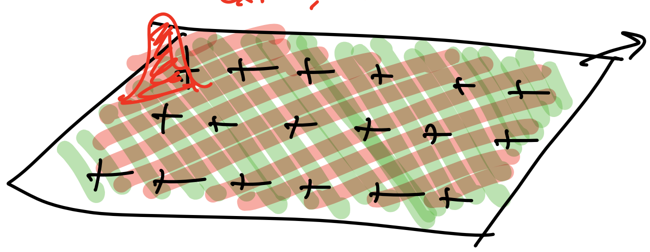
Electrons in metals

Free / ideal Fermi gas

Effective picture

$\rho_{ext}(\vec{r}, t)$



positive ions form a uniform background

Mean-field picture : each electron only sees the average distribution of the other electrons

electrons feel no net V potential

Response of the electron gas in a uniform positive background to an external charge distribution :

theory of dielectric screening

$$\begin{cases} \nabla^2 \phi_{ext}(\vec{r}, t) = -\dfrac{\rho_{ext}(\vec{r}, t)}{\epsilon_0} \\ \nabla^2 \phi(\vec{r}, t) = -\dfrac{\rho(\vec{r}, t)}{\epsilon_0} \end{cases}$$

FT

$$\rho = \rho_{ext}(\vec{r}, t) + \rho_{ind}(\vec{r}, t)$$

Assumption $\phi = 0$ if $\phi_{ext} = 0$

linear relationship between ϕ_{ext} and ϕ

$$\phi_{ext}(\vec{r}, t) = \int d^3\vec{r}\,' \int dt' \underbrace{\epsilon(\vec{r}-\vec{r}\,', t-t')}_{\text{dielectric function}} \phi(\vec{r}\,', t')$$

$$\phi_{ext}(\vec{q}, \omega) = \epsilon(\vec{q}, \omega)\, \phi(\vec{q}, \omega)$$

$$\boxed{\phi(\vec{q}, \omega) = \frac{1}{\epsilon(\vec{q}, \omega)} \phi_{ext}(\vec{q}, \omega)}$$

$$\begin{cases} -q^2 \phi_{ext}(\vec{q}, \omega) = -\dfrac{\rho_{ext}(\vec{q}, \omega)}{\epsilon_0} \\[2ex] -q^2 \phi(\vec{q}, \omega) = -\dfrac{\rho(\vec{q}, \omega)}{\epsilon_0} \end{cases}$$

$$\rho(\vec{q},\omega) = \frac{\rho_{ext}(\vec{q},\omega)}{\epsilon(\vec{q},\omega)}$$

$$\frac{1}{\epsilon(\vec{q},\omega)} = \frac{\rho_{ext}(\vec{q},\omega) + \rho_{ind}(\vec{q},\omega)}{\rho_{ext}(\vec{q},\omega)}$$

$$\phi(\vec{q},\omega) = \frac{1}{q^2 \epsilon_0}\rho(\vec{q},\omega)$$

$$= \frac{1}{\epsilon_0 q^2}\left[\rho_{ind} + q^2 \epsilon_0 \phi_{ext}\right]$$

$$= \phi_{ext}(\vec{q},\omega) + \frac{1}{\epsilon_0 q^2}\rho_{ind}$$

$$= \frac{1}{\epsilon(\vec{q},\omega)}\phi_{ext}$$

$$\boxed{\frac{1}{\epsilon(\vec{q},\omega)} = 1 + \frac{1}{\epsilon_0 q^2}\frac{\rho_{ind}}{\phi_{ext}}}$$

Thomas–Fermi theory of dielectric screening

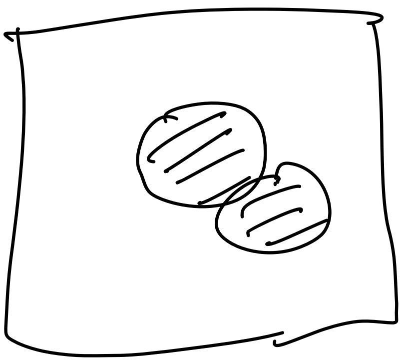
$$-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r}) - e\phi(\vec{r}) = \varepsilon\psi(\vec{r})$$

static potential ≡ immediate response of the electron gas / no retardation effect

