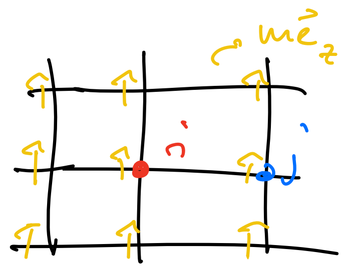
Magnetism: Ferromagnetism, mean-field approximation

$$\hat{\mathcal{H}} = -J \sum_{\langle ij \rangle} \hat{\vec{S}}_i \cdot \hat{\vec{S}}_j$$

n.n. interaction

invariant under rotations



$$= -\frac{J}{2} \sum_i \left(\sum_{j\ \text{n.n.}\ i} \vec{S}_j \right) \cdot \vec{S}_i$$

↳ quantum spins $|\vec{S}|^2 = S(S+1)$

$$\vec{m} = g\mu_B \vec{S}$$

assume the existence of a "mean field"

$$\frac{m}{g\mu_B} \vec{e}_z = \langle \vec{S}_i \rangle \neq 0$$

this state breaks the rotational symm. of $\hat{\mathcal{H}}$

SSB (spontaneous symmetry breaking)

$$\hat{\vec{S}}_i = \langle \vec{S}_i \rangle + \delta \hat{\vec{S}}_i$$

$$\langle (\delta \vec{S}_i)^2 \rangle \ll \langle \vec{S}_i \rangle^2$$

$$\hat{\mathcal{H}} \underset{MF}{\simeq} -J \sum_i \left(\sum_{j\ \text{n.n.}\ i} \langle \vec{S}_j \rangle \right) \cdot \vec{S}_i$$

↳ $z \left(\frac{m}{g\mu_B} \right) \vec{e}_z$ $\quad z$ = coordination number

$$= - B_{eff}\, g\mu_B \sum_i \vec{S}_i$$

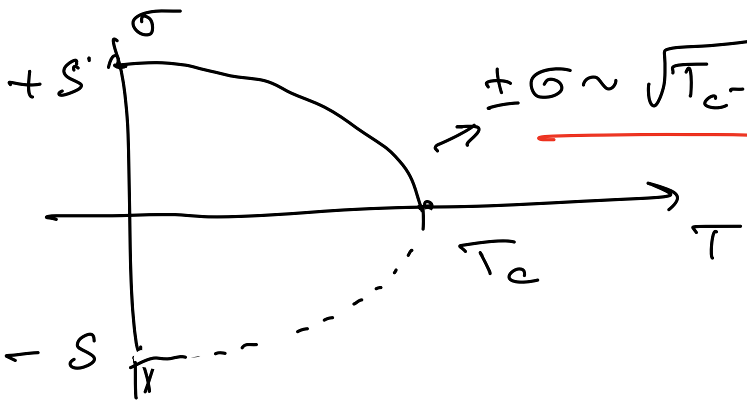
$$g\mu_B B_{eff} = J z \left(\frac{m}{g\mu_B} \right)$$

