Sheet 2 - Banach-Steinhaus, open map and closed graph

Exercise 1. Let E and F be two Banach spaces. Let us consider a sequence of continuous linear mappings $(T_n : E \to F)$ such that for every $x \in E$, $(T_n x)$ converges towards a limit denoted by Tx.

1. Show that $x \mapsto Tx$ is linear.
2. Show that $\sup_n \|T_n\| < +\infty$. Deduce that T is continuous.
3. Show that
$$\|T\| \leqslant \liminf_{n\to+\infty} \|T_n\|.$$

Exercise 2. Let E be a Banach space. Let us consider two closed subspaces F and G such that $F+G$ is closed too.

1. Prove the existence of a finite constant $C > 0$ such that, for every $z \in F+G$, there exists $x \in F$ and $y \in G$ satisfying
$$\|x\| \leqslant C\|z\|, \qquad \|y\| \leqslant C\|z\| \qquad \text{and} \qquad z = x + y.$$
2. Deduce that there exists a finite constant $C' > 0$ such that, for all $x \in E$
$$d(x, F\cap G) \leqslant C'\Big[d(x,F) + d(x,G)\Big].$$

Exercise 3 (A space with two norms)**.** 1. Let E be a vector space endowed with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ such that both $(E, \|\cdot\|_1)$ and $(E, \|\cdot\|_2)$ are Banach spaces. Assume the existence of a finite constant $C > 0$ such that
$$\forall x \in E, \quad \|x\|_1 \leqslant C\|x\|_2.$$
Prove that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

2. Consider the classical norms $\|\cdot\|_{L^1}$ and $\|\cdot\|_{L^2}$ over $L^2([0,1])$. Consider also the linear map
$$T: \begin{array}{ccc} (L^2, \|\cdot\|_{L^1}) & \to & (L^2, \|\cdot\|_{L^2}) \\ f & \mapsto & f. \end{array}$$
   (a) Verify that $\|\cdot\|_{L^1}$ is indeed a norm on L^2.
   (b) Show that the graph of T is closed.
   (c) Prove that T is not continuous.
   (d) What can we deduce from above ?

Exercise 4 (Fourier series of continuous functions)**.** We denote by $C(\mathbb{T})$ the Banach space of continuous functions on the torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. We are going to address the following question: *Is it true that for all $f \in C(\mathbb{T})$, the Fourier series of f converges pointwise to f?*
Recall that if $f \in C(\mathbb{T})$, the Fourier series' n^{th} partial sum at a point x is given by
$$s_n(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_n(x-t)dt,$$
where D_n is the Dirichlet kernel, defined by
$$D_n(t) = \sum_{k=-n}^{n} e^{ikt}.$$
Observe that D_n is an $\mathbb{R}$-valued function. For each $n \in \mathbb{N}$ and f in $C(\mathbb{T})$, we define $\Lambda_n f = s_n(f;0)$.

1. Let us prove that the norm of Λ_n in the dual of $(C(\mathbb{T}), \|\cdot\|_\infty)$ is equal to
$$\|D_n\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_n(t)|\,dt.$$
   (a) Prove that $\|\Lambda_n\| \leq \|D_n\|_1$.
   (b) We fix n and define $g = \mathbb{1}_{\{D_n \geq 0\}} - \mathbb{1}_{\{D_n < 0\}}$. Show that $\Lambda_n(f_j) \to \|D_n\|_1$ for a suitable family $f_j \in C(\mathbb{T})$ satisfying $f_j D_n \to g D_n$ pointwise.

2. Prove that
$$D_n(t) = \frac{\sin\left(\left(n+\frac{1}{2}\right)t\right)}{\sin\left(\frac{t}{2}\right)}, \quad \forall t \in (-\pi, \pi).$$
Hint: Consider $e^{it/2}D_n - e^{-it/2}D_n$.

3. Prove that $\|D_n\|_1 \to \infty$ as $n \to \infty$.
Hint: Recall that $|\sin(t)| \leq |t|$.

4. Conclude.

Exercise 5 (Fourier coefficients of L^1 functions)**.** For f in $L^1(\mathbb{T})$, we define $\hat{f} : \mathbb{Z} \to \mathbb{C}$ by
$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt.$$

We denote by c_0 the space of complex valued functions on $\mathbb{Z}$ tending to 0 at $\pm\infty$.

1. Prove that $(c_0, \|\cdot\|_\infty)$ is a Banach space.

2. Prove that, for all $f \in L^1(\mathbb{T})$, $\hat{f} \in c_0$.
Hint: Recall that the trigonometric polynomials ($\sum_{k=-n}^{n} a_k e^{ikt}$) *are dense in* $C(\mathbb{T})$.

Now we want to study the converse. In other words, is every element of c_0 the sequence of Fourier coefficients of a function in $L^1(\mathbb{T})$?

3. Prove that $\Lambda : f \to \hat{f}$ defines a bounded linear map from $L^1(\mathbb{T})$ to c_0.

4. Prove that Λ is injective.
Hint: You may use the fact that, for every measurable set $A \subseteq \mathbb{T}$, *there exists a sequence of continuous functions* $g_n : \mathbb{T} \to [0,1]$ *such that* $g_n \to \mathbb{1}_A$ *almost everywhere.*

5. Show that $f \to \hat{f}$ is not onto.
Hint: You may use the Dirichlet kernel defined in the previous exercise.