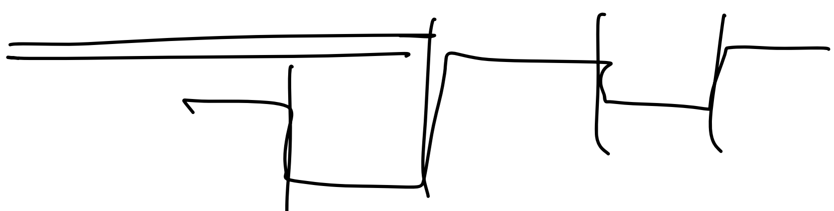
assuming that $\phi(\vec{r})$ varies slowly in space

$$-\frac{\hbar^2}{2m}\nabla^2\psi = \underbrace{(\varepsilon + e\phi(\vec{r}))}_{\frac{\hbar^2k^2}{2m}}\psi$$

$$\psi = e^{i\vec{k}\cdot\vec{r}}$$



$$\varepsilon = \frac{\hbar^2k^2}{2m} - e\phi(\vec{r})$$



T

$$\underbrace{-e\, n(\vec{r})}_{\text{total charge density}} \simeq \frac{-e}{V} \sum_{\vec{k}} \frac{\overset{\nearrow \text{spin deg.}}{2}}{e^{\beta\left(\frac{\hbar^2 k^2}{2m} \underbrace{- e\phi(\vec{r}) - \mu}_{-\mu_{\text{eff}}}\right)} + 1}$$

$$\mu_{\text{eff}} = \mu + e\,\phi(\vec{r})$$

$\longrightarrow$ (local density approximation)

$$= -e \int \frac{d^3k}{(2\pi)^3} \frac{2}{e^{\beta\left[\frac{\hbar^2 k^2}{2m} - (e\phi(\vec{r}) + \mu)\right]} + 1}$$

$$= -e\, \underbrace{n_0}_{\text{density of the ideal Fermi gas}} (\mu + e\phi(\vec{r})) = \rho_{el}(\vec{r})$$

$$\rho_{ind}(\vec{r}) = \overset{(\phi \neq 0)}{\rho_{el}(\vec{r})} - \underbrace{\overset{(\phi = 0)}{\rho_{el}}}$$

$$= -e\, n_0(\mu + e\,\phi(\vec{r})) + e\, n_0(\mu)$$

$$= \cancel{-e n_0(\mu)} + \cancel{e n_0(\mu)}$$

$$- e^2 \phi(\vec{r}) \frac{\partial n_0}{\partial \mu}(\mu)$$

$$\rho_{ind}(\vec{q},\omega) = -e^2 \phi(\vec{q},\omega) \frac{\partial n_0}{\partial \mu}$$

$$= -e^2 \frac{\phi_{ext}(\vec{q},\omega)}{\epsilon(\vec{q},\omega)} \frac{\partial n_0}{\partial \mu}$$

$$\frac{\rho_{ind}}{\phi_{ext}} = -e^2 \frac{\partial n_0}{\partial \mu} \frac{1}{\epsilon(\vec{q},\omega)}$$

$$\frac{1}{\epsilon(\vec{q},\omega)} = 1 - \overbrace{\frac{e^2}{\epsilon_0 q^2} \frac{\partial n_0}{\partial \mu}}^{q_{TF}^2} \frac{1}{\epsilon(\vec{q},\omega)}$$

