Sheet 3 - Locally convex topological vector spaces and analytic Hahn-Banach theorem

Exercise 1. Let E be a locally convex topological vector space (l.c.t.v.s.) whose topology is induced by a (separating) countable family of semi-norms $(p_n)_{n\in\mathbb{N}}$. We define

$$d(x,y) = \sum_{n\in\mathbb{N}} \frac{1}{2^n} \frac{p_n(x-y)}{1+p_n(x-y)}$$

Let us prove that the topology induced by d and the topology induced by the family of seminorms coincide.

1. Show that $g : [0,\infty) \to \mathbb{R}$ defined by $g(t) = \frac{t}{1+t}$ is an increasing sub-additive function and give its image. Deduce that d is a translation invariant distance on E.
2. Give a basis of neighbourhoods of 0_E for the topology induced by the family of semi-norms, and show that every neighbourhood of 0_E contains an open ball for the distance d.
3. Show that every open ball for the distance d centered on 0_E contains a neighbourhood of 0_E for the topology induced by the family of semi-norms.
4. Conclude.

Recall the following definitions of bounded sets:

- We say that a subset A of a topological vector space X is bounded if, for every neighbourhood V of 0_X, there exists $\lambda > 0$ such that $A \subset \lambda V$.
- We say that a subset A of a metric space X is bounded if there exists $r > 0$ such that $A \subset B(0_X, r)$.

5. Are both definitions equivalent? *Take a look at Exercise 2 for a concrete example.*

Exercise 2. Let Ω be an open subset of $\mathbb{R}^N$ and $(K_n)_{n\in\mathbb{N}}$ a sequence of compact subsets of Ω such that $K_n \subseteq \mathring{K}_{n+1}$ and $\bigcup K_n = \Omega$. The space $C(\Omega)$ of continuous functions on Ω is a l.c.t.v.s. for the topology induced by the family of seminorms

$$p_n(f) = \sup_{x\in K_n} |f(x)|, \quad n \in \mathbb{N}.$$

1. Prove that $(C(\Omega), (p_n))$ is a Fréchet space (i.e. complete for the distance defined in Exercise 1).
Hint: Fix $n_0 \in \mathbb{N}$ and prove that every Cauchy sequence in $C(\Omega)$ defines a Cauchy sequence in $C(K_{n_0})$ by restriction.
2. Recall the first definition of bounded set from Exercise 1. Prove that if B is a subset of equibounded functions of $C(\Omega)$ (*i.e.* $\sup_{f\in B} \|f\|_\infty < \infty$), B is bounded.
3. Take f_n a sequence of continuous function on Ω such that $f_n : \Omega \to [0,n]$, $f_n = 0$ on K_n and $f_n = n$ on $\Omega \setminus K_{n+1}$. Show that $\cup_n \{f_n\}$ is a bounded subset of $C(\Omega)$.
4. Prove that $C(\mathbb{R})$ is not locally bounded, that is, the origin does not have a bounded neighbourhood. *Hence the subsets of the previous questions are not neighbourhoods of the origin !*

Now consider the space $C^\infty(\Omega)$ of smooth functions on Ω. This is also a l.c.t.v.s. for the family of seminorms

$$p_{n,\alpha}(f) = \sup_{x\in K_n} |D^\alpha f(x)|, \quad n \in \mathbb{N},\ \alpha \in \mathbb{N}^N,$$

where $D^\alpha f = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_N}^{\alpha_N}$.

5. Prove that the derivation $f \mapsto \partial_{x_i} f$ defines a continuous linear operator on $C^\infty(\Omega)$ for all $i = 1, ..., N$.
6. Deduce that every linear differential operator (i.e. $P(\partial_{x_1}, ..., \partial_{x_N})$ with $P \in \mathbb{R}[X_1, ..., X_N]$) is continuous on $C^\infty(\Omega)$.

Exercise 3. Let E be a normed space.

