon_F)}}\right) \tilde{E} = +2\epsilon_c$$

$$\tilde{E} = E - 2\epsilon_F = \frac{2\epsilon_c}{1 - \underbrace{e^{2/V_0 g(\epsilon_F)}}_{\gg 1}}$$

$$V_0 g(\epsilon_F) \ll 1$$

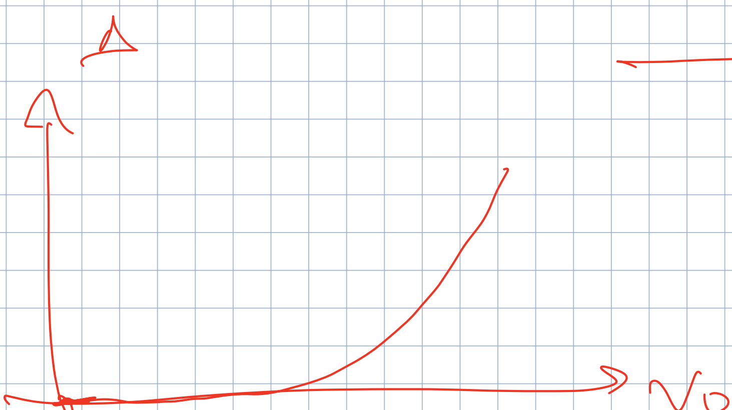
$$\simeq -2\epsilon_c \, e^{-\frac{2}{V_0 g(\epsilon_F)}} < 0 \quad = -\Delta$$

