Weakly interacting BEC : Bogolyubov theory

$$\mathcal{H}-\mu N=\int d^dr\ \psi^\dagger\left(-\frac{\hbar^2}{2m}\nabla^2-\mu\right)\psi+\frac{g}{2}\int d^dr\ \psi^\dagger\psi^\dagger\psi\psi$$

$$\Rightarrow \quad V(\vec r-\vec r\,')=g\ \delta(\vec r-\vec r\,')$$

$$g=\frac{4\pi a_s\hbar^2}{m} \Leftarrow$$

a_s = s-wave scattering length

$$\hat\psi=\underline{\Psi_0(\vec r)}+\delta\hat\psi$$

$$\mathcal{H}-\mu N=E_{GP}[\Psi,\Psi_0^*]+\mathcal{H}^{(2)}+\cancel{\dots}$$

$$\hat\psi(\vec r)=\sum_{\vec k}\frac{e^{-i\vec k\cdot\vec r}}{\sqrt V}\hat a_{\vec k} \qquad \Psi_0=\sqrt{n_0}$$

n_0 = condensate density

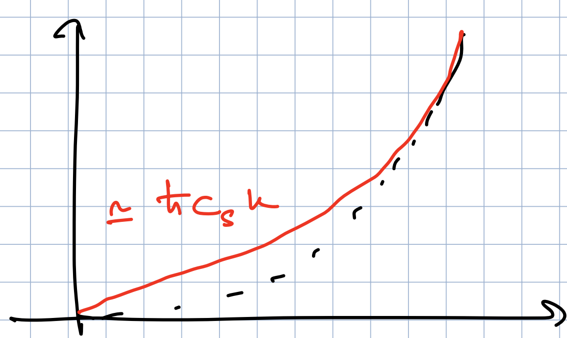
$$\Rightarrow \boxed{\mathcal{H}^{(2)}=\sum_{\vec k\neq 0}E_{\vec k}\,b_{\vec k}^\dagger b_{\vec k}+\text{const.}} \leftarrow$$

$$\rightarrow b_{\vec k}=u_{\vec k}a_{\vec k}+v_{\vec k}a^\dagger_{-\vec k}$$ Bogolyubov transformation

related to "quasi-particles": collective excitations

$$E_{\vec k}=\sqrt{\frac{\hbar^2k^2}{2m}\left(\frac{\hbar^2k^2}{2m}+2gn_0\right)}$$

$$c_s=\sqrt{\frac{gn_0}{m}}$$



$$E_{\vec k}>0 \qquad |\vec k|>0$$

Ground state of the "Bogolyubov gas"

$\equiv$ vacuum of b quasi-particles

$\vec{k} \neq 0$) $\langle a^\dagger_{\vec{k}} a_{\vec{k}} \rangle_{vac} = \langle 0 | (u_k \cancel{b^\dagger_k} - v_k b_{-k}) (u_k \cancel{b_k} - v_k b^\dagger_{-k}) | 0 \rangle$

$a_k = u_k b_k - v_k b^\dagger_{-k}$

$$= v_k^2 \langle 0 | \underbrace{b_{-k} b^\dagger_{-k}}_{1 + \cancel{b^\dagger_{-k} b_{-k}}} | 0 \rangle = v_k^2$$

we assumed that

$$N - N_0 = \int d^d r \langle \delta\psi^\dagger \delta\psi \rangle$$

$$= \sum_{\vec{k} \neq 0} \langle a^\dagger_k a_k \rangle = \sum_{\vec{k} \neq 0} v_k^2$$

$$\ll N$$

$$v_k^2 = \frac{1}{2} \left(\frac{A_k}{E_k} - 1 \right) \qquad A_k = \frac{\hbar^2 k^2}{2m} + g n_0$$

