BCS Theory of superconductivity

hypothesis of fermionic pair condensation (SSB)

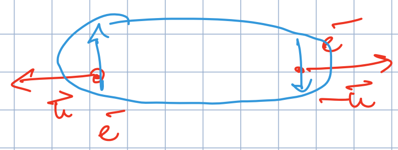
$$0 \neq \langle \psi^\dagger_\uparrow(\vec r)\, \psi^\dagger_\downarrow(\vec r') \rangle$$

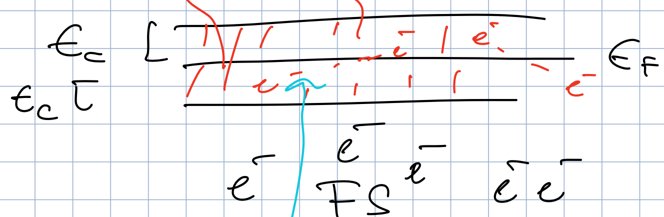
$$= \sum_{\vec k \vec q} \frac{e^{-i(\vec k\cdot\vec r + \vec q\cdot\vec r')}}{V} \langle c^\dagger_{k\uparrow}\, c^\dagger_{q\downarrow} \rangle$$

$\vec q = -\vec k$

$|\epsilon_{\vec k} - \epsilon_F| \le \epsilon_c$

$|\epsilon_q - \epsilon_F| \le \epsilon_c$





$$= \sum_{\vec k}{}' \frac{e^{-i\vec k\cdot(\vec r - \vec r')}}{V} \langle c^\dagger_{k\uparrow}\, c^\dagger_{-k\downarrow} \rangle \to \neq 0$$

postulate this on the basis of Cooper's instability

⟹ check whether our many-body calculations are consistent with this assumptions

$$\boxed{\Delta_k = H\ \Delta}$$

gap function

$$\Delta_{\vec k} = \left(-\frac{1}{V} \sum_{\vec q}{}' V_{\vec k - \vec q} \langle c_{-q\downarrow}\, c_{q\uparrow} \rangle \right) H \quad |\epsilon_{\vec k} - \epsilon_F| \le \epsilon_c$$

$\sigma = \uparrow, \downarrow$

$\mu = \epsilon_F$

$$H_{BCS} - \mu N = \sum_{k\sigma} \xi_{\vec k}\, c^\dagger_{\vec k\sigma}\, c_{\vec k\sigma} \qquad \xi_{\vec k} = \epsilon_{\vec k} - \epsilon_F$$

$$- \sum_k \Big(\Delta_k\, c^\dagger_{k\uparrow}\, c^\dagger_{-k\downarrow} + h.c. \underbrace{- \Delta_k \langle c^\dagger_{k\uparrow}\, c^\dagger_{-k\downarrow} \rangle}_{\text{const}} \Big)$$

$$\left(AB \simeq \langle A \rangle B + \langle B \rangle A - \langle A \rangle \langle B \rangle \right)$$

mean-field approximation

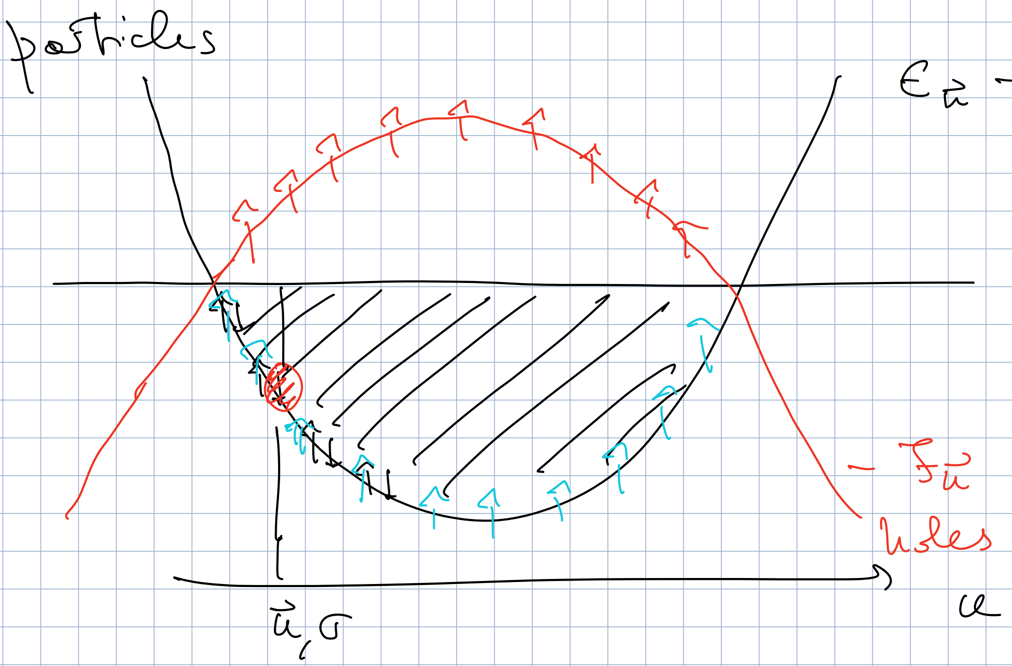
Diagonalize the BCS Hamiltonian (quadratic Ham.)