1. Let G be a subspace and $g : G \to \mathbb{R}$ a continuous linear form. Show that there exists a continuous linear form f over E that extends g, such that
$$\|f\|_{E^*} = \|g\|_{G^*}.$$
2. In this question and only in this one, we assume that E is a Hilbert space. Show that such an extension is unique.
Hint: recall that $E = \bar{G} \oplus G^\perp$.

3. Show that for every $x \in E$, there exists $f \in E^*$ such that $\|f\|_{E^*} = 1$ and $f(x) = \|x\|$.
4. Deduce that, for every $x \in E$,
$$\|x\| = \max_{\substack{f \in E^* \\ \|f\|_{E^*} \leqslant 1}} |f(x)|.$$
Remark: In general it is not true that
$$\|f\|_{E^*} = \max_{\substack{x \in E \\ \|x\| \leqslant 1}} |f(x)|.$$
In fact, James' theorem asserts that, for Banach spaces, this characterizes reflexivity.
5. Suppose that E is a Banach space. Let B^* be a subset of E^* such that
$$\forall x \in E, \quad \sup_{f \in B^*} f(x) < +\infty.$$
Show that B^* is bounded.
6. (Continuing.) Let B be a subset of E such that
$$\forall f \in E^*, \quad \sup_{x \in B} f(x) < +\infty.$$
Prove that B is bounded.

Exercise 4. Let $p \in]0,1[$ (mind that this range is unusual !). One denotes L^p the set of real-valued measurable functions f defined over $[0,1]$, modulo almost everywhere vanishing functions, for which the following quantity is finite:
$$\|f\|_p = \left(\int_0^1 |f|^p \mathrm{d}x \right)^{\frac{1}{p}}.$$

1. (a) For every $a, b \geqslant 0$, show that $(a+b)^p \leqslant a^p + b^p$.
   (b) Let $f \in L^p$ and $n \in \mathbb{N}^*$ be given. Prove that there exists a partition of $[0,1]$ in n intervals $I_1, \ldots, I_n$ such that
$$\int_{I_j} |f|^p \mathrm{d}x = \frac{1}{n} \|f\|_p^p,$$
   and compute $\|f \mathbb{1}_{I_j}\|_p$.
2. Show that L^p is a vector space and that $d(f,g) = \|f-g\|_p^p$ is a distance. Prove that (L^p, d) is complete.
3. Let $q < 0$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $f, g : [0,1] \to \mathbb{R}$ be two measurable functions such that, almost everywhere, $f \geqslant 0$ and $g > 0$. Show that
$$\int_0^1 |fg| \mathrm{d}x \geqslant \|f\|_p \left\| \frac{1}{g} \right\|_{|q|}^{-1}.$$
4. Let $f_1, \ldots, f_n$ be L^p-functions. Prove the following inequalities
$$\sum_{i=1}^n \|f_i\|_p \leqslant \left\| \sum_{i=1}^n |f_i| \right\|_p \quad \text{and} \quad \left\| \sum_{i=1}^n f_i \right\|_p \leqslant n^{\frac{1}{p}-1} \sum_{i=1}^n \|f_i\|_p.$$
Hint: For the latter inequality, one may first prove that if $\theta \geqslant 1$ and $a_1, \ldots, a_n \geqslant 0$, then one has
$$\left(\sum a_i \right)^\theta \leqslant n^{\theta - 1} \sum a_i^\theta.$$
5. Verify that the constant $n^{\frac{1}{p}-1}$ is accurate in the latter inequality.
6. Prove that the only convex open domain in L^p containing $f \equiv 0$ is L^p itself. Deduce that the Fréchet space L^p is not locally convex.
7. Show that the (topological) dual space of L^p reduces to $\{0\}$.
8. Let N, a semi-norm over L^p, be continuous for the topology associated with d.
   (a) Show that there exists $C > 0$ such that for every $f \in L^p$
$$N(f) \leqslant C \|f\|_p.$$
   (b) Deduce that for all $f \in L^p$, $N(f) = 0$. *Hint: One might consider the smallest constant C in the previous question.*