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TD9: Superconductors in a magnetic field

Useful formulae

- Gradient in cylindrical coordinates (on the cylinder surface):

$$\boldsymbol{\nabla} = \frac{1}{R}\frac{\partial}{\partial\theta}\hat{e}_\theta + \frac{\partial}{\partial z}\hat{e}_z. \tag{1}$$

- Closest integer n to a real number a

$$n = \text{int}(a + \text{sign}(a)\ 1/2). \tag{2}$$

- For a closed path γ encircling the surface S_γ

$$\oint_\gamma \boldsymbol{A}\cdot d\boldsymbol{l} = \int_{S_\gamma} \boldsymbol{B}\cdot\hat{n}\ dS_\gamma = \Phi_\gamma(\boldsymbol{B}) = \text{flux of } \boldsymbol{B} \text{ through } S_\gamma \tag{3}$$

1 Flux quantization in a superconducting cylinder

In this exercise we wish to describe the fundamental phenomenon of quantization of the magnetic flux which threads a superconducting cylinder. For this purpose, we will start with a description of the problem of a single electron confined in a cylinder of radius R, height L, immersed in a uniform magnetic field $\boldsymbol{B} = (0, 0, B)$ parallel to the axis of the cylinder (see Fig. 1). We will assume periodic boundary conditions along the z axis. The cross section of the cylinder forms a ring, whose thickness will be neglected for the moment.

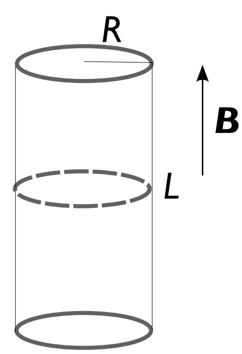

Figure 1:

The Hamiltonian of an electron in a magnetic field reads

$$\mathcal{H} = \frac{(-i\hbar\boldsymbol{\nabla} + e\boldsymbol{A})^2}{2m} \tag{4}$$

We take for the vector potential the symmetric gauge $\boldsymbol{A} = \frac{B}{2}(-y, x, 0)$. Passing to cylindrical coordinates (r, θ, z), the vector potential on the cylindrical surface reads $\boldsymbol{A} = (BR/2)\hat{e}_\theta$.

1.1

Justify that the eigenvectors of the Hamiltonian have the form

$$\psi_{n,k_z}(\theta, z) = \mathcal{N} \exp(in\theta) \exp(ik_z z) \tag{5}$$

where $\mathcal{N}$ is a normalization constant to be determined, and n is an *integer*.
Show that the eigenvalues of the Hamiltonian take the form

$$E_{n,k_z} = \frac{\hbar^2}{2mR^2} \left(n + \frac{\Phi}{\tilde{\Phi}_0} \right)^2 + \frac{\hbar^2 k_z^2}{2m} \tag{6}$$

where Φ is the flux of the magnetic field through the cylinder, and $\tilde{\Phi}_0 = h/e$ is the so-called (normal) flux quantum.

1.2

Find the ground state quantum numbers n_0 and $k_{z,0}$. Show that the ground-state energy is a periodic function of Φ, with period $\tilde{\Phi}_0$. (*Suggestion*: look at the mathematical appendix!).

1.3

The current density associated with a wavefunction ψ reads

$$\boldsymbol{j} = -\frac{\hbar e}{2mi} \left(\psi^* \boldsymbol{\nabla} \psi - \psi \boldsymbol{\nabla} \psi^* \right) - \frac{e^2}{m} |\psi|^2 \boldsymbol{A} \ . \tag{7}$$

Calculate the current density associated with the ground state ψ_0.
Show that the ground state carries a persistent electrical current, and that this current is a periodic function of the applied flux Φ with period $\tilde{\Phi}_0$. Considering that the cylinder is equivalent to a solenoid, for which values of $\Phi/\tilde{\Phi}_0$ is the magnetic field generated by the solenoid parallel/antiparallel to the applied field $\boldsymbol{B}$?

1.4

Persistent currents in normal metals can only be observed in very special conditions. Cite at least two reasons for the decay of such currents in a normal metal. If l is the mean free path of an electron in a metal, how large does l need to be for the persistent current to be observable?

