Superconductivity

→ Free Fermi gas for electrons in a metal

(robust to repulsive el-el Coulomb interactions)

→ Electron-phonon interactions

screening from the el gas

$$\left[\ \underbrace{V_{el-el}(\vec{q},\omega)} = \underbrace{\frac{e^2}{\epsilon_0\,(q^2+q_{TF}^2)}}\left(1+\frac{\omega_q^2}{\omega^2-\omega_q^2}\right)\right.$$

screening from the ionic background

$$\underset{q\to 0}{\omega_q \sim q}$$

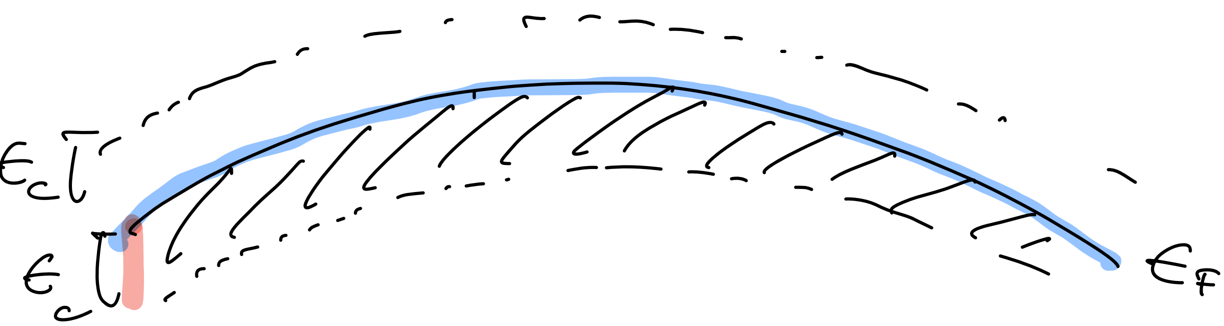
$$\underbrace{\omega^2 < \omega_q^2}$$

$$V_{el-el} < 0$$

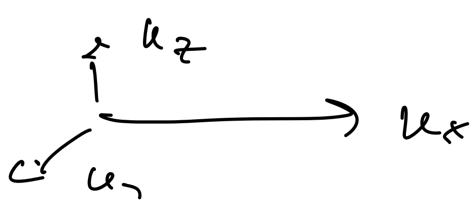
Two fundamental ingredients

1) interesting only involve a thin shell of electrons close to the FS

V_{el-el} small compared to the Fermi energy



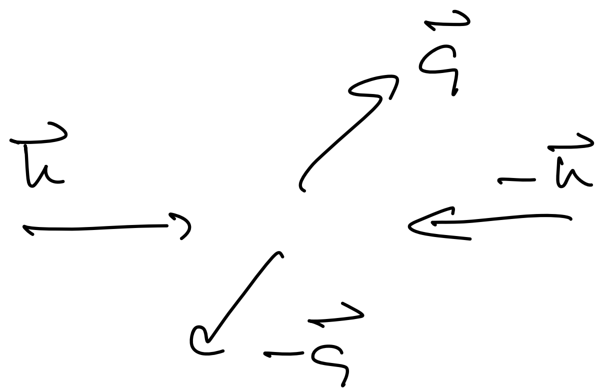
$\epsilon_c \ll \epsilon_F$



$\epsilon_c \simeq \hbar\omega_D$ → Debye frequency

2) Interactions involve primarily

= counter propagating electrons



$$(\vec{r}_1 - \vec{r}_2)$$

$$\langle \underbrace{\vec{k}, -\vec{k}}\, |\, V_{eff}\, |\, \vec{q}, -\vec{q} \rangle$$

$$\vec{r}$$

$$= \int d^3 r_1 \int d^3 r_2 \; \frac{e^{-(i \vec{k}\cdot(\vec{r}_1 - \vec{r}_2))}}{V} \; V_{eff}(\vec{r}_1 - \vec{r}_2) \; \frac{e^{i \vec{q}\cdot(\vec{r}_1 - \vec{r}_2)}}{V}$$

volume

$$= \frac{1}{V} \underbrace{\int d^3 r \; e^{-i(\vec{k}-\vec{q})\cdot\vec{r}} \; V(\vec{r})}_{V_{\vec{k}-\vec{q}}}$$

$$V_{\vec{k}\vec{q}} = V \langle \vec{k}, -\vec{k} \,|\, V_{\text{eff}} \,|\, \vec{q}, -\vec{q} \rangle$$