$$\frac{m}{g\mu_B} = S \, B_S\left(\beta g \mu_B B_{eff} S\right)$$

$$= S \, B_S\left(\beta J z \sigma S\right) = \sigma$$

in real materials

$\beta \; (\neq \frac{1}{2})$



$\pm\sigma \sim \sqrt{T_c - T} \sim (T_c - T)^{\beta}$

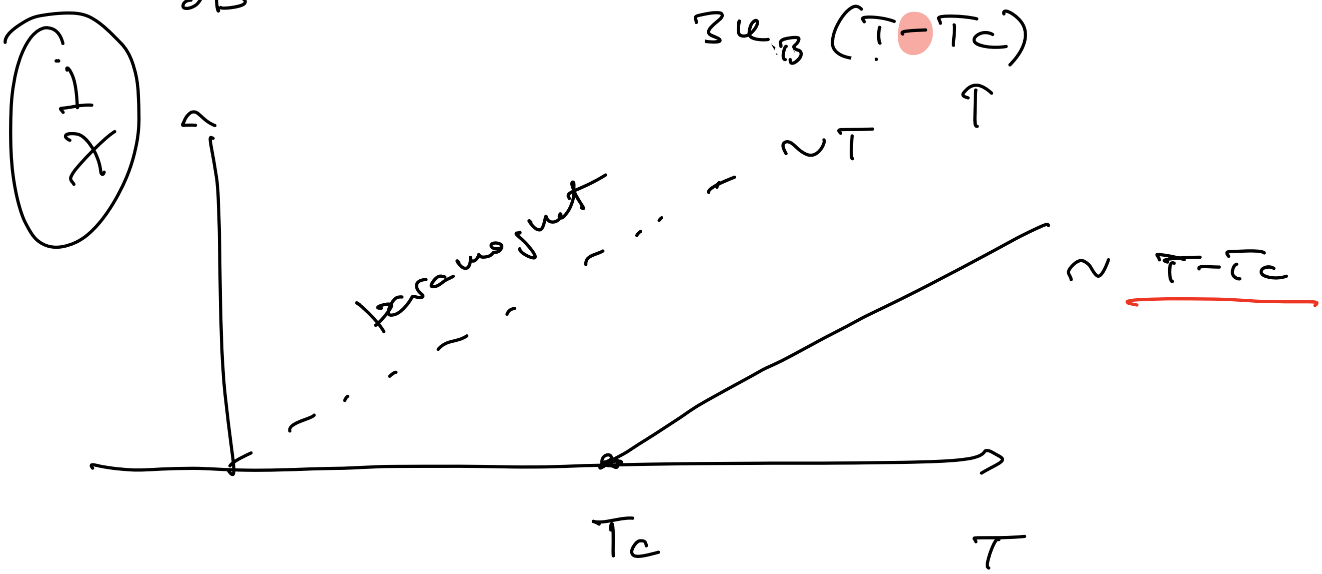
Curie critical point

$T < T_c$

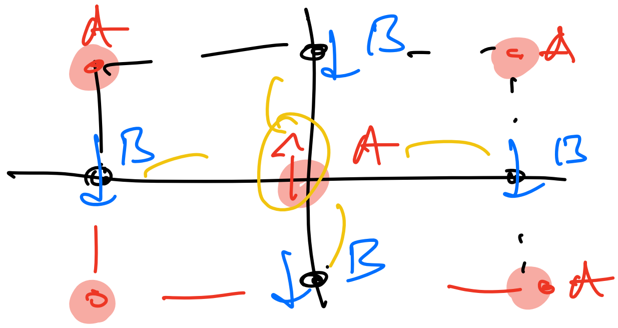
$$k_B T_c = \frac{z S(S+1) J}{3}$$

$\underline{T > T_c}$ apply a uniform field B

$$\chi = \frac{\partial m}{\partial B} = \dots = \frac{(g\mu_B)^2 S(S+1)}{3k_B(T - T_c)}$$



Antiferromagnetism



Bipartite lattice

SSB picture $\langle \vec{S}_i \rangle = (-1)^i \, m \, \vec{e}_z / g\mu_B$

$$\hat{H} = (+) J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$$

$J > 0$

$$\langle \vec{S}_i \cdot \vec{S}_j \rangle = -\left(\frac{m}{g\mu_B}\right)^2 \quad i \text{ n.n. } j$$

$$= (-) J \frac{z m \vec{e}_z}{g\mu_B} \sum_i \vec{S}_i$$

$\vec{B} = 0$

$i \in A : \langle \vec{S}_i \rangle = + \frac{m}{g\mu_B} \vec{e}_z$

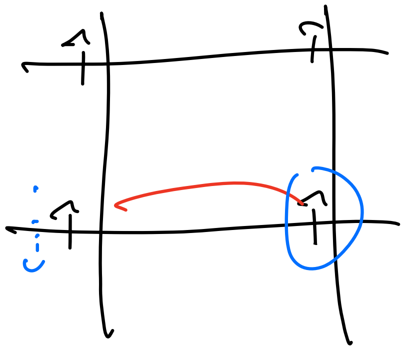
$j \in B : \langle \vec{S}_j \rangle = (-) \frac{m}{g\mu_B} \vec{e}_z$

$$= J \sum_{i \in A} \Big(\sum_{\substack{j \text{ n.n. } i \\ j \in B}} \vec{S}_j \Big) \cdot \vec{S}_i$$

$$\underset{MF}{\approx} J \sum_{i \in A} \left(-z \frac{m}{g\mu_B} \vec{e}_z\right) \vec{S}_i + J \sum_{i \in B} \left(-z \frac{m}{g\mu_B} \vec{e}_z\right) \vec{S}_i$$

Transition : Néel transition $T_N \left(\overset{MF}{=} T_c \right)$

$\boxed{T > T_N}$: apply an external uniform field $\vec{B} = B \vec{e}_z$



induced magn.