$\hookrightarrow$ free quasi-particles

$$\Longrightarrow \quad \sum_{\vec{k},p} E_{\vec{k},p}\, \hat{\gamma}^{\dagger}_{\vec{k},p}\, \hat{\gamma}_{\vec{k},p}$$

$\hookrightarrow$ fermionic operators

Holes in the Fermi sea



$$\begin{cases} c_{\vec{k}\sigma} = h^{\dagger}_{-k,\bar{\sigma}} & \bar{\sigma} = -\sigma \\ c^{\dagger}_{k\sigma} = h_{-k,\bar{\sigma}} & \end{cases}$$

$$h_{k\uparrow} h^{\dagger}_{k\uparrow} = -h^{\dagger}_{k\uparrow} h_{k\uparrow} + 1$$

$$H_{BCS} - \mu N = \sum_{\vec{k}} \xi_{\vec{k}} \left(c^{\dagger}_{k\uparrow} c_{k\uparrow} + c^{\dagger}_{-k\downarrow} c_{-k\downarrow} \right)$$

$$- \sum_{\vec{k}} \left(\Delta_k\, c^{\dagger}_{k\uparrow} \underset{\substack{\downarrow \\ c^{\dagger}_{-k\downarrow}}}{h_{k\uparrow}} + h.c. \right) + C$$

$$\mathcal{H}_{BCS} - \mu N = \sum_{\vec{u}} \Big[\xi_{\vec{u}} \left(c^{+}_{u\uparrow} c_{u\uparrow} - h^{+}_{u\uparrow} h_{u\uparrow} \right) - \left(\Delta_u \, c^{+}_{u\uparrow} h_{u\uparrow} + \Delta^{*}_{u} \, h^{+}_{u\uparrow} c_{u\uparrow} \right) \Big] + C^{t}$$

$$= \sum_{\vec{u}} \begin{array}{c} p \\ h \end{array} \begin{pmatrix} c^{+}_{u\uparrow} \\ h^{+}_{u\uparrow} \end{pmatrix}^{T} \begin{pmatrix} \xi_{\vec{u}} & -\Delta_u \\ -\Delta^{*}_{u} & -\xi_u \end{pmatrix} \begin{pmatrix} c_{u\uparrow} \\ h_{u\uparrow} \end{pmatrix} + C^{t}$$

Diagonalize a 2×2 matrix $\quad \Delta = |\Delta| e^{i\phi}$

$$\begin{array}{c} p \quad |\uparrow\rangle \\ h \quad |\downarrow\rangle \end{array} \begin{pmatrix} \xi & -\Delta \\ -\Delta^{*} & -\xi \end{pmatrix} = \xi \sigma^{z} - |\Delta| \cos\phi \, \sigma^{x} + |\Delta| \sin\phi \, \sigma^{y}$$

$$= + \vec{H} \cdot \vec{\sigma}$$

spin $\frac{1}{2}$ in a magnetic field

$$\vec{H} = \left(-|\Delta| \cos\phi, \; |\Delta| \sin\phi, \; \xi \right)$$

eigenvalues :
$$E^{(\pm)}_{u} = \pm |\vec{H}_u| = \pm E_u$$
$$= \pm \sqrt{|\Delta_u|^2 + \xi_u^2}$$

eigenvectors :

$$\begin{cases} |\psi^{(+)}\rangle = \cos\frac{\vartheta}{2} \overset{p}{|\uparrow\rangle} - e^{-i\phi} \sin\frac{\vartheta}{2} \overset{h}{|\downarrow\rangle} \\ |\psi^{(-)}\rangle = \sin\frac{\vartheta}{2} |\uparrow\rangle + e^{-i\phi} \cos\frac{\vartheta}{2} |\downarrow\rangle \end{cases}$$

$$\vartheta = \arctan \frac{|\Delta|}{\xi_u}$$

$$\{ \gamma_{u\uparrow}, \gamma^{+}_{u\uparrow} \} = \{ \cos\frac{\vartheta_u}{2} c_{u\uparrow} - e^{+i\phi} \sin\frac{\vartheta}{2} c^{+}_{-u\downarrow}, \; \cos\frac{\vartheta_u}{2} c^{+}_{u\uparrow} - e^{-i\phi} \sin\frac{\vartheta_u}{2} c_{-u\downarrow} \}$$

$$\cos\frac{\theta}{2} = u_k = \sqrt{\frac{1}{2}\left(1+\frac{\xi}{E}\right)} \qquad \left[= \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1 \right]$$