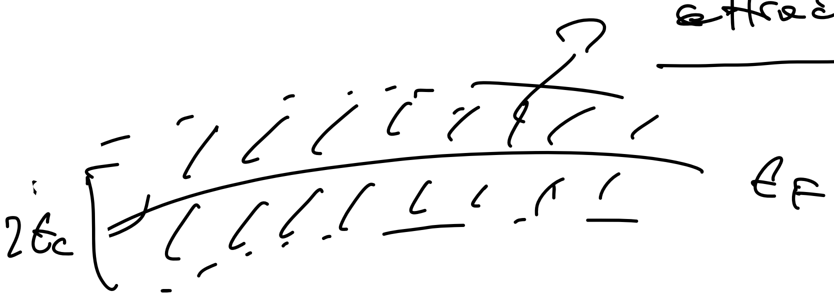
$$V(\vec{r}_1 - \vec{r}_2) = \frac{1}{V} \sum_{\vec{q}} V_{\vec{q}} \, e^{i\vec{q}\cdot\vec{r}}$$

Empirical potential (Cooper 1956)

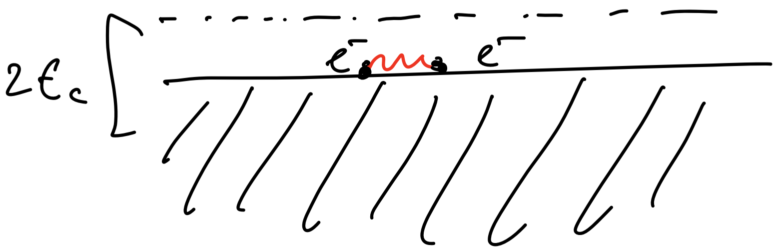
$$[V_0] = \varepsilon \cdot V$$

$$V_{\vec{k}\vec{q}} = \begin{cases} -V_0 & |\epsilon_{\vec{k}} - \epsilon_F| \leq \epsilon_c \\ & |\epsilon_{\vec{q}} - \epsilon_F| \leq \epsilon_c \\ 0 & \text{otherwise} \end{cases}$$

attractive interaction



Cooper instability → bound states among pairs of electrons



ϵ_F = Fermi sea = Pauli blocking the two electrons.

$$\left[-\frac{\hbar^2}{2m} \left(\nabla_1^2 + \nabla_2^2 \right) + V_{eff}(\vec{r}_1 - \vec{r}_2) \right] \psi(\vec{r}_1, \vec{r}_2) = E \, \psi(\vec{r}_1, \vec{r}_2)$$

~~Assumptions~~

$$\psi(\vec{r}_1, \vec{r}_2) = \cancelto{\text{const.}}{\Psi_{com}\left(\frac{\vec{r}_1 + \vec{r}_2}{2} \right)} \; \psi(\vec{r}_1 - \vec{r}_2)$$

$$= \sum_{\vec{k}} \psi_{\vec{k}} \, e^{i\vec{k}\cdot(\vec{r}_1 - \vec{r}_2)}$$

$$= V \sum_k \psi_{\vec{k}} \langle \vec{r}_1, \vec{r}_2 | \vec{k}, -\vec{k} \rangle$$

Assumption : s-wave symmetry for $\psi(\vec{r}_1 - \vec{r}_2)$

$$\psi_{\vec{k}} = \psi_{|\vec{k}|}$$

$$\psi(\vec{r}_1 - \vec{r}_2) = \sum_{\vec{k}} \psi_{|\vec{k}|} e^{i \vec{k} \cdot (\vec{r}_1 - \vec{r}_2)}$$

$$= \psi(|\vec{r}_1 - \vec{r}_2|)$$

even under exchange of the electrons

Anti-symmetric spin state

$$\underset{\text{symm.}}{\psi(|\vec{r}_1 - \vec{r}_2|)} \underset{\text{anti-symm.}}{|\chi\rangle_{12}} \longrightarrow \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}$$

Is there a bound state with this form?

$$E < 2\epsilon_F$$

$$\sum_{\vec{k}} \Big(\underbrace{2\frac{\hbar^2 k^2}{2m}}_{\epsilon_{\vec{k}}} + \frac{1}{V} \sum_{\vec{q}} V_{\vec{q}}\, e^{i\vec{q}\cdot(\vec{r}_1-\vec{r}_2)} - E \Big) \psi_{(\vec{k})}\, e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)}$$

$$V_{eff}(\vec{r}_1 - \vec{r}_2)$$

$$= 0$$

$$\sum_{\vec{k}} \underbrace{\Big((2\epsilon_{\vec{k}} - E)\psi_{(\vec{k})} + \frac{1}{V}\sum_{q} V_{\vec{k}-\vec{q}}\, \psi_{(\vec{q})} \Big)}_{= 0} e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)} = 0$$

