Sheet 1 - Topology and function spaces

Let us recall some definitions. Let $(E, \mathfrak{T})$ be a topological space.

- The elements of $\mathfrak{T}$ are the open sets of the topology.
- $\mathfrak{B}$ is a basis of the topology if $\mathfrak{B}$ is a subset of $\mathfrak{T}$ such that every open set writes as the union of elements of $\mathfrak{B}$.
- For $x \in E$, U is a neighbourhood of x if it exists $\mathcal{U} \in \mathfrak{T}$ such that $x \in \mathcal{U} \subset U$.
- For $x \in E$, a collection $\mathfrak{B}(x)$ of neighbourhood of x is a basis of neighbourhoods of x if every open set $\mathcal{U}$ containing x, contains an $U \in \mathfrak{B}(x)$.
- Warning ! A neighbourhood of x does not need to be an open set. It can even be closed. The collection of closed balls $\overline{B(0; \frac{1}{m})}$ forms a basis of (closed) neighbourhoods of 0 in $\mathbb{R}^d$.

Exercise 1. 1. Let $f : E \to F$ map a topological space into another one. Recall that f is continuous at x if, for every open $\mathcal{V} \ni f(x)$, there exists an open set $\mathcal{U}$ of E containing x such that $f(\mathcal{U}) \subset \mathcal{V}$. Prove that this definition can be restated equivalently by replacing *open* by *neighbourhood.* Actually, it is enough to verify the property when U is taken in a basis $\mathfrak{B}(x)$.

2. Let X be a set and $(F_i)_{i \in I}$ be topological spaces. Let $f_i : X \to F_i$ be a list of maps.
   (a) Prove that there exists a "coarsest topology over X which makes continuous the f_i's"; describe a basis. *We shall use this topology in the sequel.*
   (b) Let E be a topological space and $g : E \to X$ be a map. Prove that g is continuous if, and only if, for every $i \in I$, $f_i \circ g$ is continuous.
   (c) Let (x_n) be a sequence in X. Prove that (x_n) converges towards $x \in X$ if, and only if, for every $i \in I$, $(f_i(x_n))$ converges towards $f_i(x)$.
3. Let A be a set and B be a topological space. In the previous question, we make the following choices:
$$X = \mathcal{F}(A; B) \quad \text{(functions from } A \text{ to } B\text{)}, \qquad I = A, \qquad \forall a \in A, \quad f_a(g) := g(a).$$
What does it mean that a sequence of functions $g_n : A \longrightarrow B$ converges towards a function g ? Which topology do you recognize ?

Exercise 2. Let $p, q \in [1, +\infty]$ be such that $p < q$. One denote as usual
$$\ell^p = \left\{ (x_n)_n \in \mathbb{R}^{\mathbb{N}} \;\middle|\; \sum_{n=0}^{+\infty} |x_n|^p < +\infty \right\},$$
a vector space which we endow with the norm $\|(x_n)\|_p = (\sum |x_n|^p)^{\frac{1}{p}}$.

1. Which inclusion is valid between ℓ^p and ℓ^q ? Is this inclusion continuous ?
2. Let (X, μ) be a measurable space, where μ is a finite measure. Which inclusion is valid between the spaces L^p and L^q over (X, μ) ? Is this inclusion continuous ?
3. Show that there exists a measure μ such that $L^p(]0, 1[, \mu)$ and $L^p(\mathbb{R}, dx)$ are isometric.
4. Construct a linear subspace of $L^p(\mathbb{R}, dx)$ isometric to ℓ^p.

Exercise 3 (A Hörmander's theorem)**.** A bounded operator $T : (L^p(\mathbb{R}^n), \|\cdot\|_p) \to (L^q(\mathbb{R}^n), \|\cdot\|_q)$ is said to be *translation invariant* if it commutes with every translation: $\tau_h T = T\tau_h$ for all h in $\mathbb{R}^n$. The aim of this exercise is to prove the following statement: if $q < p < \infty$, a translation invariant bounded operator $T : L^p \to L^q$ is trivial.

1. Let u be in L^p. Show that $\|u + \tau_h u\|_p \to 2^{1/p}\|u\|_p$ when $\|h\| \to \infty$. *Hint: you may decompose u as the sum of a compactly supported function and of a function with arbitrarily small L^p norm.*
2. Show that if C is a constant of continuity for $T : L^p \to L^q$, and T is translation invariant, then $2^{1/p-1/q}C$ is also a constant of continuity. Conclude.
3. Can you think of an example of translation invariant bounded operator when $p \geq q$?

Exercise 4. Let Ω be a subset of $\mathbb{R}^d$. For $\alpha > 0$ and $f : \Omega \to \mathbb{R}$, one defines

$$|f|_\alpha := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

One considers next the Hölder space $C^{0,\alpha}(\Omega) = \Big\{ f \in C^0(\Omega) \mid |f|_\alpha + \|f\|_\infty < +\infty \Big\}$.

1. Let $f \in C^{0,\alpha}(\Omega)$. Prove that f extends in a unique way as a continuous fonction $\overline{f}$ over $\overline{\Omega}$, and that $\overline{f} \in C^{0,\alpha}(\overline{\Omega})$.
2. Let us assume that Ω is bounded.
   (a) Show that $\alpha < \alpha'$ implies $C^{0,\alpha'}(\Omega) \subset C^{0,\alpha}(\Omega)$.
   (b) Suppose in addition that Ω is convex. Show that if $f \in C^1(\Omega)$ and Df is bounded, then $f \in C^{0,\alpha}(\Omega)$ for every $\alpha \leq 1$.
3. Let Ω be connected. If $f \in C^{0,\alpha}(\Omega)$ for some $\alpha > 1$, prove that f is constant.
4. One denotes $\|\cdot\|_{C^{0,\alpha}} = |\cdot|_\alpha + \|\cdot\|_\infty$. Show that $(C^{0,\alpha}(\Omega), \|\cdot\|_{C^{0,\alpha}})$ is a Banach space.
5. One assumes again that Ω is bounded. Show that the imbedding $(C^{0,\alpha}(\Omega), \|\cdot\|_{C^{0,\alpha}}) \hookrightarrow (C^0(\overline{\Omega}), \|\cdot\|_\infty)$ is compact, namely that every bounded sequence in $(C^{0,\alpha}(\Omega), \|\cdot\|_{C^{0,\alpha}})$ contains a subsequence which converges in the norm $\|\cdot\|_\infty$).

Exercise 5. Let E be a topological vector space and F be a normed vector space. Let $f : E \to F$ be a linear map. Prove the equivalence between the following statements:

- f is continuous,
- f is continuous at 0,
- there exists a neighbourhood of 0, on which f is bounded.