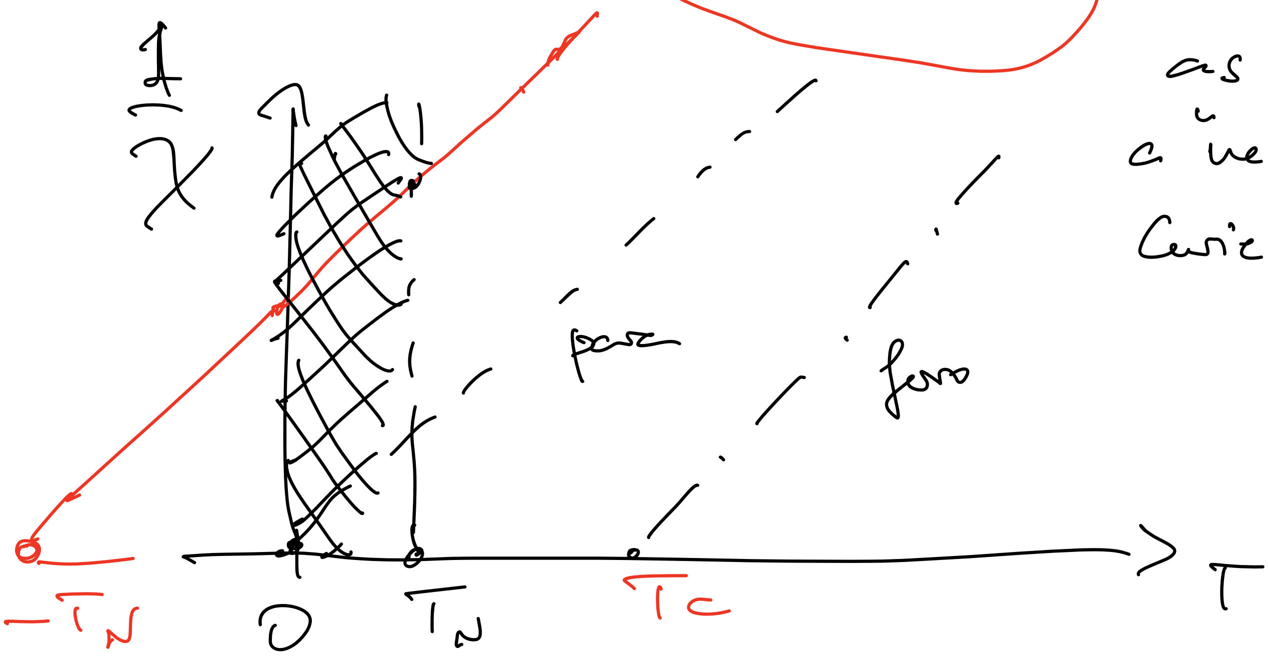
$$\frac{m}{g\mu_B} = \langle \vec{S}_i \rangle$$

$$\mathcal{H}^{MF} \simeq -g\mu_B B \sum_i S_i^z + J \sum_i \Big(\underbrace{\sum_{j\,n.n.\,i} \langle S_j^z \rangle}_{\frac{z\,m}{g\mu_B}} \Big) S_i^z$$

