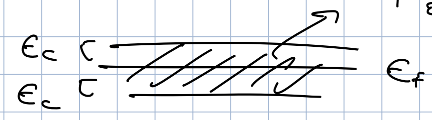
BCS theory of superconductivity

Free energy of BCS theory

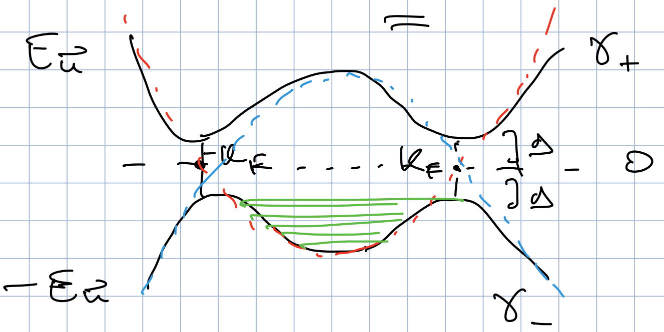
$$\Delta_k = \begin{cases} \Delta & \text{shell} \\ 0 & \text{otherwise} \end{cases}$$



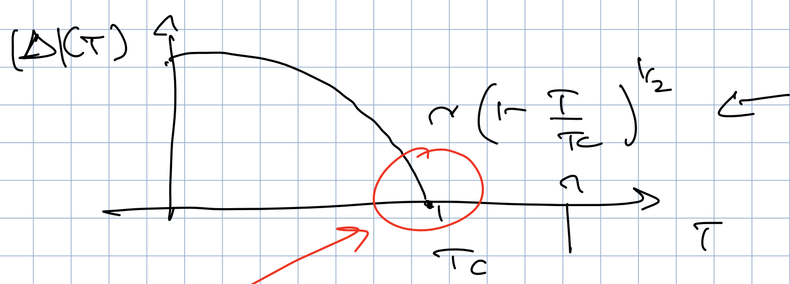
$$F(T)^{\Delta} = \langle H - \mu N \rangle_{BCS} - TS$$

$$= \nu \frac{|\Delta|^2}{V_0} - 2k_B T \sum_{\vec{k}} \log\left[2\cosh\left(\frac{\beta E_{\vec{k}}}{2}\right)\right] + \text{const.}$$

$$E_{\vec{k}} = \sqrt{|\Delta_k|^2 + \xi_k^2} \qquad \xi_k = \epsilon_{\vec{k}} - \epsilon_F$$



$$\frac{\partial F}{\partial |\Delta|} = 0 \Rightarrow \text{gap equation @ } T > 0$$



$|\Delta| \to 0$ expand $F(T, |\Delta|)$

even function of $|\Delta|$

$T \lesssim T_c$

$$F(T, |\Delta|) \underset{|\Delta| \to 0}{\simeq} \text{const.} + \frac{1}{2} A |\Delta|^2 + \frac{1}{4} B |\Delta|^4 + \dots$$

$$\frac{\partial F}{\partial |\Delta|} = 0 \qquad A|\Delta| + B|\Delta|^3 = 0$$

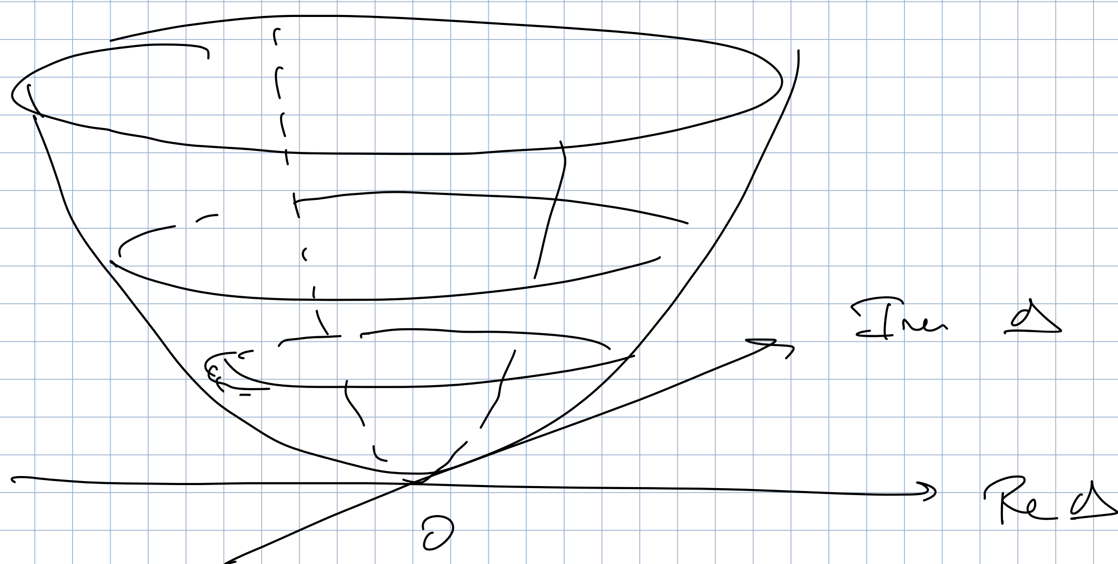
$$|\Delta|^2 = -\frac{A}{B} \qquad |\Delta| \simeq \sqrt{-\frac{A}{B}}$$