$$(2\epsilon_k - E)\psi_k + \frac{1}{V}\sum_{\vec{q}} V_{\vec{k}-\vec{q}}\, \psi_q = 0$$

$$\psi_k = -\frac{1}{V(2\epsilon_k - E)} \sum_{\vec{q}} V_{\vec{k}-\vec{q}}\, \psi_q$$

$$|\epsilon_k - \epsilon_F| \leq \epsilon_c$$

$$V_{\vec{k}-\vec{q}} = \begin{cases} -V_0 & |\epsilon_k - \epsilon_F| \leq \epsilon_c \\ & |\epsilon_q - \epsilon_F| \leq \epsilon_c \\ 0 & \text{otherwise} \end{cases}$$

$$= \frac{N_0}{(2\epsilon_k - E)} \frac{1}{V} \sum_{\vec{q}} \psi_{\vec{q}}$$

$$|\epsilon_q - \epsilon_F| \leq \epsilon_c$$

$$\psi_k = \frac{V_0}{2\epsilon_k - E} \int_{|\epsilon_q - \epsilon_F| \leq \epsilon_c} \frac{d^3 q}{(2\pi)^3} \psi_q$$

$$\psi_{\epsilon_k}$$

$$= \frac{V_0}{2\epsilon_k - E} \int_{\epsilon_F}^{\epsilon_F + \epsilon_c} d\epsilon' \, \underbrace{g(\epsilon')}_{\text{density of states per unit volume (for spin-polarized particles)}} \psi_{\epsilon'}$$

$$g(\epsilon') \simeq g(\epsilon_F)$$

$$\psi_\epsilon = \frac{V_0\, g(\epsilon_F)}{2\epsilon - E} \int_{\epsilon_F}^{\epsilon_F+\epsilon_c} d\epsilon' \, \psi_{\epsilon'}$$

$$\cancel{\int_{\epsilon_F}^{\epsilon_F+\epsilon_c} d\epsilon \, \psi_\epsilon} = \int_{\epsilon_F}^{\epsilon_F+\epsilon_c} d\epsilon \, \frac{V_0\, g(\epsilon_F)}{2\epsilon - E} \cancel{\int_{\epsilon_F}^{\epsilon_F+\epsilon_c} d\epsilon' \, \psi_{\epsilon'}}$$

$$1 = V_0\, g(\epsilon_F) \int_{\epsilon_F}^{\epsilon_F+\epsilon_c} \frac{d\epsilon}{2\epsilon - E}$$

$$\epsilon = \epsilon' + \epsilon_F$$

$$1 = V_0\, g(\epsilon_F) \int_0^{\epsilon_c} \frac{d\epsilon'}{2\epsilon' - \tilde{E}}$$

$$\tilde{E} = E - 2\epsilon_F$$

$$1 = \frac{V_0 g(\epsilon_F)}{2} \ln\left(\frac{\tilde{E} - 2\epsilon_c}{\tilde{E}}\right)$$

$$\tilde{E} - 2\epsilon_c = \tilde{E}\, e^{2/V_0 g(\epsilon_F)}$$

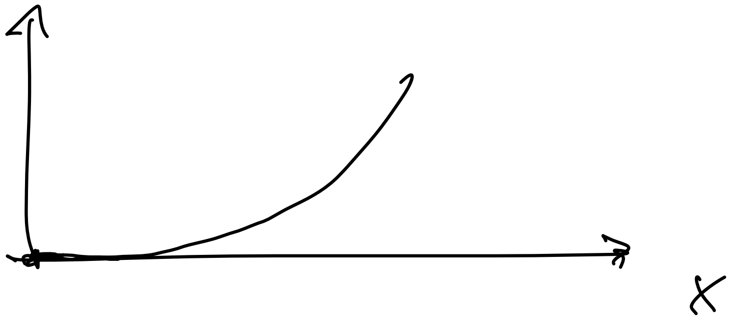
$$\tilde{E} = \frac{-2\epsilon_c}{\left[\cancel{-1} + e^{2/V_0 g(\epsilon_F)}\right]}$$

