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Superconductivity, superfluidity and magnetism
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TD1: Coherent states, phase and number operators

1 Harmonic oscillator and coherent states

Consider a one-dimensional harmonic oscillator with Hamiltonian $\hat{\mathcal{H}} = \hat{p}^2/(2m)+(1/2)\, m\omega^2\hat{x}^2$.

1.1

Introducing the dimensionless variables:

$$\hat{X} = \sqrt{\frac{m\omega}{\hbar}}\hat{x} \qquad \hat{P} = \frac{1}{\sqrt{m\hbar\omega}}\,\hat{p} \tag{1}$$

and the transformation

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{X} + i\hat{P}) \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{X} - i\hat{P}) \tag{2}$$

show the following results:

$$\hat{\mathcal{H}} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2) \qquad [\hat{a}, \hat{a}^\dagger] = 1 \qquad [\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0 \tag{3}$$

knowing that $[\hat{x}, \hat{p}] = i\hbar$.

1.2

Verify that the Hamiltonian eigenstates admit the form

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle \tag{4}$$

and find the corresponding eigenvalue. Show that the position-momentum uncertainty relation for the Hamiltonian eigenstates reads:

$$(\Delta X \Delta P)_n = n + 1/2 \tag{5}$$

1.3

We introduce the *coherent states* as eigenstates of the destruction operator

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \tag{6}$$

where α is a *complex* variable.
Calculate the uncertainty product $(\Delta X \Delta P)_\alpha$: what can you conclude?
Represent the "uncertainty volume" $\Delta X \Delta P$ in the complex plane of the α variable. Given the expectation values $\langle\alpha|\hat{X}|\alpha\rangle$, $\langle\alpha|\hat{P}|\alpha\rangle$ and the uncertainty relation, justify why these states are called "semi-classical".

2 Coherent states and number statistics

2.1

We now wish to write the coherent states on the *number* basis $|n\rangle$ as $|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$. Show that the c_n coefficients satisfy the recursion relation

$$c_{n+1} = \frac{\alpha}{\sqrt{n+1}} c_n. \tag{7}$$

Starting from the c_0 coefficient, show that this relation is satisfied by

$$c_n = \frac{\alpha^n}{\sqrt{n!}}\, c_0\ . \tag{8}$$

Show that $c_0 = \exp(-|\alpha|^2/2)$.

2.2

Show that the number distribution $P_\alpha(n) = |\langle n|\alpha\rangle|^2$ associated with a coherent state $|\alpha\rangle$ is the Poissonian distribution

$$P_\alpha(n) = e^{-\langle \hat{n}\rangle} \frac{\langle \hat{n}\rangle^n}{n!} \tag{9}$$

where $\langle \hat{n}\rangle = |\alpha|^2$. Show that variance is $\Delta n^2 = \langle \hat{n}\rangle$.

2.3

Justify that the coherent states can be written in the form

$$|\alpha\rangle = e^{-|\alpha|^2/2}\, e^{\alpha \hat{a}^\dagger} |0\rangle \tag{10}$$

where the exponential of an operator is defined via the Taylor expansion

$$e^{\alpha \hat{a}^\dagger} = \sum_{n=0}^{\infty} \frac{(\alpha \hat{a}^\dagger)^n}{n!}\quad . \tag{11}$$

3 Phase operator and phase-number uncertainty

In analogy with complex numbers, we introduce the polar decomposition of the destruction operator

$$\hat{a} = e^{i\hat{\phi}} \hat{n}^{1/2} \tag{12}$$

where

$$\hat{n}^{1/2} = \sum_{n=0}^{\infty} n^{1/2} |n\rangle\langle n| \qquad e^{i\hat{\phi}} = \sum_{n=0}^{\infty} |n\rangle\langle n+1| \tag{13}$$

3.1

Show that $\hat{\phi}$ is not Hermitian, because $e^{i\hat{\phi}}$ is not a unitary operator, $(e^{i\hat{\phi}})^\dagger e^{i\hat{\phi}} \neq 1$. What would happen if the sum started from $n = -\infty$?

3.2

Show that

$$|\phi\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} e^{in\phi} |n\rangle \tag{14}$$

is an eigenstate of the operator $e^{i\hat{\phi}}$ with eigenvalue $e^{i\phi}$. Can you draw an analogy between the $|n\rangle$ and $|\phi\rangle$ states on the one side, and momentum vs. position eigenstates on the other side? What can you guess for the commutation relation $[\hat{n}, \hat{\phi}]$?

3.3

Show that

$$[\hat{n}, e^{i\hat{\phi}}] = -e^{i\hat{\phi}} \tag{15}$$

By Taylor-expanding the exponential on both sides, show that above commutator is compatible with the commutation relation

$$[\hat{n}, \hat{\phi}] = i \quad . \tag{16}$$

This implies the phase-number uncertainty relation $\boxed{\Delta n \Delta\phi \gtrsim 1/2}$, which will be *fundamental* for this course.

4 Phase-number uncertainty for coherent states

We now consider the phase statistics associated with a coherent state.