$$\frac{1}{\epsilon(\vec{q},\omega)} \left(1 + \frac{q_{TF}^2}{q^2}\right) = 1$$

$$\boxed{\epsilon(\vec{q},\omega) = 1 + \frac{q_{TF}^2}{q^2}}$$

$$\rho_{ext}(\vec{r}) = Q\,\delta(\vec{r})$$

$$\phi_{ext}(\vec{r}) = \frac{Q}{4\pi\epsilon_0 r}$$

$$\Rightarrow \phi_{ext}(\vec{q}) = \frac{Q}{\epsilon_0 q^2}$$

$$FT\left(\frac{1}{r}\right) = \frac{4\pi}{q^2}$$

$$\phi(\vec{q}) = \frac{1}{\epsilon(\vec{q})}\,\frac{Q}{\epsilon_0 q^2}$$

$$= \frac{\cancel{q^2}}{q^2 + q_{TF}^2}\,\frac{Q}{\epsilon_0 \cancel{q^2}}$$

$$= \frac{Q}{\epsilon_0 (q^2 + q_{TF}^2)}$$

$$\frac{1}{q^2 + x^2}$$

Lorentzian

$$\boxed{\phi(\vec{r}) = \frac{Q}{4\pi\epsilon_0 r}\, e^{-q_{TF} r}}$$

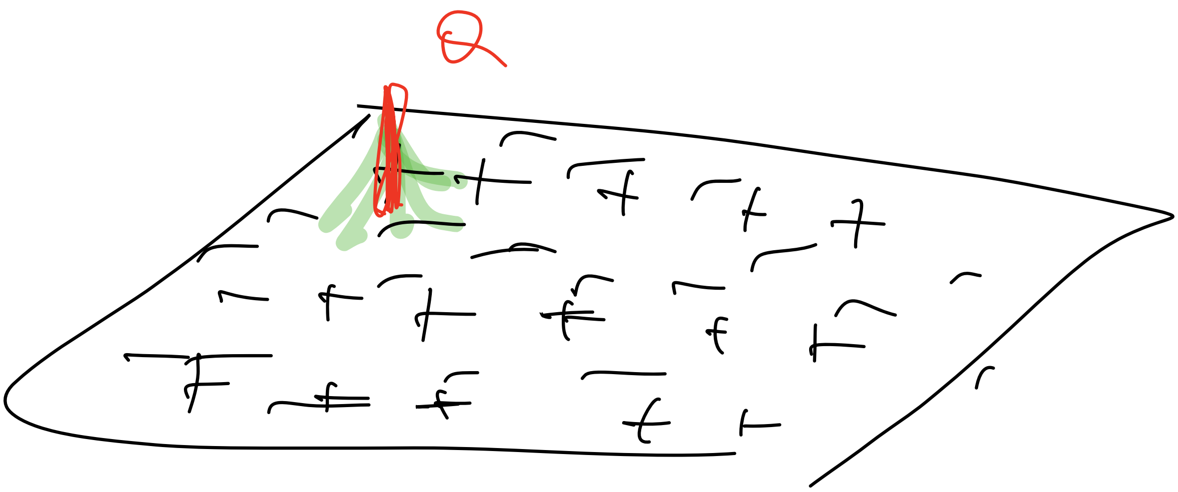
$$q_{TF}^2 = \frac{e^2}{\epsilon_0}\frac{\partial n_0}{\partial \mu} \approx \frac{e^2}{\epsilon_0} g(\epsilon_F)$$

$$\kappa = g(\epsilon_F) \quad @\, T=0$$

$$g(\epsilon_F) \to \frac{3}{2}\frac{n}{\epsilon_F}$$

$$\epsilon_F = \mu$$

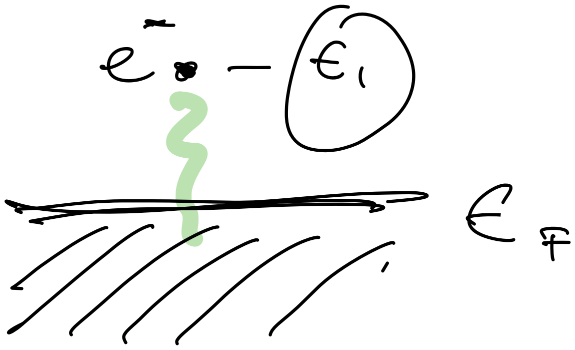
$$\lambda_{TF} = \frac{1}{q_{TF}} = \sqrt{\frac{\epsilon_0}{e^2 g(\epsilon_F)}} \sim 1\,\text{Å}$$



Landau theory of the Fermi liquid

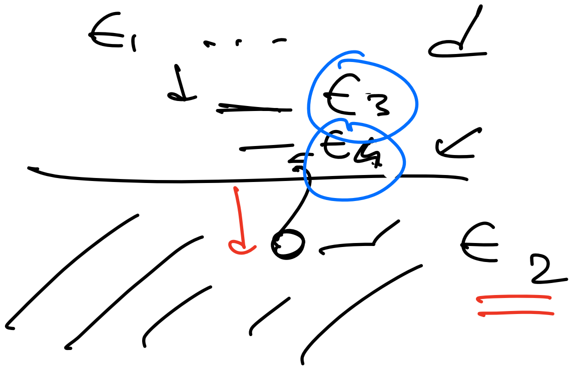
→ ground state of the electron gas is uniform, similar to a Fermi-Dirac distribution of non-interacting fermions

⇒ Fermi sea + neutralizing background



eigenstate without interactions

with interactions



$$\epsilon_1 + \epsilon_2 = \epsilon_3 + \epsilon_4$$

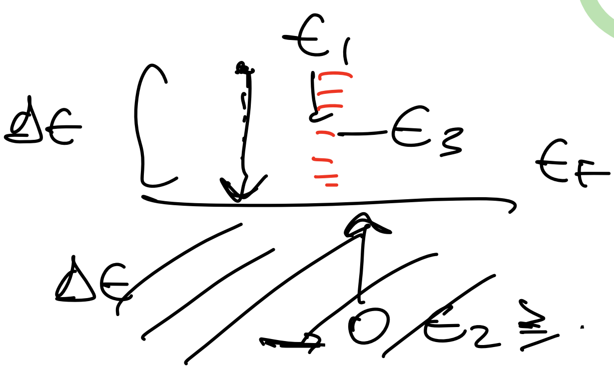
How many possibilities for the choices of ϵ_2, ϵ_3, ϵ_4 ?