$$E_k$$

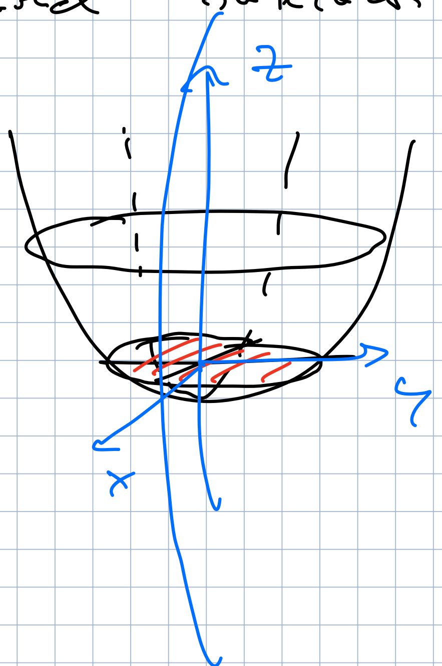
$k \to 0 \quad \sim \quad \frac{g n_0}{k}$

$$\frac{N - N_0}{V} = \frac{1}{V} \sum_{\vec{k} \neq 0} v_k^2$$

$$\sim \int \frac{d^d k}{(2\pi)^d} \frac{1}{k} \sim \int_0 dk\, k^{d-2}$$

$d = 1$ $\quad \frac{N-N_0}{V} \to \infty$

No BEC in the ground state for the one-dimensional interacting Bose gas



$$\Psi(x_1, y_1, z_1, x_2, y_2, z_2, \ldots)$$

$$= \prod_{i=1}^{N} \phi_\perp(x_i, y_i) \quad \Psi(z_1, z_2, \ldots, z_N)$$

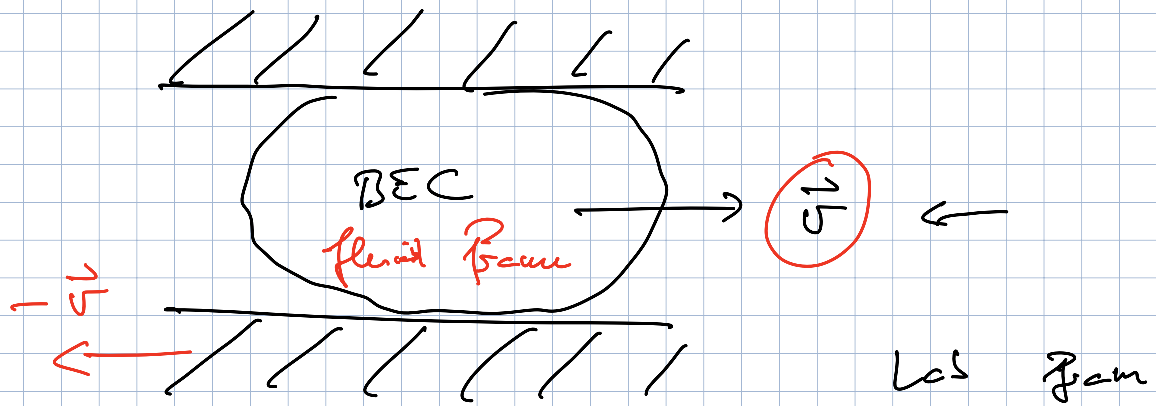
(2) $\underline{d=3}$: $\quad \frac{N-N_0}{V}$ finite

$$\Rightarrow \frac{N-N_0}{N} = \ldots = \frac{8}{3\sqrt{\pi}} (n a_s^3)^{1/2} \ll 1 \qquad n = \frac{N}{V}$$

BEC survives interactions in $d = 2, 3$

BEC with interactions : superfluidity

Superfluidity : ability to flow without friction



Landau's criterion for superfluidity

Fluid frame

$$\mathcal{H} = \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m} + V_{int}$$