On the other hand, as you know, persistent currents are quite stable in superconductors. In the following we will assume that the Cooper pairs appearing in a superconductor can be described by a *macroscopic wavefunction* $\Psi(\boldsymbol{r})$ (analogous to that of condensed bosons) which gives the amplitude of finding the whole condensate of Cooper pairs at point $\boldsymbol{r}$.
Moreover we will now consider a finite thickness for the cylinder, and we will assume that the superconductor develops persistent currents on the inner and outer surface of the cylinder, which screen completely the magnetic field in the bulk of the cylinder (Meissner effect). Hence the bulk of the cylinder has no magnetic field (and, consequently, zero current) – see Fig. 2.

1.5

We will assume that the macroscopic wavefunction satisfies a similar equation to Schrödinger's equation for single particles in a magnetic field (the so-called Ginzburg-Landau equation), but this time the particle charge is $2e$ (because we have Cooper pairs). In the boundary regions, in which the magnetic field penetrates into the superconductor, we will use the results we found for the ground state of a single electron confined to a cylinder and immersed in a magnetic field. There the macroscopic wavefunction will take the form

$$\Psi(r, z, \theta) = \Psi_r(r) \ \Psi_z(z) \ \frac{1}{\sqrt{2\pi}} \exp(in\theta) \tag{8}$$

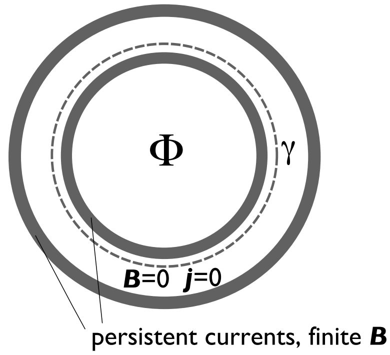

Figure 2: Cross section of the superconducting cylinder.

Justify that, in the ground state

$$n = n_0(\Phi) = -\mathrm{int}\left(\frac{\Phi}{\Phi_0} + \mathrm{sign}(\Phi)\frac{1}{2}\right) \tag{9}$$

where $\Phi_0 = h/(2e)$ is the so-called superconducting flux quantum.

1.6

Going to the bulk region with zero magnetic field and current, by continuity with the boundary region we have to assume that the macroscopic wavefunction reads:

$$\Psi(r, z, \theta) = \mathcal{A}\,\frac{1}{\sqrt{2\pi}}\exp(in\theta) \tag{10}$$

where $\mathcal{A}$ is a constant. The current carried by the macroscopic wavefunction reads

$$\boldsymbol{j} = -\frac{\hbar(2e)}{2mi}\left(\Psi^*\boldsymbol{\nabla}\Psi - \Psi\boldsymbol{\nabla}\Psi^*\right) - \frac{(2e)^2}{m}|\Psi|^2\boldsymbol{A}\ . \tag{11}$$

Imposing that $\boldsymbol{j} = 0$ all along a loop γ entirely contained in the bulk region (see Fig. 3), demonstrate that the magnetic flux threading the loop obeys the *quantization condition*:

$$\Phi_\gamma = -n_0(\Phi)\Phi_0\ . \tag{12}$$

1.7

Plot the magnetic flux Φ_γ as a function of Φ/Φ_0, and compare it with the experimental results (first obtained by Deaver/Fairbank and Doll/Näbauer in 1961) for the flux trapped in the hollow of a superconducting cylinder – Fig. 3. What is the analogous phenomenon occurring in He^4? Which aspect do the two systems share, which is responsible for both phenomena?

2 Superconducting quantum interference device (SQUID)

We are used by now to the idea that the Cooper pairs in a superconductor can be described via a macroscopic wavefunction, $\Psi(\boldsymbol{r})$, such that the (super-)current flowing in a superconductor can be obtained from it as in Eq. (11).