binding energy

bound state

Cooper pair

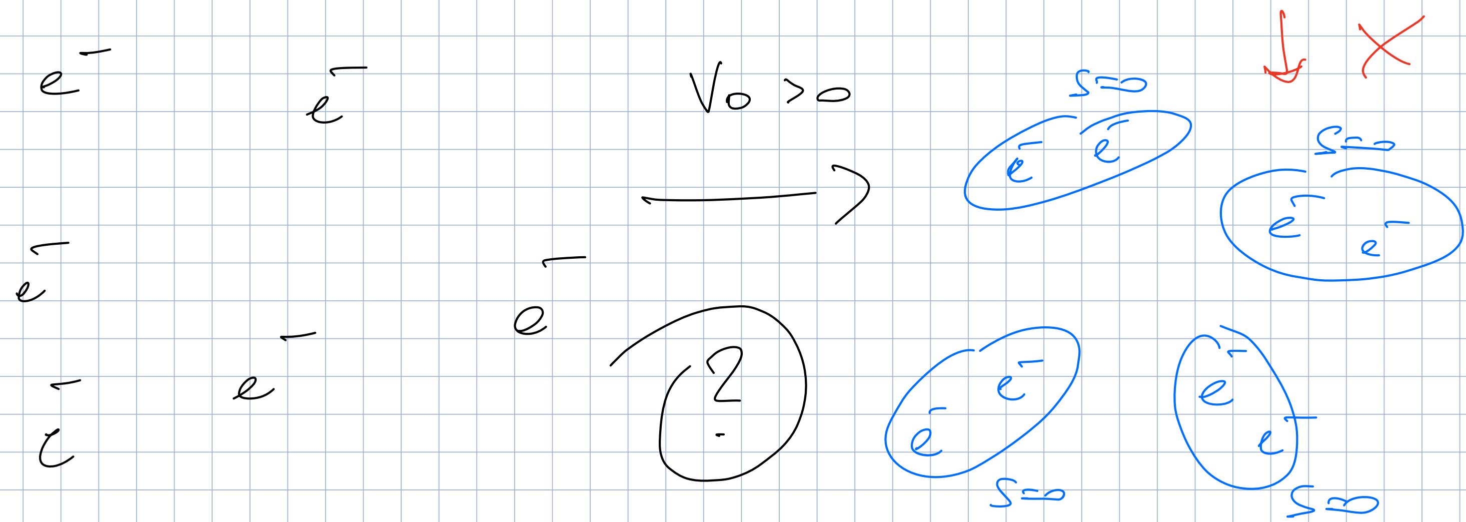
$\forall \; V_0$



$\Delta \simeq k_B T_0$ at which something new

$T \quad =$ is going to the electrons

in a metal



$$\frac{\Delta}{k_B} \simeq 1 \div 10 \text{ K}$$

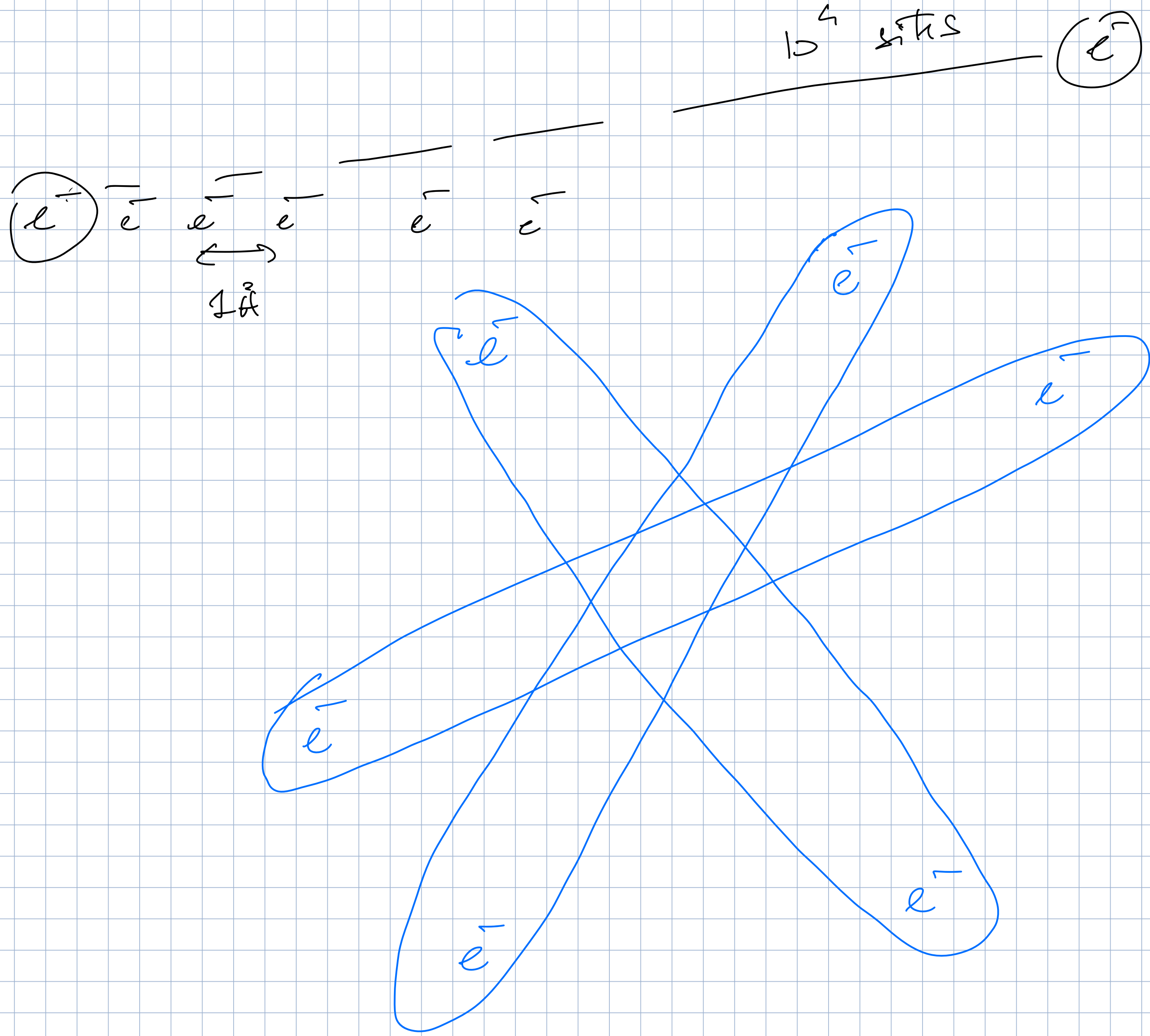
$$\psi_{\vec{k}} \to \psi(\vec{r}_1 - \vec{r}_2) = \psi(\vec{r})$$

$$\sqrt{\langle r^2 \rangle} \sim \frac{\hbar v_F}{\Delta} = \xi \simeq 10^4 \div 10^5 \text{ Å}$$

$\nearrow$ Pippard's coherence length

$$v_F = \frac{\hbar k_F}{m}$$

distance travelled by an electron at speed v_F in a time $\frac{\hbar}{\Delta} = t$



Many-body theory of electron pairing in metals:

BCS theory

(Bardeen - Cooper - Schrieffer).