$\vec{p}_i$: momentum in the fluid frame

$$\vec{P} = \sum_i \vec{p}_i$$

Lab frame

$$\vec{p}_i \to \vec{p}_i + m\vec{v}$$

$$\mathcal{H}' = \sum_{i=1}^{N} \frac{(\vec{p}_i + m\vec{v})^2}{2m} + V_{int}$$

$$= \underbrace{\sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m} + V_{int}}_{\to \mathcal{H}} + \sum_i \vec{p}_i \cdot \vec{v} + \frac{N}{2} m \vec{v}^{\,2}$$

$$\Rightarrow \vec{P} \cdot \vec{v}$$

Let us assume that our system can be described in terms of free quasi-particles

$$H = \sum_{\vec p} \varepsilon(\vec p)\, b^{\dagger}_{\vec p} b_{\vec p} + \dots$$

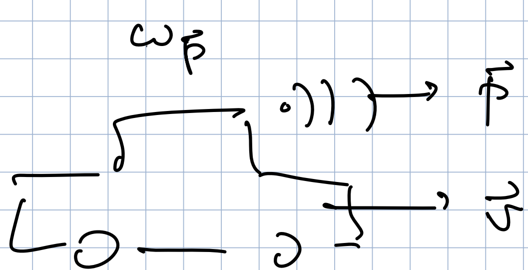
$$\vec P = \sum_{\vec p} \vec p\, b^{\dagger}_{\vec p} b_{\vec p}$$

$$H' = \sum_{\vec p} \left(\varepsilon(\vec p) + \vec p \cdot \vec v\right) b^{\dagger}_{\vec p} b_{\vec p} + \dots$$

(1)

$$\tilde\varepsilon(\vec p) = \varepsilon(\vec p) + \vec p \cdot \vec v$$

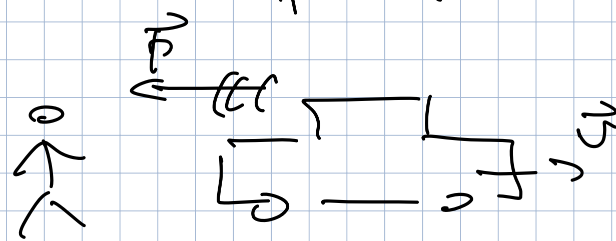
Doppler shift



$\varepsilon' > \varepsilon$

$\omega' > \omega$

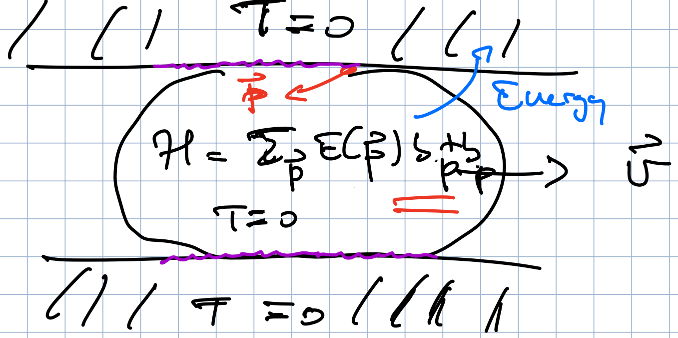
$\vec p \cdot \vec v > 0$



$\varepsilon' < \varepsilon$

$\vec p \cdot \vec v < 0$

Boundary walls are @ $\underline{T=0}$



@ $t=0$

vacuum of
b quasi-particles

only possibility to perturb the flow of the fluid :

emission of a quasi-particle $(\vec{p}\cdot\vec{v} < 0)$

$$\Rightarrow \quad E^{(1)}(\vec{p}) = \underbrace{E(\vec{p}) + \underline{\vec{p}\cdot\vec{v}}}_{d} \ \underline{\leq} \ 0 \qquad \leftarrow$$