$$= -g\mu_B \left(B - \frac{J z m}{(g\mu_B)^2} \right) \sum_i S_i^z$$

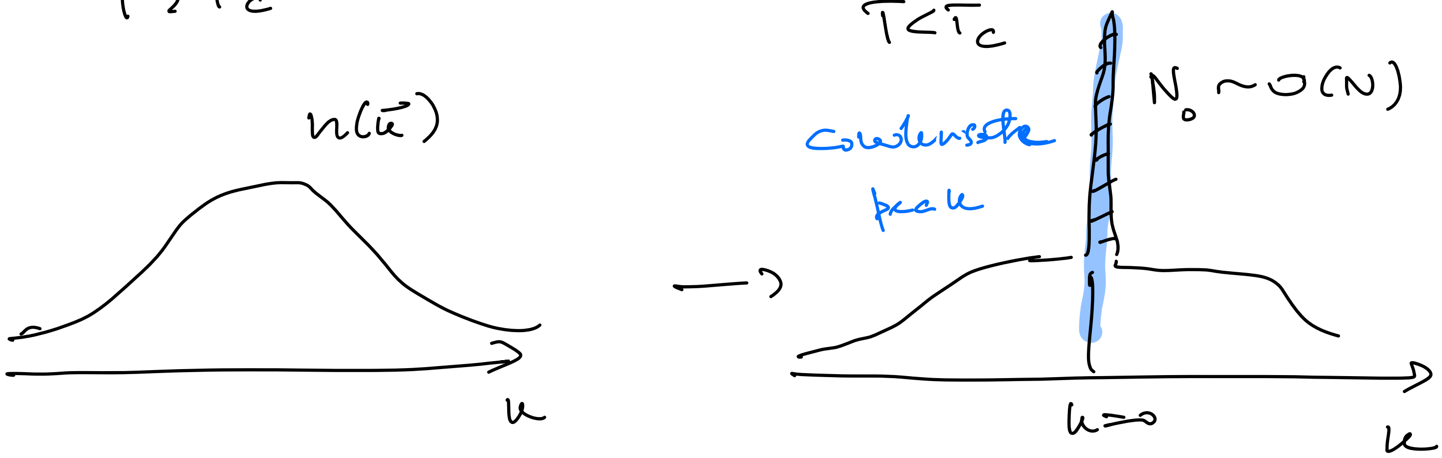
$$\chi = \frac{\partial m}{\partial B} = \dots = \frac{(g\mu_B)^2 S(S+1)}{3 k_B (T + T_N)}$$

as if a "negative Curie temp."



Bose-Einstein condensation : real-space consequences

$T > T_c$ $\qquad\qquad$ $T < T_c$



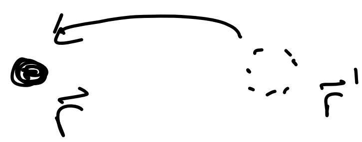
$$n(\vec{k}) \sim e^{-\beta \frac{\hbar^2 k^2}{2m}} \qquad\qquad n(\vec{k}) = \langle \hat{a}^{+}_{\vec{k}} \, \hat{a}_{\vec{k}} \rangle$$

one-body density matrix (first-order coherence function)

Bosons : $[\hat{\psi}(\vec{r}), \hat{\psi}^{+}(\vec{r}\,')] = \delta(\vec{r}-\vec{r}\,')$

$$g^{(1)}(\vec{r}, \vec{r}\,') = \langle \hat{\psi}^{+}(\vec{r}) \; \hat{\psi}(\vec{r}\,') \rangle$$

→ calculated @ thermal equilibrium



$$\hat{\psi}(\vec{r}) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} \, \hat{a}_{\vec{k}}$$

$$g^{(1)}(\vec{r}, \vec{r}\,') = \frac{1}{V} \sum_{\vec{k}\vec{k}'} e^{-i\vec{k}\cdot\vec{r}} \, e^{i\vec{k}'\cdot\vec{r}\,'} \langle \hat{a}^{+}_{\vec{k}} \, \hat{a}_{\vec{k}'} \rangle$$

translational invariance $\Rightarrow$ momentum is conserved

$\uparrow$

$$\hat{\rho} = \sum_{|\psi\rangle} |\psi\rangle\langle\psi| \, e^{-\beta E_\psi}$$

Hamiltonian eigenstates (are also eigenstates of momentum

$[\hat{H}, \hat{\vec{P}}] = 0$

$\vec{P} = \sum_{i=1}^{N} \hat{\vec{p}}_i \Rightarrow \sum_{\vec{k}} \hbar\vec{k} \, \hat{a}^\dagger_{\vec{k}} \hat{a}_{\vec{k}}$)

$$\langle a^\dagger_{\vec{k}} a_{\vec{k}'} \rangle = \delta_{\vec{k},\vec{k}'} \langle a^\dagger_{\vec{k}} a_{\vec{k}} \rangle$$

$$= \frac{1}{Z} \sum_{|\psi\rangle} \langle\psi| a^\dagger_{\vec{k}} a_{\vec{k}'} |\psi\rangle \, e^{-\beta E_\psi}$$