$$\underline{A = a(T - T_c)} \qquad \uparrow \quad \underline{B > 0}$$

$$F(T, |\Delta|) \simeq \text{const.} + \frac{1}{2} a(T - T_c)|\Delta|^2 + \frac{1}{4} B |\Delta|^4 + \dots \quad \leftarrow$$

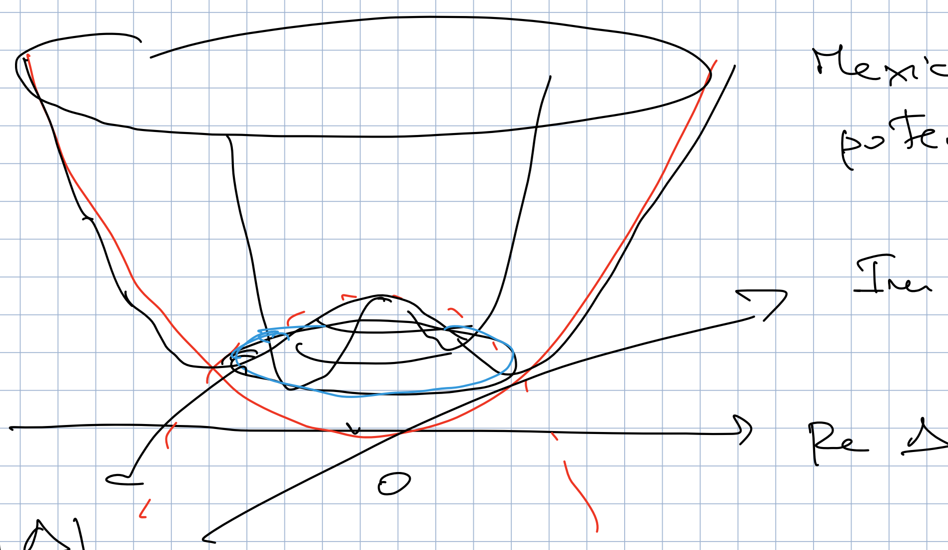
$\underline{T > T_c}$

$|\Delta|$ = order parameter



Mexican-hat potential

$T < T_c$

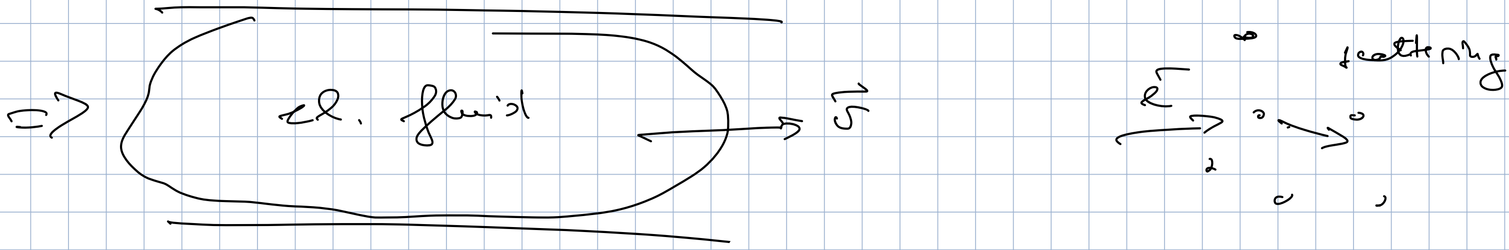


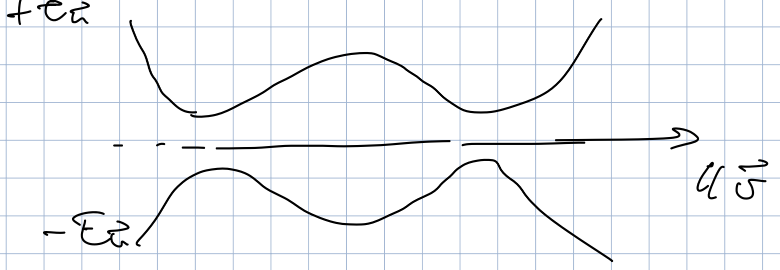
$0 \neq |\Delta|$

$$|\Delta| \neq 0 \implies \langle \psi_\downarrow(\vec{r})\, \psi_\uparrow(\vec{r}\,') \rangle \neq 0$$

Superconductivity ? Absence of resistance

Landau criteria for superconductivity

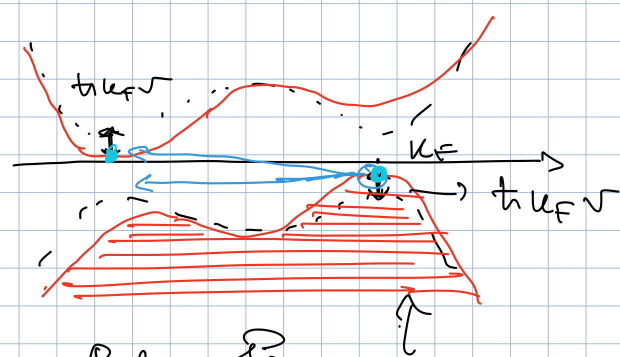




dispersion relation of the rest frame of the electrons

$\pm E_{\vec k}$

$\rightarrow$

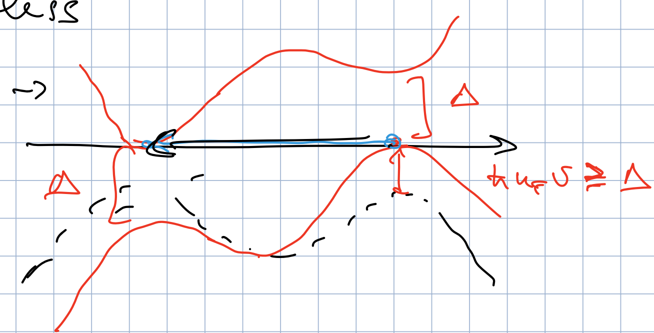


lab frame

$$\pm E_{\vec k} \underbrace{+ \hbar \vec k \cdot \vec v}_{\text{Doppler shift}}$$