4.1

Show that the phase distribution $P_\alpha(\phi) = |\langle\phi|\alpha\rangle|^2$ is given by

$$P_\alpha(\phi) = \frac{e^{-|\alpha|^2}}{2\pi} \left| \sum_n \frac{(e^{-i\phi}\alpha)^n}{\sqrt{n!}} \right|^2 \tag{17}$$

and justify why the maximum probability is associated with $\phi = \theta$, where $\alpha = |\alpha|e^{i\theta}$.

4.2

Using the number representation of the phase operator

$$\langle n|\hat{\phi}|\alpha\rangle = -i\frac{\partial}{\partial n}\langle n|\alpha\rangle \tag{18}$$

and considering $|\alpha| \gg 1$, show that

$$\langle\alpha|\hat{\phi}|\alpha\rangle = \theta - \frac{i}{2}(\log\langle n\rangle - \langle\log n\rangle) \tag{19}$$

[Use the Stirling formula $n! \approx \sqrt{2\pi n}\, n^n\, e^{-n}$, and convince yourself that its use is justified by the fact that $|\alpha| \gg 1$]. For $|\alpha| \gg 1$ the Poissonian distribution for the number statistics has a vanishing relative uncertainty, namely $P_\alpha(n) \to \delta_{n,\langle n\rangle}$. Justify in this limit that $\langle\hat{\phi}\rangle \approx \theta$.

4.3

Using the same assumptions as before, show that

$$\langle\alpha|\hat{\phi}^2|\alpha\rangle \approx \theta^2 + \frac{1}{2\langle n\rangle} \tag{20}$$

Conclude that, for a coherent state

$$\Delta\phi^2 \approx \frac{1}{2\langle n\rangle} \tag{21}$$

and, therefore, $\Delta\phi\ \Delta n \approx 1/\sqrt{2}$. Looking at the uncertainty volume in the complex plane α, as at Question 1.3), can one justify this result (roughly) with a simple geometrical argument valid in the limit $|\alpha| \gg 1$?

5 Phase correlations

Imagine to have a system of N independent harmonic oscillators. We are interested in the correlation function $\langle a_i^\dagger a_j \rangle$ associated with the i-th and j-th harmonic oscillator.

5.1

Imagine that each harmonic oscillator is in a different coherent state $|\alpha_i\rangle$, $i = 1, ..., N$, with $|\alpha_i| = |\alpha|$ for all i. Consider the quantity

$$I = \frac{1}{N} \sum_{ij} \langle a_i^\dagger a_j \rangle \quad . \tag{22}$$

Show that I is extensive, $I = N|\alpha|^2 = N_{\text{tot}}$ (total average boson number), if the phases θ_i of the α_i's are all the same (*phase coherence*), while I is intensive, $I = |\alpha|^2$ (average boson number density), if the phases are completely random (*phase incoherence*).

5.2

Imagine now that all harmonic oscillators are in a Fock state, $|n_i\rangle$. Calculate I and show that it corresponds to the phase incoherent case. Can you justify this result from the number-phase uncertainty?

6 Coherent states of fermionic pairs

Consider a fermion which can occupy two single-particle spin states, $|\uparrow\rangle$ and $|\downarrow\rangle$, with associated fermionic creation and destruction operators $\hat{a}_\sigma$, $\hat{a}_\sigma^\dagger$ ($\sigma = \uparrow, \downarrow$), satisfying fermionic anti-commutation relations $\{\hat{a}_\sigma, \hat{a}_{\sigma'}^\dagger\} = \delta_{\sigma,\sigma'}$, $\{\hat{a}_\sigma, \hat{a}_{\sigma'}\} = \{\hat{a}_\sigma^\dagger, \hat{a}_{\sigma'}^\dagger\} = 0$.

6.1

We introduce the *pair operators* $b_p = \hat{a}_\downarrow \hat{a}_\uparrow$, $b_p^\dagger = \hat{a}_\uparrow^\dagger \hat{a}_\downarrow^\dagger$. Show that $[b_p, b_p] = [b_p^\dagger, b_p^\dagger] = 0$, but $[b_p, b_p^\dagger] = 1 - \hat{a}_\uparrow^\dagger \hat{a}_\uparrow - \hat{a}_\downarrow^\dagger \hat{a}_\downarrow$, so that the b_p, $b_p^\dagger$ operators realize only *partially* the algebra of bosonic operators.

6.2

Consider the coherent states of fermionic pairs, defined as

$$|\alpha\rangle = \mathcal{N} e^{\alpha \hat{a}_\uparrow^\dagger \hat{a}_\downarrow^\dagger} |0\rangle \tag{23}$$

where $\mathcal{N}$ is a normalization factor to be determined. Write the state in terms of fermionic Fock states. Show that

$$|\alpha\rangle = \frac{1}{\sqrt{1 + |\alpha|^2}} (1 + \alpha\ \hat{a}_\uparrow^\dagger \hat{a}_\downarrow^\dagger) |0\rangle \tag{24}$$

6.3

Calculate the expectation values $\langle \hat{n}_\uparrow + \hat{n}_\downarrow \rangle$ and $\langle (\hat{n}_\uparrow + \hat{n}_\downarrow)^2 \rangle$ and the particle-number variance. Show that the relative uncertainty on the total particle number has the same expression as for bosonic coherent states.