$\rightarrow \delta_{\vec{k}\vec{k}'} \langle a^\dagger_{k} a_{k} \rangle_\psi$

$$= \frac{1}{Z} \mathrm{Tr}\left(\hat{\rho} \, a^\dagger_{\vec{k}} a_{\vec{k}'} \right)$$

transl. inv.

$$g^{(1)}(\vec{r}, \vec{r}') \overset{V}{=} \frac{1}{V} \sum_{\vec{k}} e^{-i\vec{k}\cdot(\vec{r}-\vec{r}')} \, n(\vec{k})$$

FT^{-1} of $n(\vec{k})$

$$= g^{(1)}(\vec{r} - \vec{r}')$$

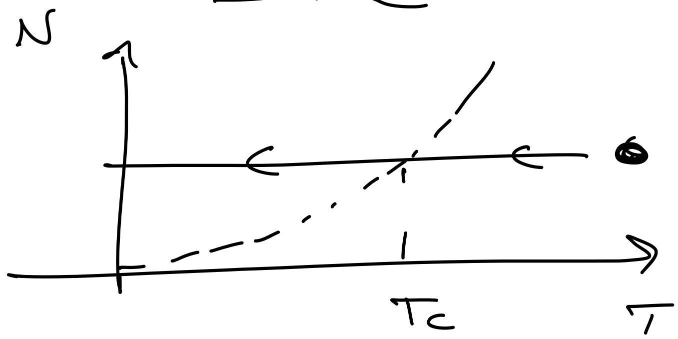
Calculation of $g^{(1)}$ for the ideal Bose gas

(grand-canonical calculation)

$T > T_c$ $\quad$ $d=3$

$$n(\vec{k}) = \frac{1}{e^{\beta \frac{\hbar^2 k^2}{2m}} e^{\beta|\mu|} - 1}$$

N fixed on average



$\mu \leq 0$

$\left[\mu \underset{T\to\infty}{\sim} - T \log T \right.$

$\beta|\mu| \gg 1$

$$n(\vec{k}) \simeq e^{-\beta \frac{\hbar^2 k^2}{2m}} e^{-\beta|\mu|}$$

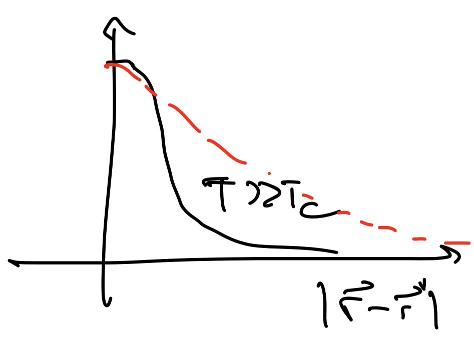
$$g^{(1)}(\vec{r}-\vec{r}') \equiv \frac{1}{V}\sum_{\vec{k}} e^{-i\vec{k}\cdot(\vec{r}-\vec{r}')} e^{-\beta\frac{\hbar^2 k^2}{2m}} e^{-\beta|\mu|}$$

$$\int \frac{d^3k}{(2\pi)^3} \left(\dots \right)$$

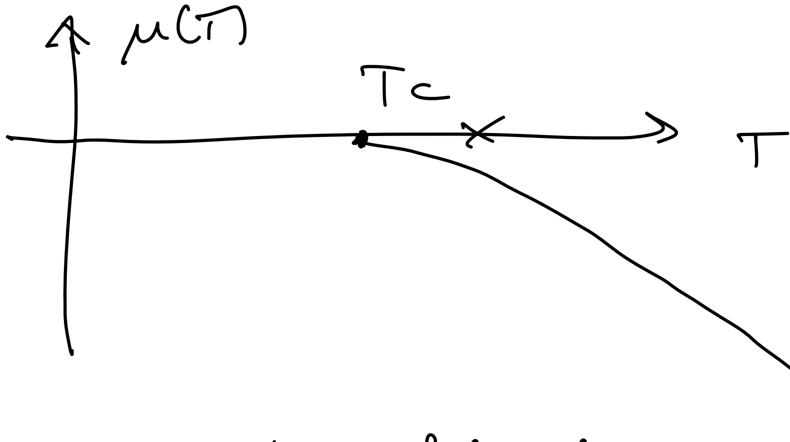
$$\left[\int_{-\infty}^{+\infty} dx \, e^{ax - bx^2} = \sqrt{\frac{\pi}{b}} \, e^{a^2/2b} \right]$$