Lab frame $\nearrow$ $\rightarrow$ energy given to the walls

$\uparrow \quad \uparrow$

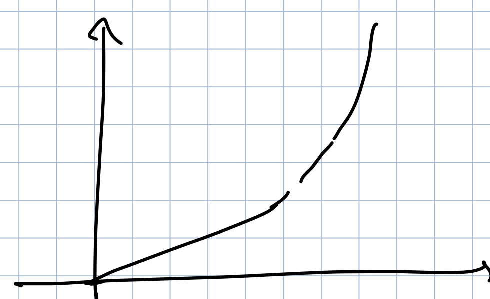
$$E(\vec{p}) \leq -\vec{p}\cdot\vec{v} \leq p\,v$$

$$\underline{v} \geq \frac{E(\vec{p})}{p} \geq \min_{\vec{p}} \frac{E(\vec{p})}{p} = \underline{\underline{v_c}} \ \underline{\underline{>0}}$$

critical value

Bogolyubov gas $\rightarrow$ superfluid

$$E(\vec{p}) = \sqrt{\frac{p^2}{2m}\left(\frac{p^2}{2m} + 2g n_0\right)} \underset{p\to 0}{\simeq} \underline{c_s}\, p$$



interactions   BEC

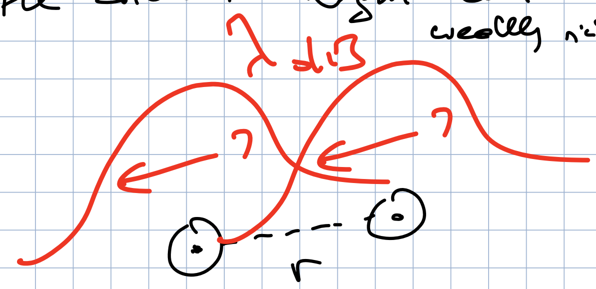
$$\min_{\vec{p}} \frac{E(p)}{p} = c_s = \sqrt{\frac{g n_0}{m}} > 0$$

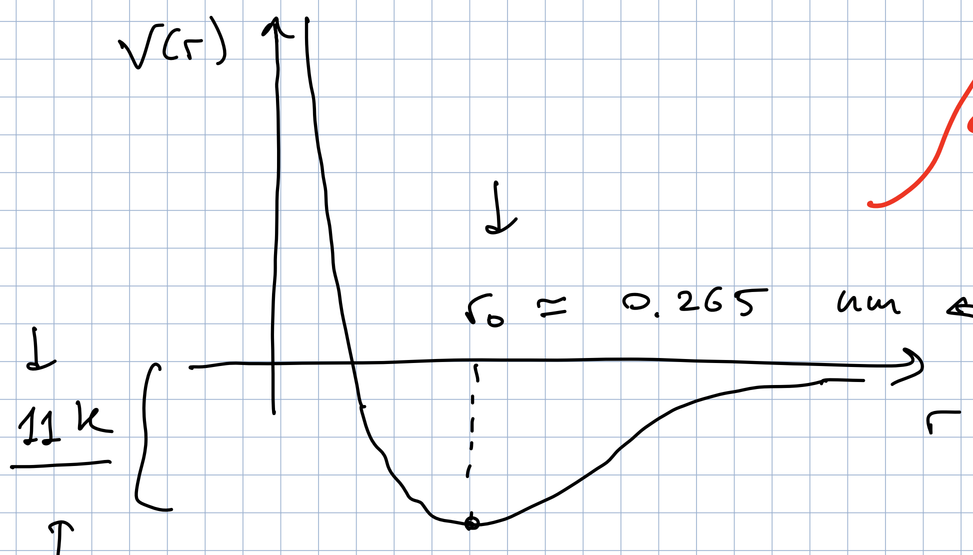
Ideal Bose gas → not superfluid

$$\min_{\vec{p}} \frac{E(p)}{p} = \min_{\vec{p}} \frac{p}{2m} = 0$$

Superfluid ^{4}He

He atom: light and weakly interacting





$$\lambda_{dB} = \frac{h}{\sqrt{2\pi m k_B T}} \underset{@\,11K}{=} 0.4 \text{ nm}$$