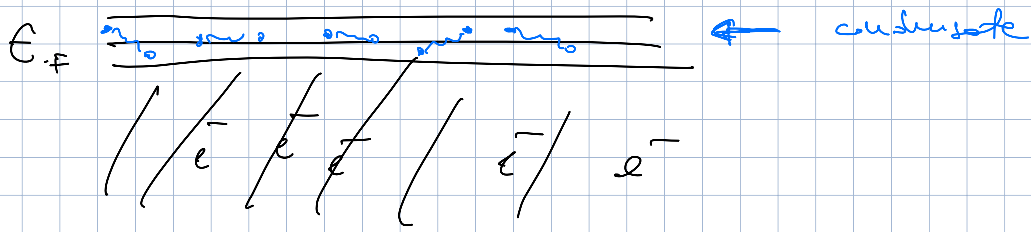
Make use of the Cooper instability to simplify the many-body problem

$\hat{\psi}_\sigma(\vec{r})$

$$\mathcal{H}^{\mu N} = \sum_\sigma \int d^3r \; \psi^\dagger_\sigma(\vec{r}) \left[\frac{\hbar^2}{2m} \nabla^2 - \mu \right] \psi_\sigma(\vec{r})$$

$$+ \frac{1}{2} \sum_{\sigma\sigma'} \int d^3r_1 \, d^3r_2 \; \psi^\dagger_\sigma(\vec{r}_1) \, \psi^\dagger_{\sigma'}(\vec{r}_2) \, V_{eff}(\vec{r}_1 - \vec{r}_2) \, \psi_{\sigma'}(\vec{r}_2) \, \psi_\sigma(\vec{r}_1)$$

Idea: many electrons forming Cooper pairs at the Fermi surface



$\Downarrow$

$$g^{(1)}_{\sigma\sigma'}(\vec{r}, \vec{r}') = \langle \psi^\dagger_\sigma(\vec{r})\, \psi_{\sigma'}(\vec{r}') \rangle \rightarrow \text{one-body DM}$$

$$G^{(2)}(\vec{r}_1 \sigma_1, \vec{r}_2 \sigma_2 ; \vec{r}_1{}' \sigma_1', \vec{r}_2{}' \sigma_2') \rightarrow \text{two-body density matrix}$$

$$G^{(2)}(i;j) = (G^{(2)}(j;i))^*$$

$$= \langle \psi^\dagger_{\sigma_1}(\vec{r}_1)\, \psi^\dagger_{\sigma_2}(\vec{r}_2)\, \psi_{\sigma_2'}(\vec{r}_2{}')\, \psi_{\sigma_1'}(\vec{r}_1{}') \rangle$$

$$= \sum_\alpha \lambda_\alpha\, \chi^*_\alpha(\vec{r}_1 \sigma_1, \vec{r}_2 \sigma_2)\, \chi_\alpha(\vec{r}_1{}' \sigma_1', \vec{r}_2{}' \sigma_2')$$

≥ 0 $\qquad \lambda_\alpha \sim O(1)$

or at most $\sim O(N)$

$\exists$ condensate of fermionic pairs if

$\exists\, \lambda_{\alpha=0} \sim O(N)$

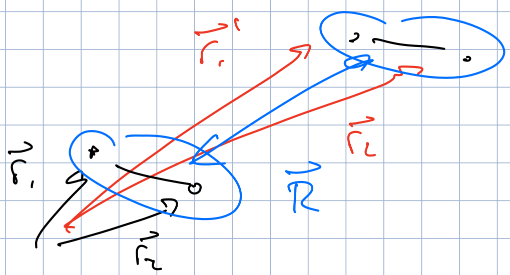
There is a macroscopic number of fermionic pairs in the same state $\chi_0(\vec{r}_1 \sigma_1 ; \vec{r}_2 \sigma_2)$

Spontaneous symmetry breaking (SSB)

$$= \langle \psi^\dagger \psi^\dagger \quad \psi \psi \rangle$$

$$G^{(2)}(\vec r_1 \sigma_1, \vec r_2 \sigma_2 ; \vec r_1' \sigma_1', \vec r_2' \sigma_2') \simeq$$

$$\downarrow \quad \vec R \to \infty$$



$$\sqrt{N_0}\, \chi_0^*(\vec r_1 \sigma_1, \vec r_2 \sigma_2) \sqrt{N_0}\, \chi_0(\vec r_1' \sigma_1', \vec r_2' \sigma_2')$$

SSB

$$\simeq \boxed{\langle \psi^\dagger_{\sigma_1}(\vec r_1)\, \psi^\dagger_{\sigma_2}(\vec r_2) \rangle} \langle \psi_{\sigma_2'}(\vec r_2')\, \psi_{\sigma_1'}(\vec r_1') \rangle$$

$$N_0 \sim O(N)$$

$$\psi^\dagger_{\sigma_1}(\vec r_1)\, \psi^\dagger_{\sigma_2}(\vec r_2) = \langle \qquad \rangle + \delta(\psi^\dagger \psi^\dagger)$$