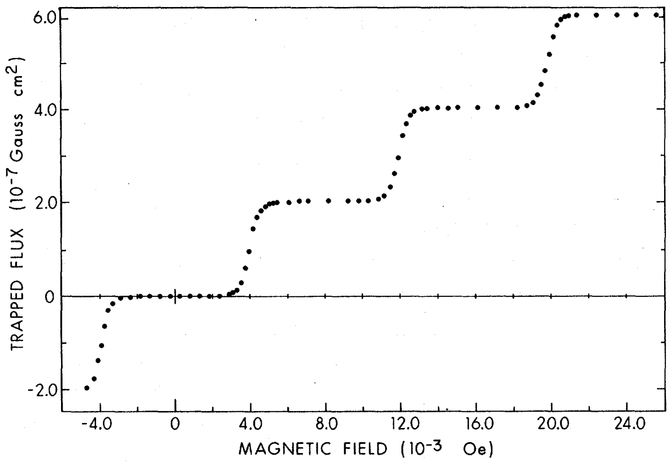

FIG. 3. Trapped flux as a function of the magnetic field in which the cylinder was cooled below its transition temperature. These data were taken with a tin cylinder, 56-μ i.d. and 24 mm long, with walls about 5000 Å thick.

Figure 3: Flux quantization experiment (from W. L. Goodman *et al.*, Phys. Rev. B **4**, 1530 (1971)).

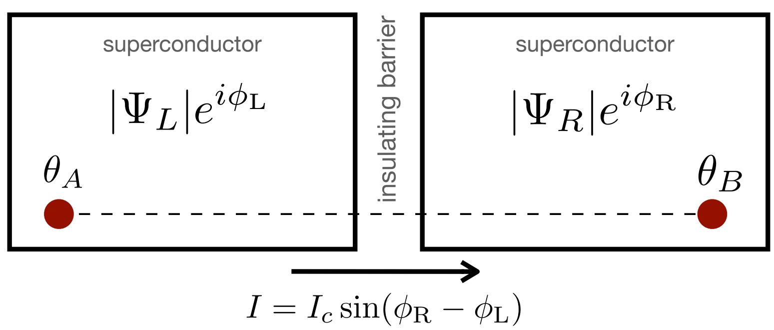

Figure 4: A superconducting Josephson junction.

2.1

Using the amplitude/phase decomposition, $\Psi(\boldsymbol{r}) = |\Psi(\boldsymbol{r})|e^{i\phi(\boldsymbol{r})}$, express $\boldsymbol{j}$ as a function of $\phi(\boldsymbol{r})$. Show that if we introduce the so-called *gauge-invariant phase*

$$\theta(\boldsymbol{r}) = \phi(\boldsymbol{r}) + \frac{2e}{\hbar}\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{A}\cdot d\boldsymbol{l} \tag{13}$$

then, assuming $\Psi(\boldsymbol{r}) = |\Psi(\boldsymbol{r})|e^{i\theta(\boldsymbol{r})}$, we obtain the same current if we eliminate the term proportional to $\boldsymbol{A}$ in Eq. (11). The line integral of the vector potential is calculated along an arbitrary line starting from the (arbitrary) point $\boldsymbol{r}_0$ – do not worry, it will become better defined later!

2.2

Let us consider now a *superconducting Josephson junction* (Fig. 4), formed by two superconducting leads separated by a thin barrier (typically a layer of insulator). It is the exact superconducting analog of the Josephson junction we explored in the case of bosons in TD no. 5. There we saw that a tunneling current $I = I_c \sin\Delta\phi$ – going, say, from left (L) to right (R) – crosses the junction when a phase difference

$\Delta\phi = \phi_R - \phi_L$ is present between the two macroscopic wavefunctions $\Psi_{L(R)} = |\Psi_{L(R)}|e^{i\phi_{L(R)}}$ describing the superconductors on both sides of the junction.
When a vector potential $\boldsymbol{A}$ is present in the system, give the expression of the gauge-invariant phase difference $\theta_B - \theta_A$ between two points A and B on opposite sides of the junction.