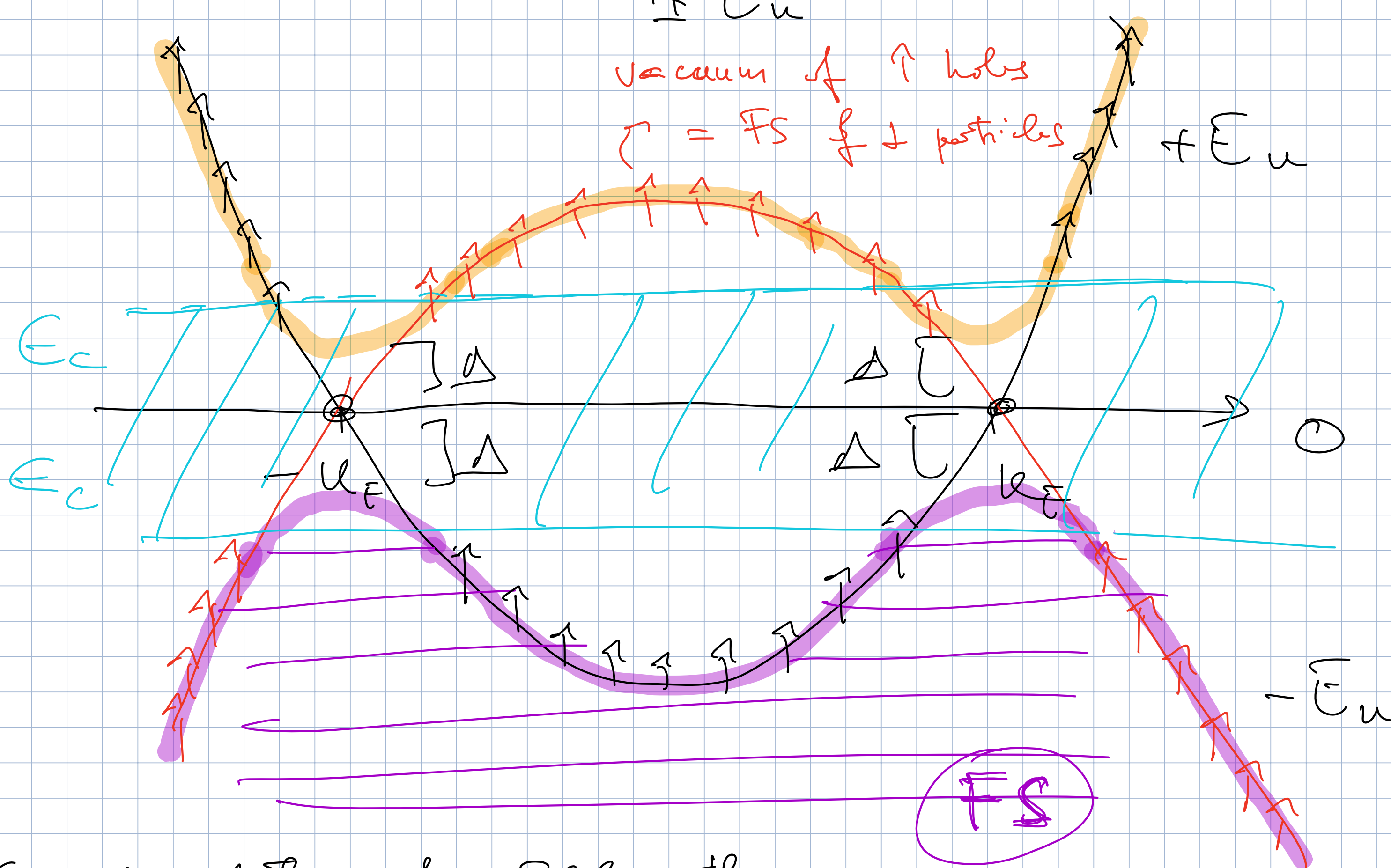
$$\sin\frac{\theta}{2} = v = \sqrt{\frac{1}{2}\left(1-\frac{\xi}{E}\right)}$$