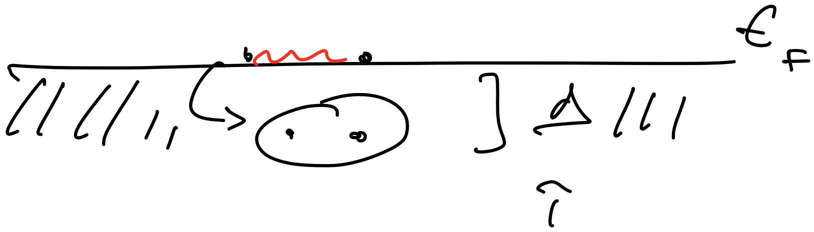
$$V_0 g(\epsilon_F) \ll 1$$

$$\boxed{\tilde{E} = E - 2\epsilon_F \simeq -2\epsilon_c\, e^{-\frac{2}{V_0 g(\epsilon_F)}}} = \underline{\Delta}$$

$$\underline{< 0} \quad \forall\, \underline{V_0}$$

$$e^{-\frac{1}{x}}$$





binding energy

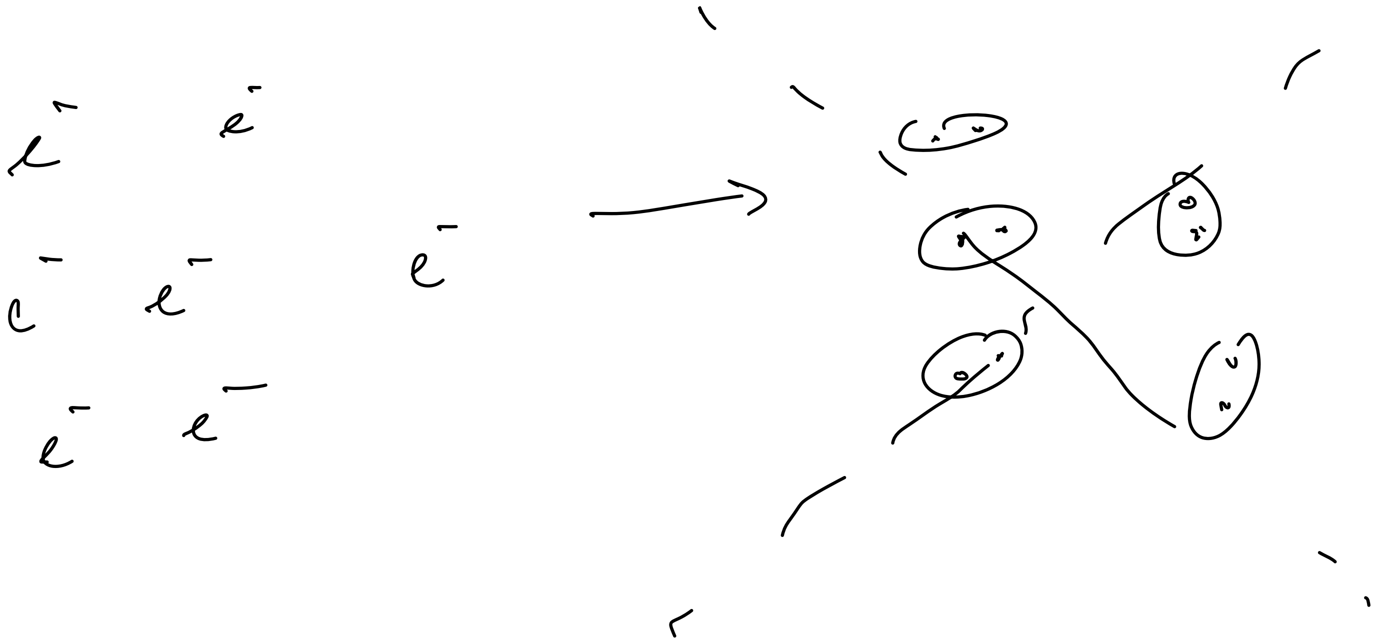
$$V_0\, g(\epsilon_F) \simeq O(1)$$

$$\frac{3}{4}\frac{n}{\epsilon_F}$$ density of electrons

$$V_0\, g(\epsilon_F) \simeq 0.2 \div 0.4$$

Cooper pairing (Cooper pairs)

$$\frac{\Delta}{k_B} \simeq 1 \div 10\ K$$



Size of a Cooper pair

$$\psi(\vec r_1 - \vec r_2) = \int_{|\epsilon_{\vec k} - \epsilon_F| \le \epsilon_c} \frac{d^3k}{(2\pi)^3}\, e^{i\vec k\cdot(\vec r_1 - \vec r_2)}\, \psi_k$$

$$\psi_k = \frac{A}{2\epsilon_k - E}$$

$$\sqrt{\langle r^2 \rangle} = \left(\int d^3r_1\, d^3r_2\; r^2\, |\psi(\vec r)|^2 \right)^{1/2}$$