@ T > 0 : not all the ^{4}He atoms are participating to the frictionless flow

Notion of superfluid fraction

ρ_s : mass density of particles that exhibit frictionless flow

$$\frac{\rho_s}{\rho} = \text{sup. fraction}$$

@ $T = 0$ $\qquad \rho_s/\rho = 1 \qquad v < v_c$

100 % superfluid

In general

superfluid

$$\rho = \rho_s(T) + \rho_n(T)$$

mass density

normal part

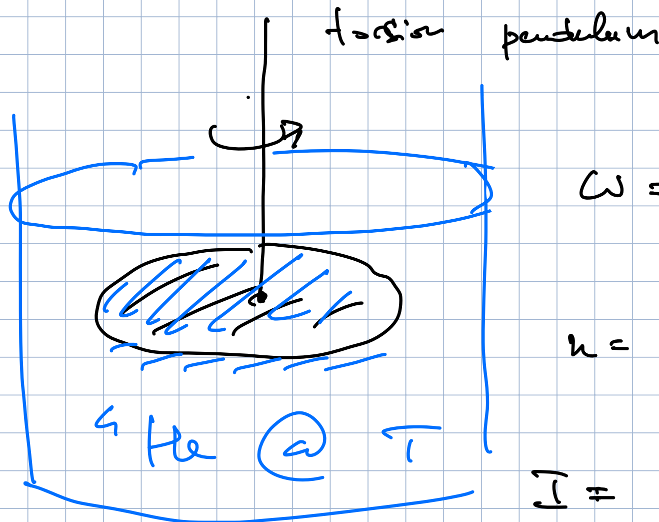
"Two-fluid" model:

$$\vec{j}(\vec{r}) = \rho_s(T)\,\vec{v}_s(\vec{r}) + \rho_n(T)\,\vec{v}_n(\vec{r})$$