Bogoliubov transformation

$$\begin{cases} \gamma^+_{k\uparrow} = \cos\frac{\theta_k}{2}\, c^+_{k\uparrow} - e^{-i\phi_k}\sin\frac{\theta_k}{2}\, h^+_{k\uparrow} \\ \gamma^+_{k-} = \sin\frac{\theta_k}{2}\, c^+_{k\uparrow} + e^{-i\phi_k}\cos\frac{\theta_k}{2}\, h^+_{k\uparrow} \end{cases}$$

$$\pm E_k = \pm\sqrt{|\Delta_k|^2 + \xi_k^2} = \begin{cases} \pm\sqrt{} & |\epsilon_k - \epsilon_F| \le \epsilon_c \\ \pm|\xi_k| & \text{otherwise} \end{cases}$$

$$H_{BCS} - \mu N = \sum_k E_k \left(\gamma^+_{k\uparrow}\gamma_{k\uparrow} - \gamma^+_{k-}\gamma_{k-}\right) + C$$



Ground-state of BCS theory:

$\Rightarrow$ Fermi sea of γ_- fermions

$=$ vacuum of γ_+ fermions

Test the assumption of pair condensation

$$\Delta_{\vec{k}} = \Theta \, \frac{V_0}{V} \sum_{\vec{q}}' \langle \underbrace{c_{-q\downarrow}}_{\to h^+_{q\uparrow}} \, c_{q\uparrow} \rangle \neq 0 \quad ?$$

$|\epsilon_{\vec{k}} - \epsilon_F| \leq \epsilon_c$

in the ground state

$$\begin{cases} c_{k\uparrow} = \cos\frac{\vartheta_k}{2}\, \gamma_{k+} + \sin\frac{\vartheta_k}{2}\, \gamma_{k-} \\ h^+_{k\uparrow} = -\sin\frac{\vartheta_k}{2}\, e^{i\phi}\, \gamma^+_{k+} + \cos\frac{\vartheta_k}{2}\, e^{i\phi}\, \gamma^+_{k-} \end{cases}$$

$$= \Theta \, \frac{V_0}{V} \sum_{\vec{q}}' \Big\langle \Big(-\sin\frac{\vartheta_q}{2}\, e^{i\phi}\, \gamma^+_{q+} + \cos\frac{\vartheta_q}{2}\, e^{i\phi}\, \gamma^+_{q-}\Big) \Big(\cos\frac{\vartheta_q}{2}\, \gamma_{q+} + \sin\frac{\vartheta_q}{2}\, \gamma_{q-}\Big) \Big\rangle$$

FS of γ_- particles
+ vacuum of γ_+ particles

$$= \Theta \, \frac{V_0}{V} \sum_{\vec{q}}' \cos\frac{\vartheta_q}{2} \sin\frac{\vartheta_q}{2}\, e^{i\phi} \, \underbrace{\langle \gamma^+_{q-} \gamma_{q-} \rangle}_{=1}$$

$$= \Theta \, \frac{V_0}{V} \sum_{\vec{q}}' \frac{|\Delta| e^{i\phi}}{2E_q} \to \Delta = \Delta$$

$$\boxed{1 = \frac{V_0}{2V} \sum_{\vec{q}}{}' \frac{1}{E_{\vec{q}}}}$$ gap equation

$E_{\vec q} \to \sqrt{\xi_q^2 + |\Delta|^2}$ $\to 2$ in Cooper's calculation

$$\Rightarrow |\Delta| \simeq 2\epsilon_c \, e^{-\frac{1}{V_0 g(\epsilon_F)}}$$ $\Rightarrow$ similar to the binding energy of a Cooper pair

$(V_0 g(\epsilon_F) \ll 1)$

Ground-state energy and variational derivation of the gap equation

$$\mathcal{H}_{BCS} - \mu N = \sum_{\vec k} E_{\vec k} \left(\gamma^{+}_{k+} \gamma_{k+} - \gamma^{+}_{k-} \gamma_{k-} \right)$$

$$+ \sum_{\vec k} \xi_{\vec k} + \sum_{\vec k}{}' \Delta^{*} \langle c_{-k\downarrow} c_{k\uparrow} \rangle$$

$$= \frac{V}{V_0} \sum_{\vec k}{}' \Delta^{*} \frac{V_0}{V} \langle c_{-k\downarrow} c_{k\uparrow} \rangle$$