scattering does not occur unless

$$\hbar v k_F \geq \Delta$$



$$v \geq v_c^{(T)} = \frac{\Delta(T)}{\hbar k_F}$$

depairing velocity

scattering from impurities / dislocations

$v < v_c$

$j_c = n_s(T)\, v_c(T)$

persistent current

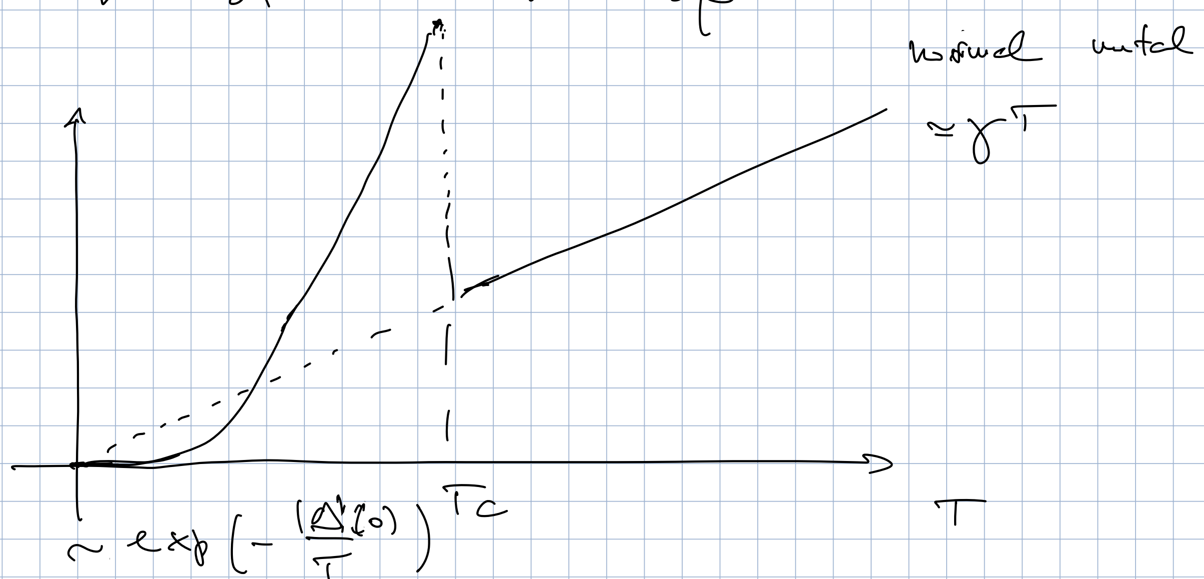
$\gamma^{+}_{-k_F^+} \gamma_{k_F^-} |\Psi_0\rangle$

Specific heat

$$S(T) = -k_B \sum_{\vec{k}} \left[n^{(+)}_{\vec{k}} \log n^{(+)}_{\vec{k}} + (1-n^{(+)}_k)\log(1-n^{(+)}_k) \right] + \left[(+ \to -) \right]$$

$$n_k^{(+)} = \frac{1}{e^{\beta E_k}+1}$$

$$C = \frac{1}{V} T \frac{\partial S}{\partial T} = -\frac{1}{V} \beta \frac{\partial S}{\partial \beta}$$



Superconductors in the presence of inhomogeneities / magnetic fields

Phenomenological description of SCs at a mesoscopic scale

$\ell \gg \xi$ (~ size a Cooper pair)

(Ginzburg-Landau theory) (1950)

population of pairs

SSB: $\langle \psi_\downarrow(\vec{r})\, \psi_\uparrow(\vec{r}') \rangle = \sqrt{n_0}\, \chi_0(\vec{r}\downarrow; \vec{r}'\uparrow)$

$$= \frac{1}{V} \sum_{\vec{k}\vec{k}'}{}' e^{i\vec{k}\cdot\vec{r} + \vec{k}'\cdot\vec{r}'} \underbrace{\langle c_{\vec{k}\downarrow} c_{\vec{k}'\uparrow} \rangle}_{\vec{k} = -\vec{k}'}$$

