Sheet 11 - Fractional Sobolev spaces and traces

Exercise 1. *Some properties of $H^s(\mathbb{R}^d)$.*

1. Show that $\delta_0 \in H^s(\mathbb{R}^d)$ for $s < -d/2$.
2. Show that if $s > d/2$, $H^s(\mathbb{R}^d)$ embeds continuously into $C_0(\mathbb{R}^d) = \{u \in C(\mathbb{R}^d) \,:\, u(x) \to 0 \text{ as } |x| \to \infty\}$.
3. Let $k \in \mathbb{N}$, and $s > k + d/2$. Show that $H^s(\mathbb{R}^d) \subset C^k(\mathbb{R}^d)$ and that if $|\alpha| \leqslant k$,
$$\lim_{\|x\|\to\infty} |D^\alpha u(x)| = 0, \quad \forall u \in H^s(\mathbb{R}^d).$$
4. Show that for every $u \in \mathcal{E}'(\mathbb{R}^d)$ there exist $s \in \mathbb{R}$ and $C > 0$ such that
$$\left|\left\langle (1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}(u), \varphi\right\rangle\right| \leq C\|\varphi\|_{L^2}, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^d).$$
5. Conclude that $\mathcal{E}'(\mathbb{R}^d) \subset \bigcup_{s\in\mathbb{R}} H^s(\mathbb{R}^d)$.
6. Show that $H^{s_1}(\mathbb{R}^d)$ embeds continuously into $H^{s_2}(\mathbb{R}^d)$ for $s_1 \geq s_2$.
7. Suppose $s \in]d/2, d/2+1[$.
   (a) Show that for all $\alpha \in [0,1]$ and all $x, y, \xi \in \mathbb{R}^d$:
   $$|e^{ix\cdot\xi} - e^{iy\cdot\xi}| \leq 2|x-y|^\alpha|\xi|^\alpha.$$
   (b) Deduce that for all $\alpha \in]0, s-d/2[$, there exists $C(\alpha)$ such that for all $u \in H^s(\mathbb{R}^d)$
   $$\frac{|u(x)-u(y)|}{|x-y|^\alpha} \leq C(\alpha)\|u\|_{H^s}, \quad \forall x, y \in \mathbb{R}^d.$$
   (c) Conclude that $H^s(\mathbb{R}^d)$ embeds continuously into $C^\alpha(\mathbb{R}^d)$.

Exercise 2. *Some examples.*

1. Let $a, b \in \mathbb{R}$ such that $a < b$. Show that the indicator function $\mathbb{1}_{[a,b]}$ belongs to $H^r(\mathbb{R})$ for all $r < 1/2$ but not to $H^{1/2}(\mathbb{R})$.
2. Let u be the function on the half-disc $\Omega = D(0,R) \cap \{(x,y), y > 0\}$ defined by
$$\begin{aligned}]0,R[\times]0,\pi[&\to \mathbb{R} \\ (r,\theta) &\mapsto \theta. \end{aligned}$$
   Show that u does not belong to $H^1(\Omega)$.

Exercise 3. *Duality.* Let $s \in \mathbb{R}$.

1. Show that every $v \in H^{-s}(\mathbb{R}^d)$ defines an element ψ_v of $H^s(\mathbb{R}^d)^*$ by
$$\langle \psi_v, u\rangle = \int_{\mathbb{R}^d} \overline{\mathcal{F}}(v)\mathcal{F}(u),$$
   and $\|\psi_v\| \leq \|v\|_{H^{-s}}$.
2. Show that for every $\psi \in H^s(\mathbb{R}^d)^*$ there exists a unique $v \in H^{-s}(\mathbb{R}^d)$ such that
$$\langle \psi, u\rangle = \int_{\mathbb{R}^d} \overline{\mathcal{F}}(v)\mathcal{F}(u),$$
   and that, moreover, it satisfies $\|v\|_{H^{-s}} = \|\psi\|$. Conclude that $H^s(\mathbb{R}^d)^*$ can be identified with $H^{-s}(\mathbb{R}^d)$.
3. Let $\Omega \subset \mathbb{R}^d$ be an open set. Recall that $H^s(\Omega)$ is defined as the space of restrictions to Ω of elements of $H^s(\mathbb{R}^d)$, with the norm
$$\|u\|_{H^s(\Omega)} = \inf_{\substack{v\in H^s(\mathbb{R}^d)\\ v|_\Omega = u}} \|v\|_{H^s(\mathbb{R}^d)}.$$
   Show that for every integer $m \geq 1$, $H_0^m(\Omega)^*$ can be identified with $H^{-m}(\Omega)$.
   Hint: Recall that if F is a closed subspace of a Banach space E, then $F^* \cong E^*/F^\perp$.

Exercise 4. Let Ω be a bounded domain of $\mathbb{R}^d$.

1. Let $u \in H^1(\Omega)$. Prove that $\Delta u \in H^{-1}(\Omega) = H_0^1(\Omega)^*$.
2. Let $u_0 \in H^1(\Omega)$ and $f \in H^{-1}(\Omega)$. Show that there exists a unique $u \in H^1(\Omega)$ such that

$$\begin{cases} \Delta u &= f & \text{in } \Omega \\ u &= u_0, & \text{on } \partial\Omega. \end{cases}$$

   Here the first identity is to be understood in the sense of distributions and the second one in the sense of traces: $u - u_0 \in H_0^1(\Omega)$.
   Hint: Consider first the case $u_0 = 0$.