$$= \dots = \frac{e^{-\beta|\mu|}}{\lambda_T^3} e^{-\pi |\vec{r}-\vec{r}'|^2/\lambda_T^2}$$

$$\lambda_T = \frac{h}{\sqrt{2\pi m k_B T}}$$



$T \gtrsim T_c$



$\beta|\mu| \lesssim 1$

FT of $n(\vec{k})$ is dominated by $k \to 0$ if I'm interested in $|\vec{r}-\vec{r}'| \to \infty$

$$n(\vec{k}) = \frac{1}{e^{\beta|\mu|}\, e^{\beta \frac{\hbar^2 k^2}{2m}} - 1}$$

$$\underset{k\to 0}{\simeq} \frac{1}{e^{\beta|\mu|}\left(1 + \beta\frac{\hbar^2k^2}{2m}\right) - 1 + O(k)^4}$$

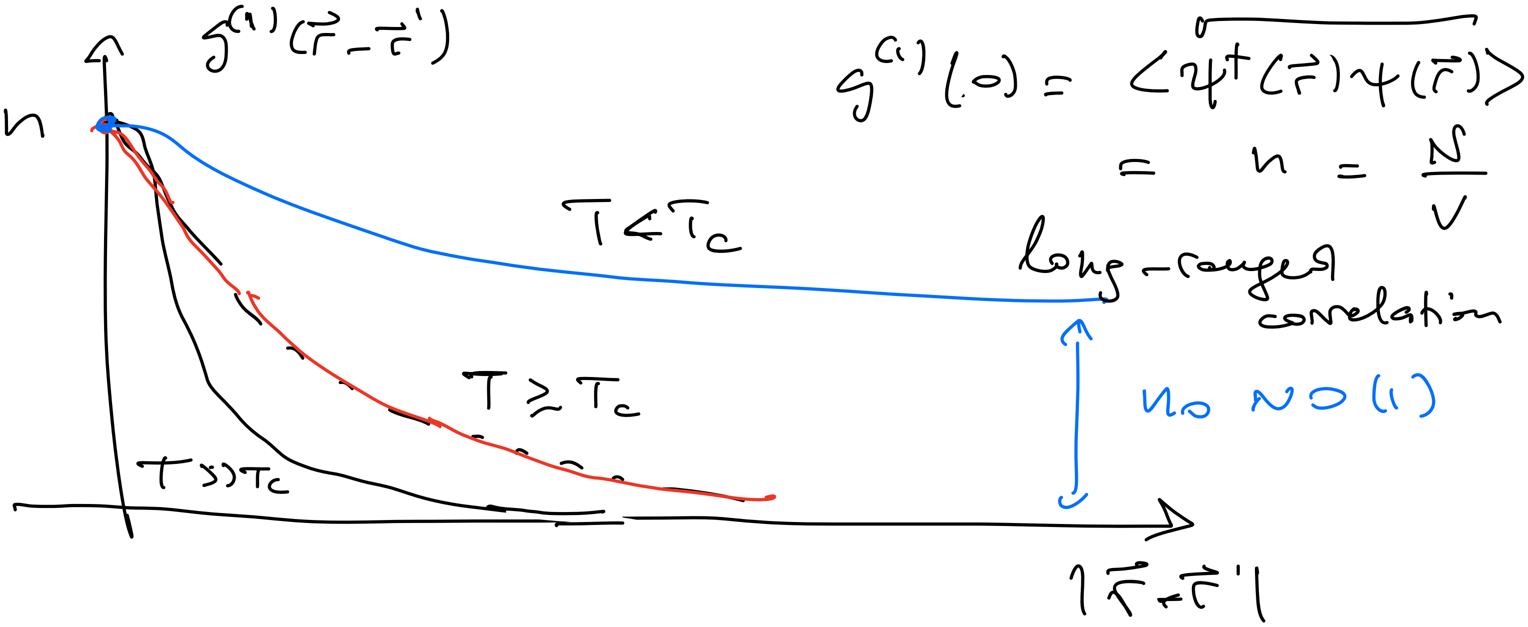
$$= \frac{1}{e^{\beta|\mu|}\left[\left(1 - e^{-\beta|\mu|}\right) + \beta\frac{\hbar^2k^2}{2m}\right]}$$