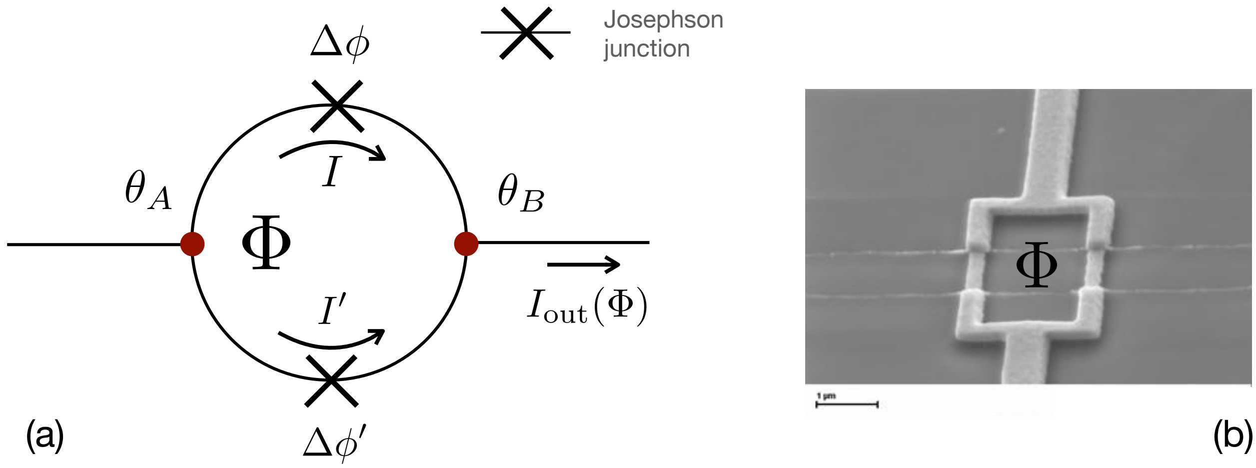

Figure 5: (a) Circuit scheme of a SQUID (all the circuit elements are superconducting); (b) an actual SQUID (the Josephson junctions are defined by the two discontinuities in the square loop).

2.3

We now consider the circuit geometry in Fig. 5(a), defining a so-called superconducting quantum interference device (SQUID): two Josephson junctions are present in a superconducting loop, which is threaded with a magnetic-field flux Φ.
Using the result of the previous question for the expression of $\theta_B - \theta_A$ – this time calculated for the two points indicated in Fig. 5(a)) – establish the relationship between the two phase differences $\Delta\phi$ and $\Delta\phi'$ across the two junctions.
Note: the line integral defining the gauge-invariant phase difference has to be taken along a circuit which runs *inside* the superconductor.

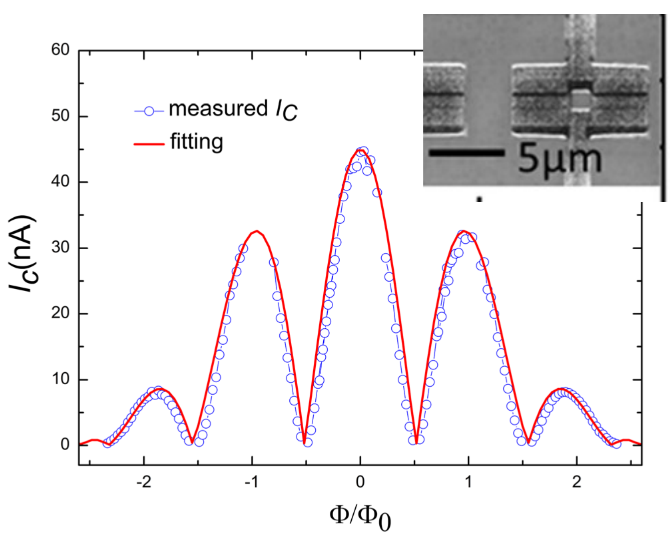

Figure 6: Critical current of a micro-SQUID (from S.-S. Yeh et al., Appl. Phys. Lett. 101, 232602 (2012)).

2.4

By using Kirchhoff's law for the current, conclude that the outgoing (super-)current in the circuit takes the form

$$I_{\text{out}}(\Phi) = 2I_c \cos\left(\frac{\pi\Phi}{\Phi_0}\right) \sin(\theta_B - \theta_A) \ . \tag{14}$$

2.5

Fig. 6 shows the measured critical current through a SQUID: can you understand this result (at least partially) from the previous formula?
This result shows that a SQUID is sensitive to magnetic fluxes of the order of Φ_0. If you have a macroscopic superconducting loop of 1mm^2, what sensitivity can you achieve on the measurement of a magnetic field?

Note: in fact, SQUIDs can achieve a sensitivity of 1fT (10^{-15} T), which is the order of magnitude of the magnetic fields associated *e.g.* with the activity of the human brain. SQUID magnetometers are among the most sensitive ones, used broadly in many fields of research – among them, the study of brain activity.