$$= \dots = \left(\frac{\hbar}{\Delta}\right) v_F$$

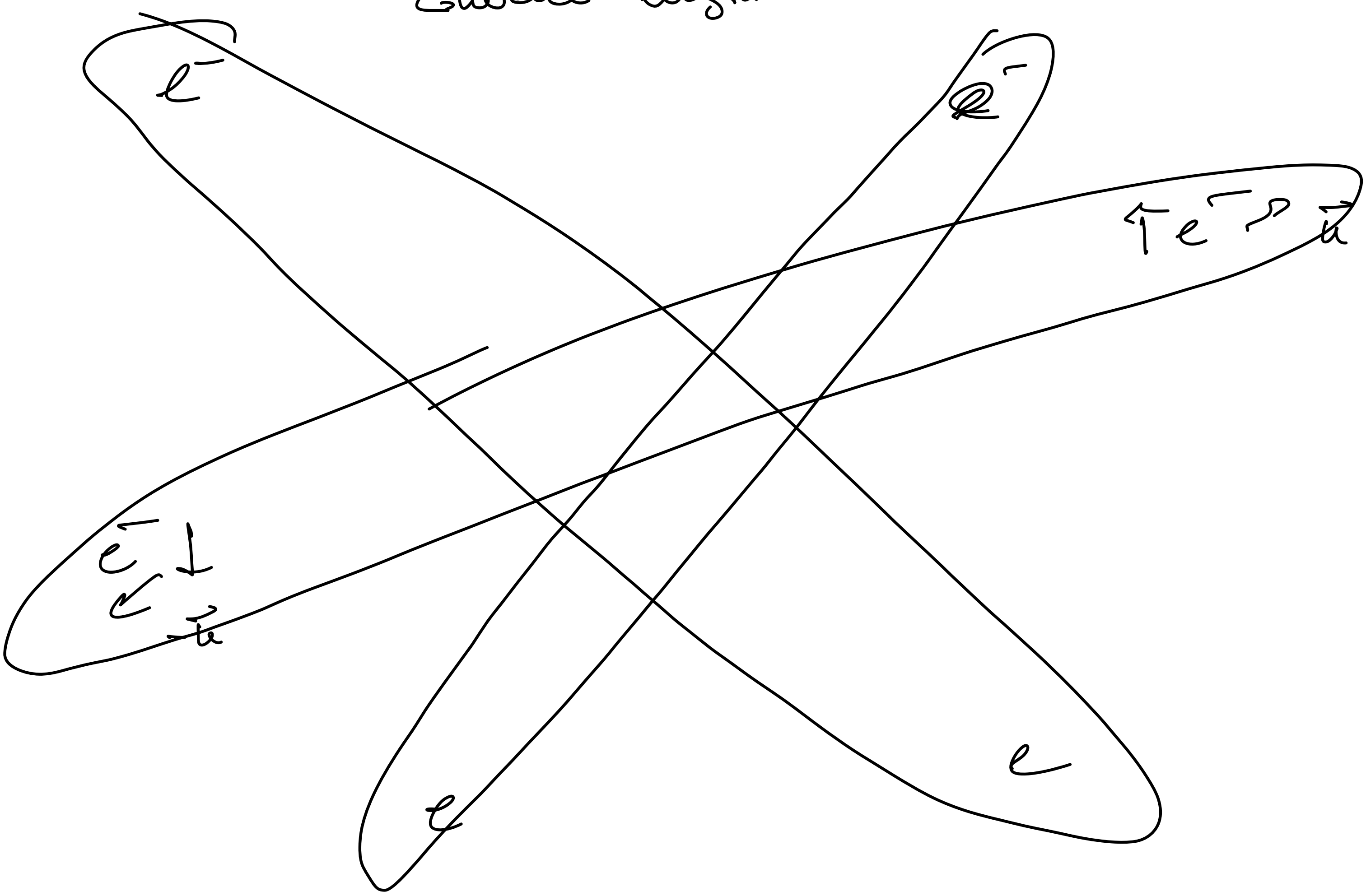
$10^6\ \mathrm{m\,s^{-1}}$

$1 \div 10\ \mathrm{K}$

$10^{-12} \div 10^{-11}\ \mathrm{s}$

$$\sqrt{\langle r^2 \rangle} \sim \underbrace{\xi = \frac{\hbar v_F}{\Delta}}_{\text{Pippard's coherence length}} \sim \underline{10^{-6} \div 10^{-5}\ m}$$

$$\underline{\underline{10^4 \div 10^5\ \mathring{A}}}$$



Many-body calculation

BCS Theory

Bardeen - Cooper - Schrieffer (1956 - 1957)

"à la Bogolyubov"

