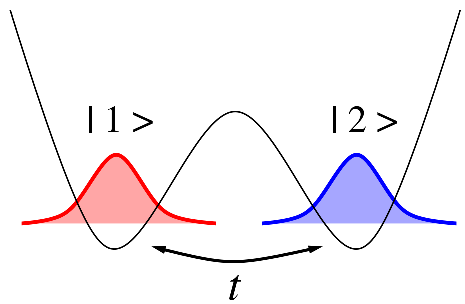
Tommaso Roscilde, Fabio Mezzacapo, Youssef Trifa

TD3: Bosons in a double-well potential

In this TD we will investigate the physics of N identical bosons trapped in a double-well potential (bosonic Josephson junction). In the limit of a very deep potential, we can consider only two orthonormal single-particle states, $|1\rangle$ and $|2\rangle$, localized on the two sides of the double-well potential.
The many-body Hamiltonian on this reduced basis takes the form:

$$\hat{\mathcal{H}} = -t\left(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1\right) + \frac{U}{2}\left[\hat{n}_1\left(\hat{n}_1 - 1\right) + \hat{n}_2\left(\hat{n}_2 - 1\right)\right] \tag{1}$$

where $\hat{a}_1, \hat{a}_1^\dagger$ and $\hat{a}_2, \hat{a}_2^\dagger$ are bosonic creation/destruction operators.



1 Many-body states: Fock states, Schrödinger's cat states and BEC states.

In this section we consider trial wave-functions to describe the ground state of the model.

1.1

Consider the Fock state

$$|N_1, N_2\rangle = \frac{\left(\hat{a}_1^\dagger\right)^{N_1}}{\sqrt{N_1!}} \frac{\left(\hat{a}_2^\dagger\right)^{N_2}}{\sqrt{N_2!}} |0\rangle \tag{2}$$

Write down its one-body density matrix $g_{ij}^{(1)} = \langle \hat{a}_i^\dagger \hat{a}_j \rangle$. For what values of N_1 and N_2 is the system showing condensation/fragmentation?

1.2

Consider then the *Schrödinger's cat* state

$$|N_1, N_2; N_2, N_1\rangle = \frac{1}{\sqrt{2}}\left(|N_1, N_2\rangle + |N_2, N_1\rangle\right) \tag{3}$$

Calculate its one-body density matrix $g_{ij}^{(1)}$ for $|N_1 - N_2| > 1$; can this state show condensation?

1.3

We then take the BEC state

$$|\alpha, \phi\rangle = \frac{1}{\sqrt{N!}} \left(\frac{\hat{a}_1^\dagger + \alpha e^{i\phi}\ \hat{a}_2^\dagger}{\sqrt{1+\alpha^2}} \right)^N |0\rangle \tag{4}$$

Write down its one-body density matrix $g_{ij}^{(1)}$. To do so, it can be useful to introduce the orthonormal states

$$|+\rangle = \frac{|1\rangle + \alpha e^{i\phi}\ |2\rangle}{\sqrt{1+\alpha^2}} \qquad |-\rangle = \frac{\alpha|1\rangle - e^{i\phi}\ |2\rangle}{\sqrt{1+\alpha^2}} \tag{5}$$

and the associated creation/destruction operators $\hat{a}_\pm$, $\hat{a}_\pm^\dagger$. Diagonalise $g_{ij}^{(1)}$ and show that the state corresponds to a perfect condensate (as it was obvious from the first definition....).

1.4

Show that the Fock state can be obtained by Fourier transformation of the BEC state with respect to ϕ. Conclude on the existence of a phase-difference/number-difference uncertainty.

2 Variational determination of the ground state

2.1

Calculate the energy expectation value for the Fock state $E(N_1, N_2) = \langle N_1, N_2|\hat{\mathcal{H}}|N_1, N_2\rangle$, and for the Schödinger's cat state. Find the combination (N_1, N_2) minimizing the energy in the case of repulsive $(U > 0)$ and attractive $(U < 0)$ interactions. Is the Fock state with $N_1 \neq N_2$ an acceptable equilibrium state of the system?

2.2

Calculate the energy expectation value for the BEC state $E(\alpha, \phi)$, and show that the energy is an explicit function of the phase ϕ. Assuming $\alpha = 1$, minimize with respect to ϕ for both attractive and repulsive interactions.

2.3

Comparing the minimum energies $E_{\min}(N_1, N_2)$ and $E_{\min}(\alpha = 1, \phi)$, determine the transition points between fragmented states and condensate states upon varying the ratio U/t. Draw the corresponding phase diagram on the U/t axis.

3 Schwinger pseudo-spin representation

3.1

Introducing the operators

$$\hat{J}_x = \frac{1}{2}\left(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1\right) \qquad \hat{J}_y = \frac{1}{2i}\left(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1\right) \qquad \hat{J}_z = \frac{1}{2}\left(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2\right) \tag{6}$$

show that they satisfy the commutation relations of angular momentum, $[\hat{J}_\alpha, \hat{J}_\beta] = i\varepsilon^{\alpha\beta\gamma}\hat{J}_\gamma$.