$$= \dots = \frac{e^{-\beta|\mu|}}{\lambda_T^2}\, \frac{4\pi}{k^2 + \ell_c^{-2}} \qquad \ell_c^2 = \frac{\lambda_T^2}{1 - e^{-\beta|\mu|}}$$

FT^{-1}

$$g^{(1)}(\vec{r}-\vec{r}') = \frac{e^{-\beta|\mu|}}{\lambda_T^2}\, \frac{e^{-|\vec{r}-\vec{r}'|/\ell_c}}{|\vec{r}-\vec{r}'|}$$

ℓ_c: correlation length



$$g^{(1)}(0) = \langle \psi^\dagger(\vec{r})\psi(\vec{r})\rangle = n = \frac{N}{V}$$

$T < T_c$ BEC

$\left(\begin{matrix} \mu \to 0^- \\ T \to T_c^+ \end{matrix} \quad l_c \to \infty \right)$

$$n(\vec{k}) = N_0 \delta_{\vec{k},0} + \frac{1}{e^{\beta \frac{\hbar^2 k^2}{2m}} - 1}$$

FT^{-1}

$$g^{(1)}(\vec{r}-\vec{r}\,') = \underbrace{\frac{N_0}{V}}_{n_0} + \frac{1}{\lambda_T^2 |\vec{r}-\vec{r}\,'|}$$