From the Cooper instability $\longrightarrow$ electrons tend to form pairs

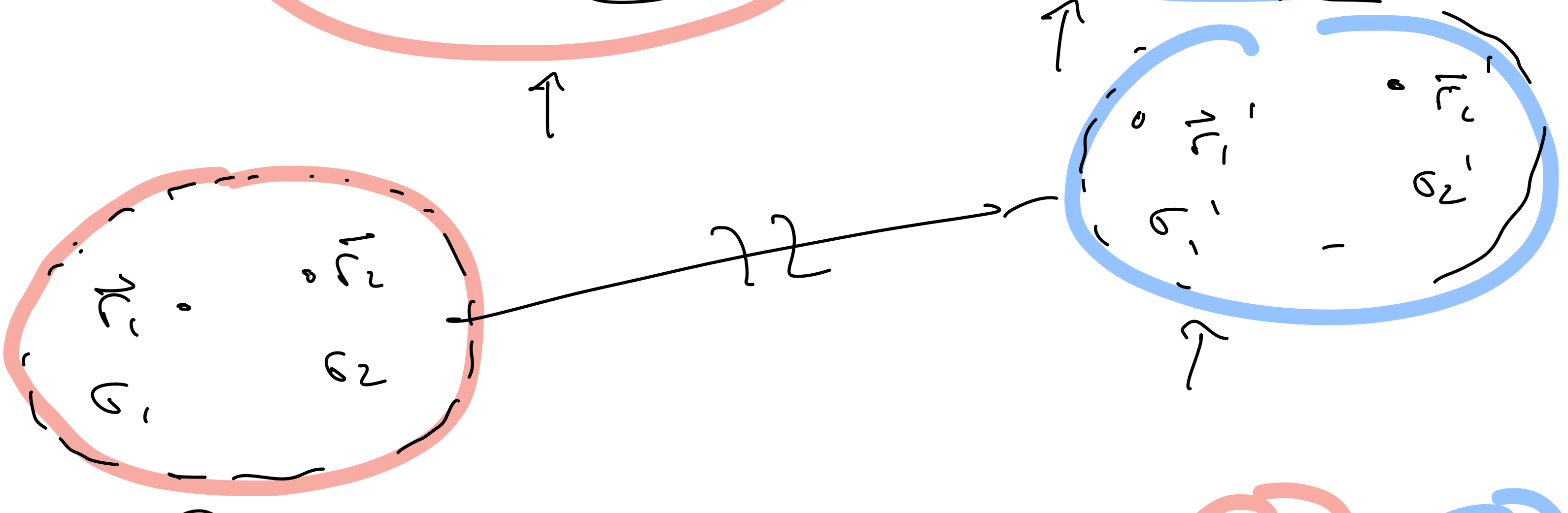
$$\psi(\vec{r}_1 - \vec{r}_2)\,|\chi\rangle = \sum_{\substack{\vec{k} \\ |\epsilon_{\vec{k}} - \epsilon_F| \leq \epsilon_c}}' \psi_{|\vec{k}|} \, \frac{e^{i\vec{k}\cdot(\vec{r}_1 - \vec{r}_2)}}{V} \, \frac{|\uparrow_1 \downarrow_2\rangle - |\downarrow_1 \uparrow_2\rangle}{\sqrt{2}}$$

$$= \sum_{\vec{k}}' \underline{\underline{\psi_{|\vec{k}|}}} \, \frac{|\vec{k}\uparrow; -\vec{k}\downarrow\rangle - |\vec{k}\downarrow; -\vec{k}\uparrow\rangle}{\sqrt{2}}$$

Electrons at the surface of the Fermi sea form a "condensate" of Cooper pairs

generalized matrix

$$G^{(2)}(\vec r_1, \sigma_1, \vec r_2, \sigma_2 ; \vec r_1{}', \sigma_1', \vec r_2{}', \sigma_2') = \langle \psi^\dagger_{\sigma_1}(\vec r_1)\, \psi^\dagger_{\sigma_2}(\vec r)\, \psi_{\sigma_2'}(\vec r_2{}')\, \psi_{\sigma_1'}(\vec r_1{}') \rangle$$



$$\longrightarrow \quad \underline{g^{(1)}(\vec r, \vec r')} = \langle \psi^\dagger(\vec r)\, \psi(\vec r') \rangle$$