$$|\epsilon_{\vec{k}} - \epsilon_F| \leq \epsilon_c$$

Cooper pairing

$$\overset{?}{\cong} \frac{1}{V} \sum_{\vec{k}}{}' e^{-i\vec{k}\cdot(\vec{r}-\vec{r}')} \langle c_{-k\downarrow} c_{k\uparrow} \rangle$$

$\vec{r} = \vec{r}'$

$$\langle \psi_\downarrow(\vec{r}) \psi_\uparrow(\vec{r}) \rangle = \sqrt{\lambda_0} \, \chi_0(\vec{r}\downarrow; \vec{r}\uparrow)$$

$$= \frac{1}{V} \underbrace{\sum_{\vec{k}}{}' \langle c_{-k\downarrow} c_{k\uparrow} \rangle}_{\frac{\Delta}{V_0}} = \frac{\Delta}{V_0}$$

$$\sqrt{\lambda_0}\,\chi_0(\vec{r}\downarrow; \vec{r}'\uparrow) =: \Psi_0\left(\vec{R} = \frac{\vec{r}+\vec{r}'}{2}\right) \varphi_0(\vec{r}-\vec{r}')$$

$$\sqrt{\lambda_0}\,\chi_0(\vec{r}\downarrow; \vec{r}\uparrow) = \frac{\Delta}{V_0} \sim \boxed{\Psi_0(\vec{R}=\vec{r})} \, \varphi_0(0)$$

macroscopic wavefunction

$$\int d^3R \, |\Psi_0(\vec{R})|^2 = \lambda_0 \sim |\Delta|^2$$

$$\int d^3r \, |\varphi_0(\vec{r})|^2 = 1$$

Cooper pairs of finite momentum

$$\Psi_0(\vec{R}) \sim e^{i\vec{K}\cdot\vec{R}} \frac{\Delta}{V_0}$$

$$E_{kin} = n_0 \frac{\hbar^2 K^2}{2m^*} = \int d^3R \; \Psi_0^* \left(-\frac{\hbar^2}{2m^*} \nabla^2 \right) \Psi_0$$

