Homework - to give back on March 7th

Exercise 1: linear reduction. Let us consider the matrix $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

1. Why is A diagonalizable?
2. Observe that $(1, 1, 1)$ is an eigenvector for A.
3. From the trace and the determinant of A, determine the eigenvalues and their multiplicity.
4. Write the characteristic polynomial $\chi_A(X) = \det(A - XI_3)$ to check the previous result.

Exercise 2: optimization without constraints - geometrical point of view.
Let $f(x, y) = \frac{x^2+y^2}{5-2y}$.

1. Give the domain of definition of f. What is the regularity of f on this domain? Show that f is not bounded above or below.
2. For which $k \in \mathbb{R}$ is the level set of f associated with k, *i.e.* $\left\{(x, y) \in \mathbb{R}^2 | f(x, y) = k\right\}$, non empty? Show that this level set is then a circle, and give its characteristics. Draw the level sets of f for $k = -6$, $k = -5$, $k = 0$, $k = 1$, $k = 4$.
3. Determine the critical points of f. What is the local behaviour of f around these points?

Exercise 3: optimization without constraints - differential point of view.
Let $f(x, y, z) = xz - x - z + \frac{1}{2}y^2$. Is f bounded below or above? Determine the critical points of f. What is the behaviour of f around these points?

Exercise 4: concavity of the value function. Let D be a convex domain of $\mathbb{R}^n$, $f : D \to \mathbb{R}$ be a concave continuous function and $g_i : D \to \mathbb{R}$ be k convex continuous constraint functions. Assume that there exists a convex domain C of $\mathbb{R}^k$ such that for all $c \in C$, the quantity $V(c) = \sup_{g(x) \le c} f(x)$ is achieved at an admissible point. Prove that V is concave.

Exercise 5: optimization with constraints. We look at the domain $D = \{x \ge 0, y \ge 0, x^2+y^2 \le 5\}$ and at a point $P = (x_P, y_P) = (2, 4)$ out of this domain. We are interested in the maximal distance between P and a point of D, and the minimal distance between P and a point of D. We define

$$M = \max_{(x,y)\in D} (x_P - x)^2 + (y_P - y)^2 \qquad m = \min_{(x,y)\in D} (x_P - x)^2 + (y_P - y)^2$$

1. Justify without computations that these two quantities are achieved.
2. Show that every point of the constraint D is qualified for the constrained optimization problem.
3. Write a Lagrangian function addressing both problems simultaneously, write the associated Kuhn-Tucker system of (in)equalities, and solve it. Which ones of the solution are candidates for the considered optimization problems? Determine M and m.
4. Draw on a figure all the candidates obtained, and the gradients of the goal function and of the active constraints (*contraintes saturées*) at those points. Discuss for each of them if they are local extrema under constraints or not.
5. Now let (x_P, y_P) be considered as parameters, staying close to $(2, 4)$, and define $m(x_P, y_P) = \min_{(x,y)\in D}(x_P - x)^2 + (y_P - y)^2$. Assuming that the conditions of the envelope theorem are met, give the partial derivatives of m at $(2, 4)$. Give a first order approximation of

$$\min_{(x,y)\in D} (2.1 - x)^2 + (3.9 - y)^2.$$