mass current

follows the boundaries like in a viscous fluid

Andronikashvili's experiment



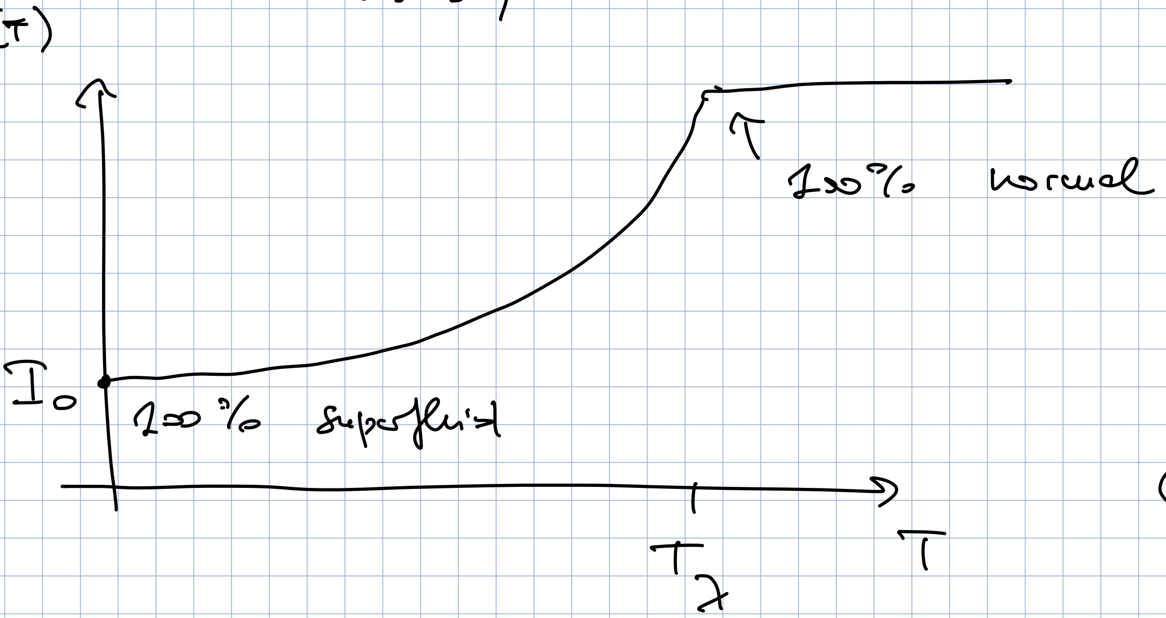
$$\omega = \sqrt{\frac{k}{I}}$$

k = elastic constant of the wire

I = moment of inertia of the disc

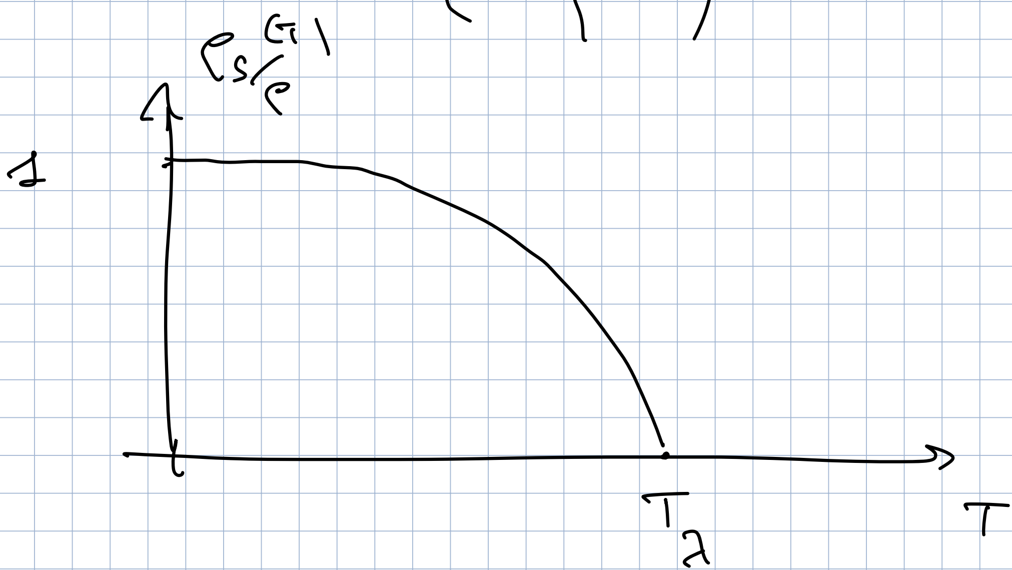
$$I^{(T)} = I_0 + I_{liq.}(T) \qquad {}^4He$$

$\uparrow$ viscosity



$$\frac{\rho_n}{\rho} = 1 - \frac{\rho_s}{\rho}$$

$$I(T) = I_0\left(1 + \frac{\rho_n(T)}{\rho}\right) = I_0\left(2 - \frac{\rho_s(T)}{\rho}\right)$$



^4He is @ $T=0$ $\begin{cases} 100\ \% & \text{superfluid} \\ \sim 7\ \% & \text{Bose-Einstein condensed} \end{cases}$

normal fraction $\neq$ condensate depletion

Even though ^{4}He is only 7% condensed

$$\vec{J}(\vec{r}) = \vec{J}_0(\vec{r}) + \underbrace{\sum_{\alpha\neq 0} \vec{J}_\alpha(\vec{r})}_{?}$$

$\vec{J}(\vec{r})$: mass current

$\vec{J}_0(\vec{r})$: condensate

$$m\, n_0^{^4He}(\vec{r})\; \underbrace{\vec{v}_s(\vec{r})}_{\frac{\hbar}{m}\vec{\nabla}\phi_0(\vec{r})}$$

macroscopic current which is irrotational

BEC + interactions $\Rightarrow$ superfluidity

$\not\Leftarrow$

One can have superfluidity without BEC

(requirement: quasi-BEC $N_0 \sim O(N^\alpha)$)

$\underline{\alpha < 1}$