$m^* = 2m$ $\qquad$ int. by parts $\overset{\text{int. by parts}}{=}$ $\displaystyle \int d^3R \; \underline{\underline{\frac{\hbar^2}{2m}}} \left| \vec{\nabla} \Psi_0 \right|^2$

Free energy for moving Cooper pairs

(c.p. described by a non-uniform $\Psi_0(\vec{R})$)

$$|\Psi_0|^2 \sim |\Delta|^2 \qquad \Delta \to \Delta(\vec{R})$$

$$|\Delta|^2 \to \int d^3R \; |\Psi_0(R)|^2$$

$$|\Delta|^4 \to \int d^3R \; |\Psi_0(\vec{R})|^4$$

Ginzburg - Landau free energy

$$F(\Psi_0, \Psi_0^*; T)$$

$$\simeq \text{const.} + \int d^3R \left[\frac{1}{2} a(T - T_c) |\Psi_0|^2 + \frac{1}{2m^*} |-i\hbar\vec{\nabla} \Psi_0|^2 + \frac{1}{4} b |\Psi_0|^4 + \dots \right]$$

remember : GP functional

Vector potential $\Rightarrow$ magnetic field

$\vec{A}(\vec{r})$ $\qquad$ $\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}$

$$-i\hbar\vec{\nabla} \to -i\hbar\vec{\nabla} + \frac{2e}{\hbar}\vec{A}$$

$$F(\Psi_0, \Psi_0^*, \vec{A}; T)$$

$$= \text{const.} + \int d^3R \left[\left(\frac{1}{2}a(T-T_c) + V_{ext}(\vec{R})\right)|\Psi_0|^2 + \frac{1}{2m^*}\left|\left(-i\hbar\vec{\nabla} + \frac{2e}{\hbar}\vec{A}\right)\Psi_0\right|^2 + \frac{1}{4}b|\Psi_0|^4 + \dots\right]$$

$$\frac{\delta F}{\delta \Psi_0^*} = 0 =$$

$$= -\frac{\hbar^2}{2m^*}\left(\vec{\nabla} + i\frac{2e}{\hbar}\vec{A}\right)^2\Psi_0 + \frac{1}{2}\left(a(T-T_c) + V_{ext}(\vec{R})\right)\Psi_0 + \frac{b}{2}|\Psi_0|^2\Psi_0 = 0$$

$$\frac{\delta F}{\delta \vec{A}} = 0 \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$$

$$\vec{j} = -2e\left[\frac{\hbar}{2m^* i}\left(\Psi_0^*\vec{\nabla}\Psi_0 - \Psi_0\vec{\nabla}\Psi_0^*\right) + \frac{2e}{m^*}|\Psi_0|^2\vec{A}\right]$$

Cooper pairs

$$\Psi_0 = |\Psi_0(\vec{R})|\, e^{i\phi(\vec{r})}$$

$$\vec j = -2e\,|\Psi_0|^2\left[\frac{\hbar}{m^*}\vec\nabla\phi + \frac{2e}{m^*}\vec A\right]$$