$$\frac{1}{\tau} \sim \left(\# \text{ of final states of the scattering} \right)$$

T

scattering rate

$$\sim (\epsilon_1 - \epsilon_F)^2$$



$$\Delta\epsilon = \epsilon_1 - \epsilon_F$$

$$\epsilon_2 \geq \epsilon_F - (\epsilon_1 - \epsilon_F) = 2\epsilon_F - \epsilon_1$$

$$\sim \underbrace{(\epsilon_1 - \epsilon_F)}_{\text{choices for } \epsilon_2} \quad \underbrace{(\epsilon_1 - \epsilon_F)}_{\text{choices for } \epsilon_3}$$

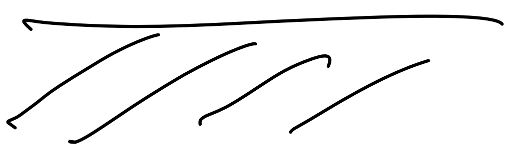
$$\frac{1}{\tau} \sim (\epsilon_1 - \epsilon_F)^2 \to 0$$

$$\epsilon_1 \to \epsilon_F$$

extra electrons are nearly free if $\epsilon_1 \to \epsilon_F$

<u>Landau Fermionic quasi-particle</u>

$\sim \frac{\hbar}{\tau} \; [- \epsilon_1$



ϵ_F

Dielectric response with dynamical ions

$$\rho(\vec{q},\omega) = \rho_{ext}(\vec{q},\omega) + \rho^{(e)}_{ind}(\vec{q},\omega) + \rho^{(i)}_{ind}(\vec{q},\omega)$$

$$\epsilon(\vec{q},\omega) = \frac{\rho_{ext}}{\rho} = \frac{\rho_{ext}}{\rho_{ext} + \rho^{(i)}_{ext} + \rho^{(e)}_{ext}}$$

From the point of view of the electrons

$$\rho^{(e)}_{ext} = \rho^{(i)}_{ind} + \rho_{ext}$$

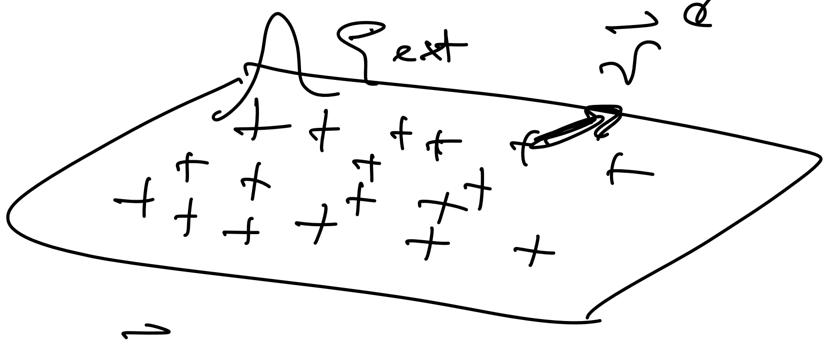
$$\epsilon_{TF}(\vec{q},\omega) = 1 + \left(\frac{q_{TF}}{q}\right)^2 = \frac{\rho_{ext} + \rho^{(i)}_{ind}}{\rho_{ext} + \rho^{(i)}_{ind} + \rho^{(e)}_{ind}}$$

$$= 1 - \frac{\rho^{(e)}_{ind}}{\rho}$$

$$\frac{\rho^{(e)}_{ind}}{\rho} = -\left(\frac{q_{TF}}{q}\right)^2$$

classical } description of ions
continuum }

$\vec{v}$



$M\dot{\vec{v}} = -Ze\vec{\nabla}\phi$ $\quad \vec{E}$

$\nabla^2\phi = -\frac{\rho}{\epsilon_0}$

Charge conservation

$$\vec{\nabla}\cdot\vec{j}_i + \frac{\partial}{\partial t}\rho^{(i)}_{ind} = 0$$

$$\vec{j} = n(Ze)\vec{v}$$

$$\frac{\partial^2}{\partial t^2}\rho^{(i)}_{ind} = -n(Ze)\,\vec{\nabla}\cdot\frac{\partial\vec{v}}{\partial t}$$