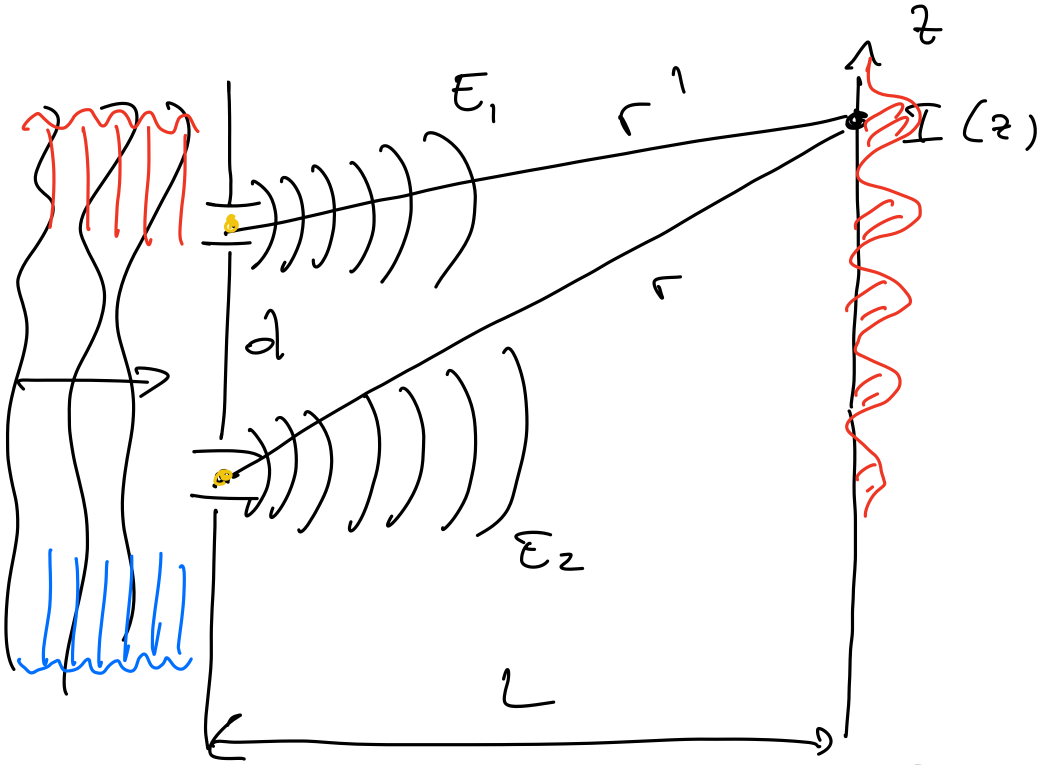
n_0 condensate density

$l_c = \infty$

Measure $g^{(1)}$ function : atom interferometry

Interferometry : e.m. fields

double slit



$$\overline{I(z)} \approx \overline{|E(z)|^2} \longrightarrow \text{photon fluctuations}$$

$$= \overline{\left| \underbrace{\frac{A}{r} e^{i(kr+\phi)}}_{E_1} + \frac{A}{r'} e^{i(kr'+\phi')} \right|^2}$$

$L \gg d$ $\qquad r \simeq r'$

$$\simeq \overline{A^2 \left[\frac{1}{r^2} + \frac{1}{(r')^2} + \frac{2}{rr'} \cos\left(\frac{kdz}{r} + \phi - \phi' \right) \right]}$$

$$\frac{2}{r r'} \left(\cos\left(\frac{k d z}{r}\right) \cos(\phi_1 - \phi'') + \sin(\quad) \sin(\quad) \right)$$

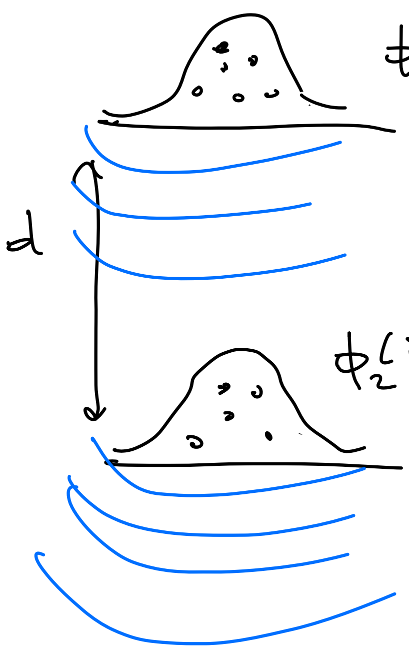
Interference $\Longleftrightarrow$ phase coherence

between "

two e.m. sources "

Atom interferometry

trap indistinguishable bosons at two locations in space



$\phi_1(\vec{r}) \rightarrow \hat{a}_1, a_1^+$

ϕ_1, ϕ_2 are orthogonal

$\phi_2(\vec{r}) \rightarrow \hat{a}_2, \hat{a}_2^+$

$t=0$

$$\psi(\vec{r}) = \phi_1(\vec{r})\,\hat{a}_1 + \phi_2(\vec{r})\,\hat{a}_2$$

$$+ \phi_3(\vec{r})\,\hat{a}_3 + \ldots$$

$$\langle \psi^\dagger(\vec r)\,\psi(\vec r)\rangle = |\phi_1(\vec r)|^2 n_1 + |\phi_2(\vec r)|^2 n_2 + \phi_1^*(\vec r)\,\phi_2(\vec r)\langle a_1^\dagger a_2\rangle + c.c.$$

↓ evolve in time under free expansion

$$\phi_{1,2}(\vec r) \to \phi_{1,2}(\vec r, t)$$

$$\psi(\vec r, t) = \phi_1(\vec r, t)\, a_1 + \phi_2(\vec r, t)\, a_2$$

$$\langle \psi^\dagger(\vec r, t)\, \psi(\vec r, t)\rangle_{t=0}$$

Heisenberg picture

$$= |\phi_1(\vec r, t)|^2 n_1 + |\phi_2(\vec r, t)|^2 n_2 + \phi_1^*(\vec r, t)\,\phi_2(\vec r, t)\,\langle a_1^\dagger a_2\rangle_{t=0} + c.c.$$

$$\langle a_1^\dagger a_2\rangle \sim \langle \psi^\dagger(\vec r_1)\,\psi(\vec r_2)\rangle$$