irrotational

$$\vec\nabla\times(\vec\nabla\phi)=0 \qquad \vec\nabla\times\vec A=\vec B$$

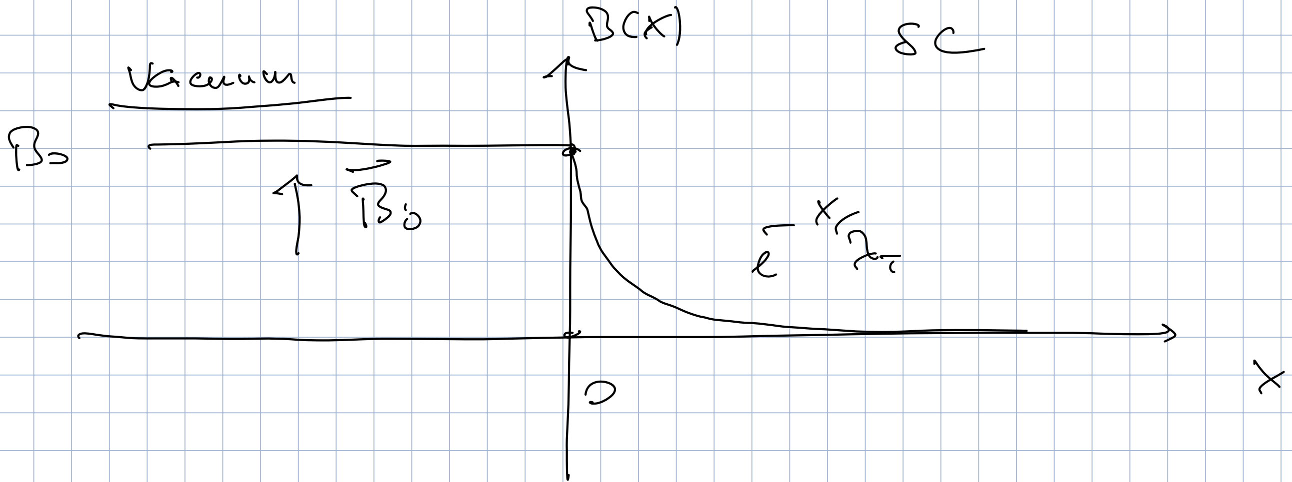
$$\vec\nabla\times\vec j = -\frac{(2e)^2|\Psi_0|^2}{m^*}\vec B$$

London brothers (1935)

$$\vec\nabla\times(\vec\nabla\times\vec B) = -\frac{(2e)^2|\Psi_0|^2\mu_0}{m^*}\vec B \qquad \vec j = \frac{1}{\mu_0}\vec\nabla\times\vec B$$

$$\vec\nabla(\cancel{\vec\nabla\cdot\vec B})^{0} + \boxed{\nabla^2\vec B = +\frac{\mu_0(2e)^2|\Psi_0|^2}{m^*}\vec B}$$

Cooper pair density



$$\frac{d^2}{dx^2}B(x) = \frac{(2e)^2|\Psi_0|^2\mu_0}{m^*}B(x) = \frac{1}{\lambda_T^2}B(x)$$

$$\lambda_T = \sqrt{\frac{m^*}{(2e)^2|\Psi_0|^2\mu_0}}$$

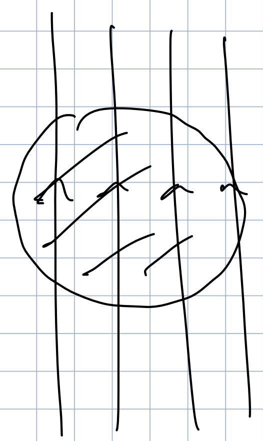
London penetration depth

$$B(x) = B_0 \, e^{-x/\lambda_T}$$

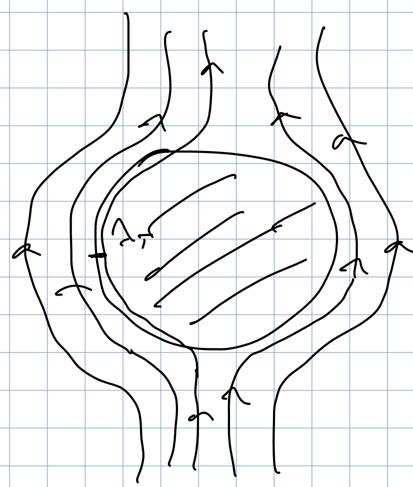
Meissner effect

(1933)