$$= \frac{n(Ze)^2}{M}\vec{\nabla}^2\phi$$

$$= -n\frac{(Ze)^2}{M\epsilon_0}\rho \qquad \omega_i^2$$

$$-\omega^2\rho^{(i)}_{ind}(\vec{q},\omega) = -n\frac{(Ze)^2}{M\epsilon_0}\rho(\vec{q},\omega)$$

$$\rho^{(i)}_{i,ind} = \frac{\omega_i^2}{\omega}\rho$$

$$\underbrace{\rho^{(i)}_{i,ind} + \rho^{(e)}_{i,ind}}_{\rho - \rho_{ext}} = \left[-\left(\frac{q_{TF}}{q}\right)^2 + \left(\frac{\omega_i}{\omega}\right)^2\right]\rho(\vec{q},\omega)$$

$$\rho_{ext} = \underbrace{\left[1 + \left(\frac{q_{TF}}{q}\right)^2 - \left(\frac{\omega_i}{\omega}\right)^2\right]}_{\epsilon(\vec{q},\omega)}\rho$$

$$\frac{1}{\epsilon(\vec{q},\omega)} = \frac{1}{\underbrace{1 + \left(\frac{q_{TF}}{q}\right)^2 - \left(\frac{\omega_i}{\omega}\right)^2}_{M \to \infty \qquad \omega_i \to 0}}$$

ϵ^{-1} has poles $\vec{q}, \omega$

$$\phi(\vec{q},\omega) = \frac{1}{\epsilon(\vec{q},\omega)} \phi_{ext}(\vec{q},\omega)$$

∞ 1

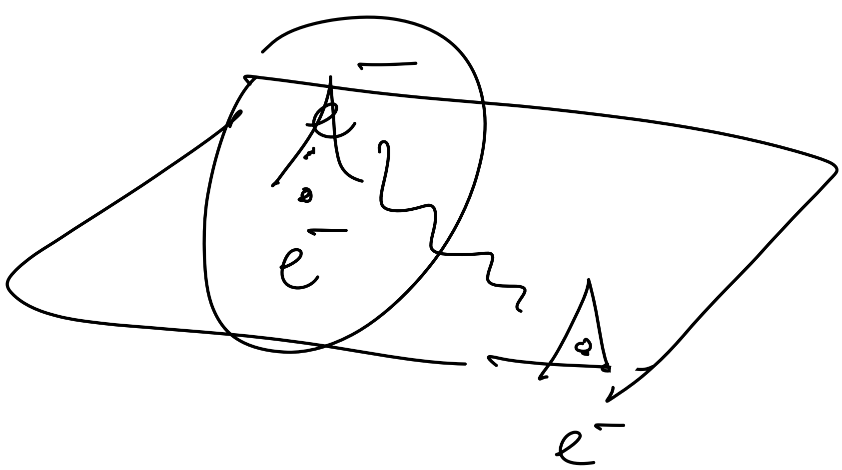
$$\frac{1}{\epsilon(\vec{q},\omega)} \overset{(\sim)}{=} \underbrace{\frac{q^2}{q^2+q_{TF}^2}} \left[1 + \frac{\omega^2}{\omega^2-\omega_q^2} \right]$$

$$\omega_q = \frac{\omega_p}{\sqrt{q_{TF}^2+q^2}}\, q \quad \overset{q \ll q_{TF}}{\simeq} \quad \rightarrow \left(\frac{\omega_p}{q_{TF}}\right) q$$

$$c$$

$$\underline{\omega \Rightarrow \omega_q^-} \qquad \underbrace{1 + \frac{\omega^2}{\omega^2-\omega_q^2}} \rightarrow -\infty$$

$$\phi(\vec{q},\omega) = \frac{1}{\epsilon(\vec{q},\omega)} \phi_{ext}(\vec{q},\omega)$$



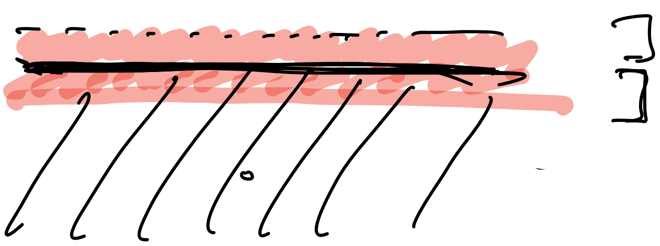
e^-

retardation effect on the time scale of the motion of the ions

$$\boxed{\tau \sim \omega_D^{-1}}$$

Electrons interacting via the linear response of the electron gas + the harmonic lattice can attract each other in a time-dependent way $\Leftrightarrow$ energy dependent

ϵ_F



$$\hbar\omega_D = \epsilon_c$$