Sheet 10 - Sobolev embeddings

Continuous embeddings. Let Ω be $\mathbb{R}^d$, $\mathbb{R}^d_+$ or a bounded domain (i.e. open and connected) of $\mathbb{R}^d$ with C^1 boundary.

1. If $1 \leq p < d$, then $W^{1,p}(\Omega) \subset L^q(\Omega)$ for all $q \in \left[p, \frac{dp}{d-p}\right]$.
2. If $p = d$, then $W^{1,d}(\Omega) \subset L^q(\Omega)$ for all $q \in [d, \infty[$.
3. If $d < p < \infty$, then $W^{1,p}(\Omega) \subset C^{1-\frac{d}{p}}(\overline{\Omega})$.

Exercise 1. Let Ω be a bounded domain of $\mathbb{R}^d$ with C^1 boundary. Show that if $p > d$, $W^{1,p}(\Omega)$ is closed under multiplication.

Exercise 2. *Optimality of the Sobolev embeddings.*
Let $1 \leq p < d$ and $\alpha \in [1, \infty]$. Show that if the inclusion $W^{1,p}(\mathbb{R}^d) \subset L^\alpha(\mathbb{R}^d)$ holds, then necessarily $p \leq \alpha \leq \frac{dp}{d-p}$.
Hint: Define $u_\lambda(x) = u(\lambda x)$ for $\lambda > 0$.

Exercise 3. Let $m \geq 1$ and $1 \leq p < \infty$. Show that

1. if $\frac{1}{p} - \frac{m}{d} > 0$, then $W^{m,p}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$, where $\frac{1}{q} = \frac{1}{p} - \frac{m}{d}$,
2. if $\frac{1}{p} - \frac{m}{d} = 0$, then $W^{m,p}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$ for all q in $[p, \infty[$,
3. if $\frac{1}{p} - \frac{m}{d} < 0$, then $W^{m,p}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ (in fact, one can even show that $W^{m,p}(\mathbb{R}^d) \subset C^k(\mathbb{R}^d)$ where $k = [m - \frac{d}{p}]$).

Exercise 4 (Fourier series and the action functional)**.** Let $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$, for $n \in \mathbb{N}$, $H^n(\mathbb{T}^d)$ will denote the usual Sobolev space of order n on $\mathbb{T}^d$. For $u \in L^2(\mathbb{T}^d)$, we will work with its Fourier series

$$u(x) = \sum_{k \in \mathbb{Z}^d} u_k e^{ik \cdot x}$$

which converges in $L^2(\mathbb{T}^d)$ with $u_k \in \mathbb{C}$.

1. (a) Show that $H^n(\mathbb{T}^d) = \left\{u \in L^2(\mathbb{T}^d) \mid \sum_{k \in \mathbb{Z}^d} |k|^{2n} |u_k|^2 < +\infty\right\}$ and that the Hermitian form
   $$(u|v)_n := u_0 \overline{v_0} + \sum_{k \in \mathbb{Z}^d} |k|^{2n} u_k \overline{v_k}$$
   defines an equivalent inner product for this Hilbert space.
   Then, we define $H^s(\mathbb{T}^d) := \left\{u \in L^2(\mathbb{T}^d) \mid \sum_{k \in \mathbb{Z}^d} |k|^{2s} |u_k|^2 < +\infty\right\}$ for all real number $s \geq 0$.
   (b) For $0 \leq s < t$, show the inclusion $H^t(\mathbb{T}^d) \hookrightarrow H^s(\mathbb{T}^d)$.
   (c) Denote by $j : H^{\frac{1}{2}} \hookrightarrow L^2$ the inclusion map and by $j^* : L^2 \to H^{\frac{1}{2}}$ its adjoint operator. Show that $j^*(L^2) \subset H^1$ and that $\|j^*(u)\|_1 \leq \|u\|_{L^2}$.
   (d) Find the relationship between $s \geq 0$ and $m \in \mathbb{N}$ providing a natural continuous injection $H^s(\mathbb{T}^d) \hookrightarrow C^m(\mathbb{T}^d)$.
2. We will now limit ourselves to the case $d = 1$ and study maps from $\mathbb{T}^1$ to $\mathbb{C}^n$. The only changes to make are to formally replace $u_k \in \mathbb{C}$ by $u_k \in \mathbb{C}^n$ and $u_k \overline{v_k}$ by $\langle u_k, v_k \rangle$ in the former questions where $\langle x, y \rangle = \mathrm{Re}(x\bar{y})$ denotes the usual inner product of the Euclidean space $\mathbb{R}^{2n} \simeq \mathbb{C}^n$.
   We are interested in the study of the functional:
   $$\Phi(u) = \int_0^1 \left(\frac{1}{2} \langle -iu', u \rangle - H \circ u\right) \mathrm{d}t.$$
   where $u : \mathbb{T}^1 \to \mathbb{C}^n$ (regularity to be specified) and $H : \mathbb{C}^n \to \mathbb{R}$ is smooth.
   (a) Show that $a(u, v) := \int_0^1 \frac{1}{2} \langle -iu', v \rangle \, \mathrm{d}t$ is a smooth bilinear form of $H^{\frac{1}{2}}(\mathbb{T}^1, \mathbb{C}^n)$.
   (b) Show that $b(x) := \int_0^1 H \circ u \mathrm{d}t$ is a C^1 function of $H^{\frac{1}{2}}(\mathbb{T}^1, \mathbb{C}^n)$ such that its gradient $\nabla b : H^{\frac{1}{2}}(\mathbb{T}^1, \mathbb{C}^n) \to H^{\frac{1}{2}}(\mathbb{T}^1, \mathbb{C}^n)$ satisfies $\nabla b(x) = j^* \nabla H(x)$.
   (c) Show that if $u \in H^{\frac{1}{2}}(\mathbb{T}^1, \mathbb{C}^n)$ is a critical point of Φ (*i.e.* $\Phi'(u) = 0$), then $u \in C^\infty(\mathbb{T}^1, \mathbb{C}^n)$ and it is a periodic solution of the Hamiltonian equation:
   $$u' = i\nabla H(u).$$