$\vec{B}$

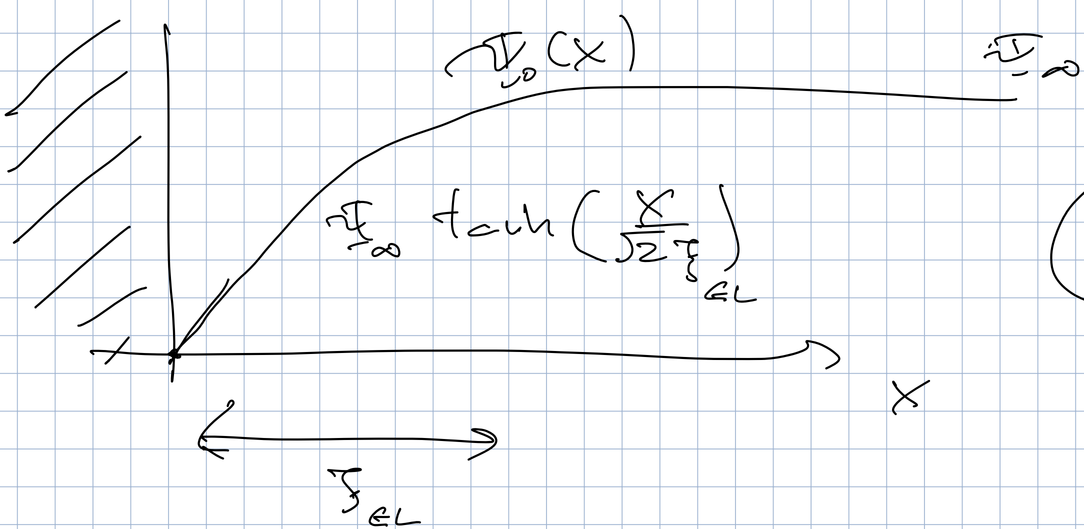


$T > T_c$



$T < T_c$

<u>Characteristic length</u> $\leftarrow$ healing length



$\vec{A} = 0$

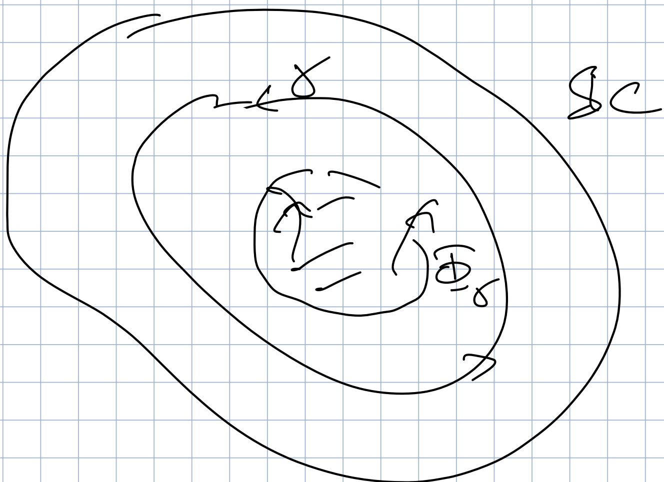
$$\xi_{GL} = \sqrt{\frac{\hbar}{m^* |a(T-T_c)|}} \underset{T \to T_c}{\sim} |T-T_c|^{-1/2}$$

compare $\lambda_T = \sqrt{\frac{m^*}{(2e)^2 \mu_0 |\Psi_0|^2}} \sim |T - T_c|^{1/2}$

$\hookrightarrow |\Delta|^2 \sim T_c - T$

$\kappa = \frac{\lambda_T}{\xi_{GL}}$

Flux quantization



$\vec{J} = 0$

$|\Psi_0|^2 \sim \text{const}$

$2\pi p$

$$\oint_\gamma \vec{J} \cdot d\vec{l} = 0 = -(2e)|\Psi_0|^2 \left[\frac{\hbar}{m^*} \oint \vec{\nabla}\phi \cdot d\vec{l} + \frac{2e}{m^*} \oint_\gamma d\vec{l} \cdot \vec{A} \right]$$

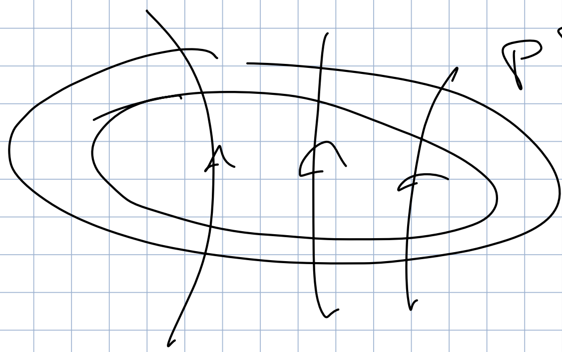
Φ_γ

$= 0$

$$\vec{J} = -2e|\Psi_0|^2 \left[\frac{\hbar}{m^*} \vec{\nabla}\phi + \frac{2e}{m^*} \vec{A} \right]$$