T

small

$$\psi^\dagger_\sigma(\vec r) = \sum_k \frac{e^{-i\vec k\cdot\vec r}}{\sqrt V}\, c^\dagger_{k\sigma}$$

$$\Downarrow$$

$$c^\dagger_{\vec k\sigma}\, c^\dagger_{\vec k'\sigma'} = \langle c^\dagger_{\vec k\sigma}\, c^\dagger_{\vec k'\sigma'} \rangle + \delta(\qquad)$$

$\sim$ FT of the Cooper-pair wavefunction

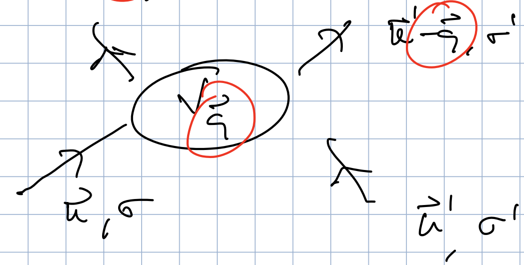
$$\mathrm{FT}\left(\psi(\vec r_1 - \vec r_2)\, \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt 2} \right)$$

$$\langle c^\dagger_{\vec k\sigma}\, c^\dagger_{\vec k'\sigma'} \rangle \sim \delta_{\vec k, -\vec k'}\, \delta_{\sigma\sigma'}$$

$\mu = \epsilon_F$

$$\mathcal{H} - \mu N = \sum_{\vec k \sigma} (\epsilon_{\vec k} - \mu)\, c^\dagger_{\vec k \sigma} c_{\vec k \sigma}$$

$$+ \underbrace{\frac{1}{2V}\sum_{\vec k \vec k' \vec q} \sum_{\sigma\sigma'}} V_{\vec q}\; c^\dagger_{\vec k+\vec q,\sigma}\, c^\dagger_{\vec k'-\vec q,\sigma'}\, c_{\vec k'\sigma'}\, c_{\vec k \sigma}$$



$$c^\dagger_{\vec k+\vec q,\sigma}\, c^\dagger_{\vec k'-\vec q,\sigma'} \xrightarrow[\substack{\vec k = -\vec k' \\ \sigma' = -\sigma = \bar\sigma}]{} c^\dagger_{\vec k+\vec q,\sigma}\, c^\dagger_{-\vec k-\vec q,\bar\sigma}$$

$$= \langle \qquad \rangle + \delta(\qquad)$$

$$\mathcal{H} - \mu N \simeq \sum_{\vec k \sigma} (\epsilon_k - \mu)\, c^\dagger_{k\sigma} c_{\vec k \sigma}$$

$$\underbrace{+ \frac{\cancel{1}}{\cancel{2}V}\sum_{\vec k \vec q} \cancel{\sum_{\sigma}}\; V_{\vec k-\vec q}\; \underbrace{c^\dagger_{k\uparrow}\, c^\dagger_{-\vec k,\downarrow}}_{\langle\ \rangle + \delta(\ )}\; \underbrace{c_{-\vec q,\downarrow}\, c_{\vec q\uparrow}}_{\langle\ \rangle + \delta(\ )}}$$

$$-\frac{V_0}{V}\sum_{\vec k}{}' \sum_{\vec q}{}' \Big[\langle c^\dagger_{\vec k\uparrow}\, c^\dagger_{-k\downarrow}\rangle\, c_{-q\downarrow}\, c_{q\uparrow} + h.c. - \langle\ \rangle\langle\ \rangle\Big]$$

$\downarrow$

$$\sum_{\vec k:\, |\epsilon_{\vec k} - \epsilon_F| \le \epsilon_c}$$

pairing function

$$\boxed{\Delta_{\vec{k}} = \frac{V_0}{V} \sum_{\vec{q}}{}' \langle c_{-\vec{q}\downarrow} \, c_{\vec{q}\uparrow} \rangle}$$

$$\boxed{\begin{aligned} \mathcal{H}_{BCS} - \mu N \simeq & \sum_{\vec{k}\sigma} (\epsilon_k - \mu) \, c^\dagger_{k\sigma} \, c_{k\sigma} \\ & - \sum_{\vec{k}} \left(\Delta_{\vec{k}} \, c^\dagger_{k\uparrow} \, c^\dagger_{-k\downarrow} + h.c. \right) + \text{const.} \end{aligned}}$$