<u>Natural-orbital decomposition</u> $\qquad [\psi] = \frac{1}{\sqrt{V}}$

$$G^{(2)}(\ldots) \qquad \left[\frac{1}{V}\right]^2$$

$$= \sum_\alpha \lambda_\alpha\, \chi^*_\alpha(\vec r_1 \sigma_1 ; \vec r_2 \sigma_2)\, \chi_\alpha(\vec r_1{}' \sigma_1' ; \vec r_2{}' \sigma_2')$$

$$\lambda_\alpha \geq 0$$

$$\longrightarrow \sum_{\sigma_1 \sigma_2} \int d^3 r_1 \int d^3 r_2\, |\chi_\alpha|^2 = 1$$

$$[\chi_\alpha] = \frac{1}{V}$$

λ_α (pure number)

$$\boxed{\lambda_\alpha \sim O(N) \text{ at most} \quad \text{fermions}}$$

N = tot. number of fermions

"Fermionic pair condensation"

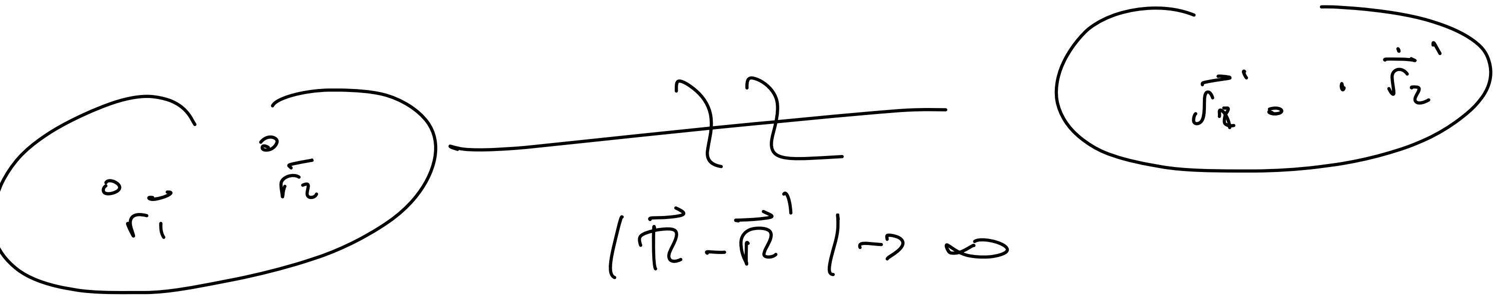
$\exists\ \lambda_0 \in O(N)$

Let's assume that a Fermi gas with attractive interaction forms a Fermionic condensate

$$\frac{\vec{r}_1 + \vec{r}_2}{2} = \vec{R} \qquad \frac{\vec{r}_1' + \vec{r}_2'}{2} = \vec{R}' \qquad \text{"CoM variables"}$$

$$\vec{r}_1 - \vec{r}_2 = \vec{r} \qquad \vec{r}_1' - \vec{r}_2' = \vec{r}'$$

$$G^{(2)}(\vec r_1\sigma_1, \vec r_2\sigma_2; \vec r_1{}'\sigma_1', \vec r_2{}'\sigma_2') \xrightarrow[|\vec R-\vec R'|\to\infty]{}$$



$$\simeq \sqrt{\lambda_0}\,\chi_0^*(\vec r_1\sigma_1;\vec r_2\sigma_2)\,\sqrt{\lambda_0}\,\chi_0(\vec r_1{}'\sigma_1';\vec r_2{}'\sigma_2')$$

Spontaneous symmetry breaking (SSB) scenario

$$= \langle \psi^\dagger_{\sigma_1}(\vec r_1)\,\psi^\dagger_{\sigma_2}(\vec r_2)\,\psi_{\sigma_2'}(\vec r_2{}')\,\psi_{\sigma_1'}(\vec r_1{}')\rangle$$

$$\underset{|\vec R-\vec R'|\to\infty}{\simeq} \langle\psi^\dagger_{\sigma_1}(\vec r_1)\,\psi^\dagger_{\sigma_2}(\vec r_2)\rangle\,\langle\psi_{\sigma_2'}(\vec r_2{}')\,\psi_{\sigma_1'}(\vec r_1{}')\rangle$$

$$\left[\, g^{(1)}(\vec r-\vec r') = \langle\psi^\dagger(\vec r)\,\psi(\vec r')\rangle \xrightarrow[|\vec r-\vec r'|\to\infty]{} \langle\psi^\dagger(\vec r)\rangle\langle\psi(\vec r')\rangle \,\right]$$

pairing field

$$\langle \psi^{+}_{\sigma_1}(\vec{r}_1)\, \psi^{+}_{\sigma_2}(\vec{r}_2) \rangle = \sqrt{\chi_0}\, \chi_0^{*}(\vec{r}_1\sigma_1, \vec{r}_2\sigma_2)$$