Φ_0 SC flux quantum

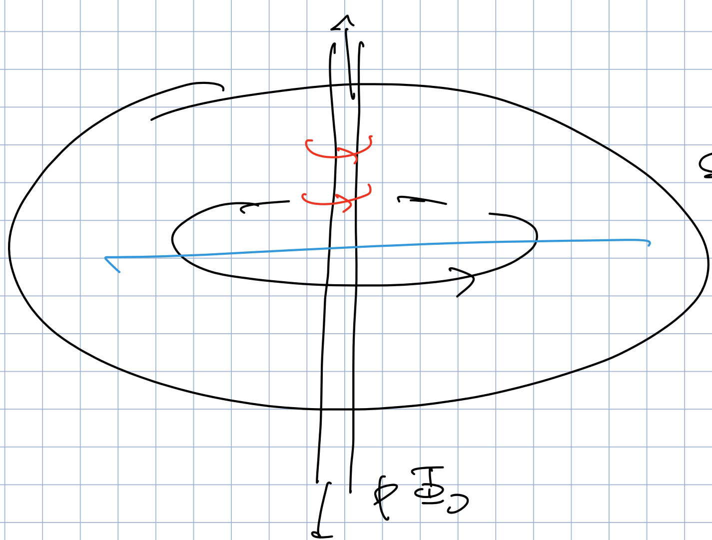
$$\Phi_\gamma = \frac{\hbar}{2e} 2\pi p = \frac{h}{2e} p$$



$p\Phi_0$

ring

If $\vec{B}$ penetrates a SC



SC

$p\Phi_0$

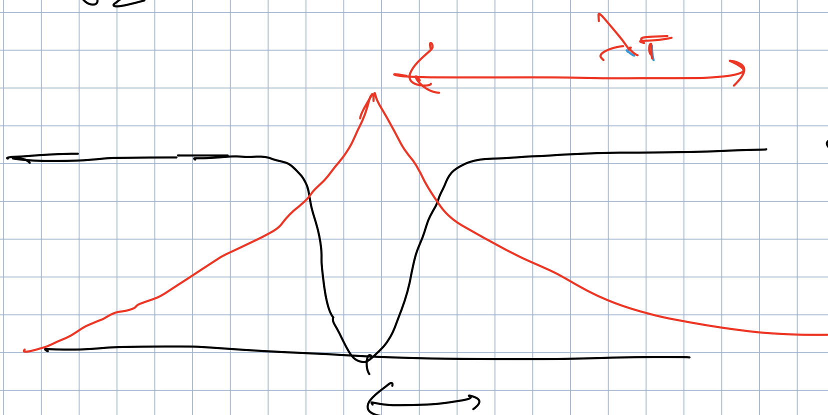
SC vortex

$\kappa \gtrsim 1$

$\kappa > \frac{1}{\sqrt{2}}$ type II SC

type I : $\kappa < \frac{1}{\sqrt{2}}$

B does not penetrate

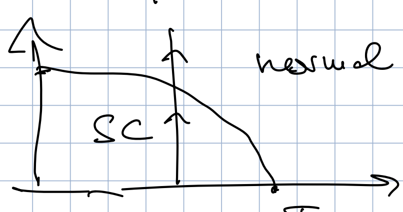


λ_T

Ψ_0

ξ_{GL}

type I SC



H

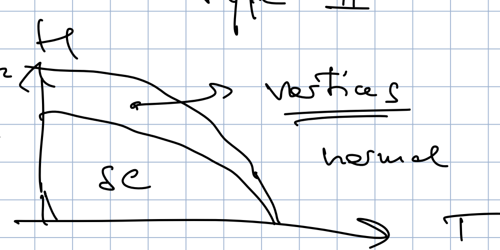
SC

normal

T_c

T

type II



H

H_{c2}

H_{c1}

vortices

normal

sc

T