Δ

$$= \frac{V |\Delta|^2}{V_0}$$

$$\langle \mathcal{H}_{BCS} - \mu N \rangle_{T=0} = \sum_k \left[\xi_k - \sqrt{\xi_k^2 + |\Delta_k|^2} \right] + \frac{V}{V_0} |\Delta|^2$$

$$= \text{const.} + \sum_k' \left[\xi_{\vec{k}} - \underbrace{\sqrt{\xi_k^2 + |\Delta|^2}}_{E_k} \right] + \frac{V}{V_0} |\Delta|^2$$

$$\frac{\partial \langle \mathcal{H}_{BCS} - \mu N \rangle}{\partial |\Delta|} = -\frac{1}{2} \sum_{\vec{k}}' \frac{2\cancel{|\Delta|}}{E_{\vec{k}}} + \frac{2V}{V_0} \cancel{|\Delta|} = 0$$

$$\Rightarrow \boxed{1 = \frac{V_0}{2V} \sum_{\vec{k}}' \frac{1}{E_{\vec{k}}}}$$

Gap equation at finite energy

From the minimization of the free energy

$$F = \mathcal{E}^{(T)} - TS(T)$$

$$\langle \mathcal{H}_{BCS} - \mu N \rangle_T = \sum_k E_k \left(n_k^{(+)} - n_k^{(-)} \right)$$

$$n_k^{(+)} = \frac{1}{e^{\beta E_k} + 1} \qquad n_k^{(-)} = \frac{1}{e^{-\beta E_k} + 1}$$

$$+ \sum_{\vec{k}} \xi_k + \frac{V}{V_0} |\Delta|^2$$

$$S(T) = -k_B \sum_{\vec{k}} \left[n_{\vec{k}}^{(+)} \log n_{\vec{k}}^{(+)} + (1 - n_{\vec{k}}^{(+)}) \log (1 - n_{\vec{k}}^{(+)}) + (+ \to -) \right]$$

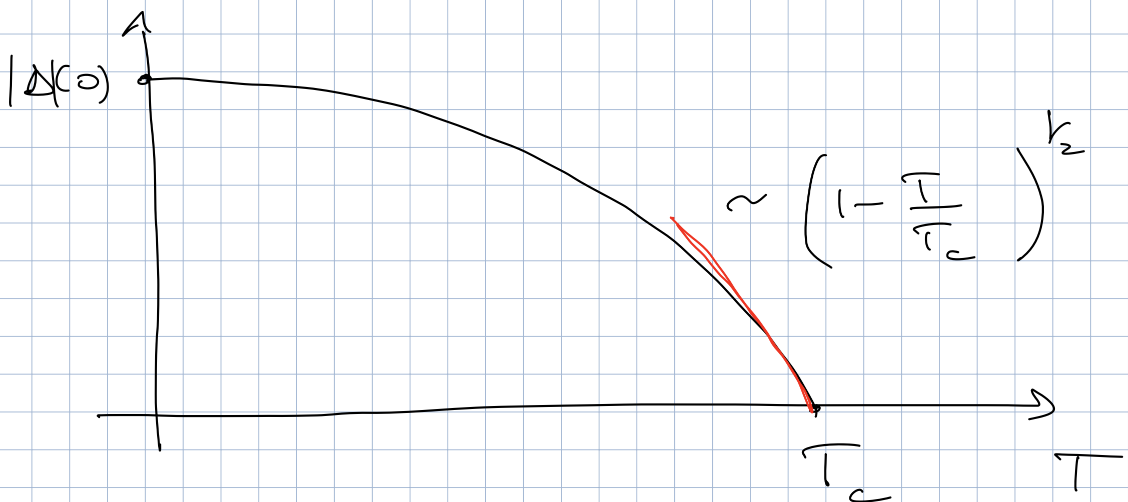
$$F(T) = C' + V\frac{|\Delta|^2}{V_0} - 2k_B T \sum_{\vec{k}}{}' \log\left[2\cosh\left(\frac{\beta E_{\vec{k}}}{2}\right)\right]$$

$$\frac{\partial F}{\partial |\Delta|} = 0 \quad \Rightarrow \quad \boxed{1 = \frac{V_0}{2V} \sum_{\vec{k}}{}' \frac{\tanh\left(\frac{\beta E_{\vec{k}}}{2}\right)}{E_{\vec{k}}}}$$

$\tanh\left(\frac{\beta E_{\vec{k}}}{2}\right) \to 1$, $\beta \to \infty$, $T \to 0$

$|\Delta|(T)$

$\Delta = \Delta(T)$



$$k_B T_c \simeq 1.13\, \epsilon_c\, e^{-\frac{1}{V_0 g(\epsilon_F)}} = \frac{|\Delta|(0)}{1.764}$$

$$\frac{2|\Delta|(0)}{k_B T_c} \cong 3.5$$