3.2

Calculating $J^2 = |\hat{\boldsymbol{J}}|^2$, determine the effective spin length. Show that the Hamiltonian of the system in the pseudospin variables takes the form

$$\hat{\mathcal{H}} = -2t\hat{J}_x + U\left(\hat{J}_z^2 + J^2 - N\right) \quad . \tag{7}$$

Treating the J_α operators as classical variables, what are the values of the spin components which minimize the energy in the opposite limits $|U|/t \ll 1$, $U/t \gg 1$ and $-U/t \gg 1$?

3.3

Calculate the vector $\langle\hat{\boldsymbol{J}}\rangle = (\langle\hat{J}_x\rangle, \langle\hat{J}_y\rangle, \langle\hat{J}_z\rangle)$ for the Fock state $|N_1, N_2\rangle$ and the BEC state $|\alpha, \phi\rangle$. Introducing the angle variable θ such that

$$\frac{1}{\sqrt{1+\alpha^2}} = \cos(\theta/2) \qquad \frac{\alpha}{\sqrt{1+\alpha^2}} = \sin(\theta/2) \tag{8}$$

show that the state vector $\langle\hat{\boldsymbol{J}}\rangle$ for the BEC state is a vector of length N with polar/azimuthal angles ϕ, θ. Represent the ground state of the system in the various phases, as determined in the previous section, via the length and orientation of the state vector.

4 Momentum distribution in the BEC state

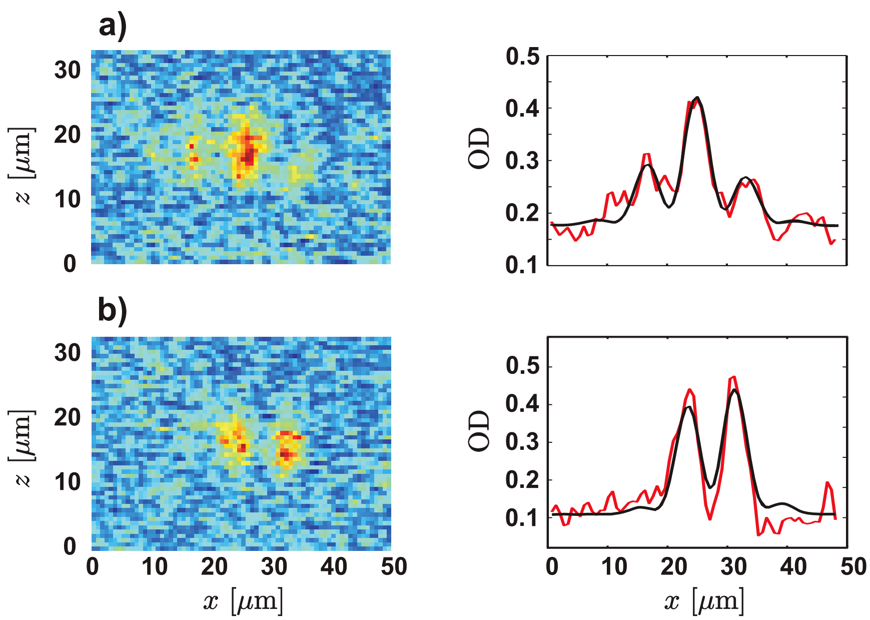


Figure 1: Absorption images of bosons released from a double-well trap. From M. Albiez's PhD thesis, University of Heidelberg (2005).

Be $\Phi_1(x)$ and $\Phi_2(x)$ the spatial wavefunction associated with the two states $|1\rangle$ and $|2\rangle$, with the property $\Phi_2(x) = \Phi_1(x-d)$ (d = separation between the two wells). The states Φ_1 and Φ_2 are part of a basis Φ_i of orthonormal wavefunctions.

4.1

Calculate $\langle\hat{\psi}^\dagger(x)\hat{\psi}(x')\rangle$ for the generic BEC state $|\alpha, \phi\rangle$.
(*Suggestion*: expand the field operators on the Φ_i basis).

4.2

Calculate the momentum distribution $n(k) = \langle \hat{a}_k^\dagger \hat{a}_k \rangle$, where

$$a_k = \int dx \, \frac{e^{-ikx}}{\sqrt{V}} \, \psi(x) \; . \tag{9}$$

You should find an expression containing N, α, ϕ, k and $\tilde{\Phi}_1(k)$ (Fourier transform of the Φ_1 wavefunction). In the case of Gaussian wavefunctions, $\Phi_1(x) \sim \exp[-x^2/(2\sigma^2)]$, sketch the form of $n(k)$.

4.3

Fig. 1 shows the measurement of the momentum distribution of bosons trapped in double well potential for two different measurement shots. Can you tell the phase difference ϕ between the two wells in the two cases?