SSB

$$\psi^{+}_{\sigma_1}(\vec{r}_1)\,\psi^{+}_{\sigma_2}(\vec{r}_2) = \langle \quad \rangle + \delta(\quad)$$

spin independent

Interaction Hamiltonian for Fermions

$$H_{int} = \frac{1}{2}\sum_{\sigma_1\sigma_2} \int d^3r_1\, d^3r_2\, \psi^{+}_{\sigma_1}(\vec{r}_1)\,\psi^{+}_{\sigma_2}(\vec{r}_2)\, V(\vec{r}_1-\vec{r}_2)\, \psi_{\sigma_2}(\vec{r}_2)\,\psi_{\sigma_1}(\vec{r}_1)$$

$$H^{-\mu N} = \sum_{\sigma} \int d^3r\, \psi^{+}_{\sigma}(\vec{r}) \left(-\frac{\hbar^2}{2m}\nabla^2 - \mu\right) \psi_{\sigma}(\vec{r}) + H_{int}$$

$$= \underbrace{\sum_{\sigma} \sum_{\vec{k}} (\epsilon_{\vec{k}} - \mu)\, c^{+}_{\vec{k}\sigma}\, c_{\vec{k}\sigma}}_{H_0} + H_{int}$$

$$H - \mu N = H_0 +$$

$$\frac{1}{2}\sum_{\sigma_1 \sigma_2}\int d^3r_1 \int d^3r_2 \Big(\big(\langle \psi^\dagger \psi^\dagger \rangle + \delta(\psi^\dagger\psi^\dagger)\big) V \big(\langle \psi\psi \rangle + \delta(\psi\psi)\big)\Big)$$

$$\langle \psi^\dagger \psi^\dagger \rangle V \psi\psi + \langle \psi\psi \rangle V \psi^\dagger \psi^\dagger$$

$$\underbrace{- \langle \psi^\dagger \psi^\dagger \rangle V \langle \psi\psi \rangle}_{\text{const.}} + O(\delta^2)$$

$$\psi_\sigma(\vec{r}) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} c_{\vec{k}\sigma}$$

$$H - \mu N = \sum_{\vec{k}\sigma} (\epsilon_{\vec{k}} - \mu)\, c^\dagger_{\vec{k}\sigma} c_{\vec{k}\sigma}$$

$$+ \frac{1}{2} \sum_{\sigma_1 \sigma_2} \sum_{\vec{k}_1 \vec{k}_2} \sum_{\vec{k}_1' \vec{k}_2'} \langle \vec{k}_1 \vec{k}_2 | V | \vec{k}_1' \vec{k}_2' \rangle$$

$$\Big(\langle c^\dagger_{\vec{k}_1\sigma_1} c^\dagger_{\vec{k}_2\sigma_2} \rangle c_{\vec{k}_2'\sigma_2} c_{\vec{k}_1'\sigma_1} + h.c. \Big)$$

$$+ \text{const.} + O(\delta^2)$$

$$\neq 0 \qquad \vec{k}_1 = -\vec{k}_2$$
$$\sigma_1 = -\sigma_2$$

$$\langle k, -k | V | \vec{k}_1', \vec{k}_2' \rangle \neq 0$$

iff $\quad \vec{k}_1' = -\vec{k}_2' = \vec{q}$

$$H - \mu N \cong \sum_{k\sigma} (\epsilon_k - \mu)\, c^\dagger_{k\sigma} c_{k\sigma}$$
$$+ \cancel{\tfrac{1}{2}} \sum_{\vec{k},\vec{q}} \langle \vec{k}, -\vec{k} | V | \vec{q}, -\vec{q} \rangle \quad \frac{1}{V} V_{\vec{k}-\vec{q}}$$
$$\Big(\langle c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} \rangle c_{-q\downarrow} c_{q\uparrow}$$
$$+ \langle c_{-k\uparrow} c_{k\downarrow} \rangle c^\dagger_{q\uparrow} c_{-q\downarrow} \Big)$$
$$+ \text{const} \ldots$$

pairing function (gap)

$$\Delta_{\vec{k}} = - \frac{1}{V} \sum_{\vec{q}} V_{\vec{k}-\vec{q}} \langle c_{-\vec{q}\downarrow} c_{\vec{q}\uparrow} \rangle$$

$$(\mathcal{H}-\mu N)_{BCS} \cong \sum_{k\sigma} (\epsilon_k - \overset{\epsilon_F}{\mu}) c^\dagger_{k\sigma} c_{k\sigma}$$

$$\xi_{\vec{k}}$$

$$- \sum_{\vec{k}} \left(\Delta^*_k c_{-k\downarrow} c_{k\uparrow} + \Delta c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} \right)$$

$$+ \text{const.}$$